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SINGULAR BGG COMPLEXES FOR THE SYMPLECTIC CASE

DOCTORAL THESIS

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SINGULARNI BGG KOMPLEKSI ZA SIMPLEKTIČKI SLUČAJ

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Za Ninu, Niku i Ivu

Summary

Let G be a semisimple Lie group and P its parabolic subgroup. It is well known that any finite-dimensional simple G-module allows a resolution by invariant differential operators acting between direct sums of homogeneous bundles over the generalized flag manifold G/P. Such a resolution is called the Bernstein-Gelfand-Gelfand (BGG for short) resolution. In the dual setting, this corresponds to the resolution of a finite-dimensional simple $\mathfrak{g}$-module by direct sums of generalized Verma modules, which is also called the BGG resolution. Modules in the resolution have a regular infinitesimal character. Using the Penrose transform, we construct analogues of such resolutions in certain singular infinitesimal characters, in the holomorphic geometric setting, for type C. We take G to be the symplectic group, P its $|1|$-graded parabolic subgroup, so that G/P is the Lagrangian Grassmannian. We explicitly describe the operators in the resolution, and determine their order. We prove the exactness of the constructed complex over the big affine cell.

Keywords: BGG; Resolution; Homogeneous bundles; Invariant differential operators; Generalized Verma modules; Symplectic group, Lagrangian Grassmannian; Penrose transform

Prošireni sažetak

Neka je G poluprosta Liejeva grupa, te P njena parabolička podgrupa. Poznato je da svaki konačno-dimenzionalan ireducibilan G-modul dozvoljava rezoluciju invarijantnim diferencijalnim operatorima koji djeluju među homogenim svežnjevima nad generaliziranom mnogostrukošću zastava G/P. Takve rezolucije zovu se Bernstein-Gelfand-Gelfand (kraće: BGG) rezolucije. U dualnom pristupu, to odgovara rezoluciji konačno-dimenzionalog ireducibilnog $\mathfrak{g}$-modula pomoću direktnih suma generaliziranih Vermaovih modula, koja se također zove BGG rezolucija. Moduli u rezoluciji su regularnog infinitezimalnog karaktera. Koristeći Penroseovu transformaciju, konstruiramo analogone takvih rezolucija u izvjesnim singularnim infinitezimalnim karakterima, u holomorfnoj geometrijskoj varijanti, za tip C. Dakle, G je simplektička grupa, P njezina $|1|$-graduirana parabolička podgrupa, te je G/P Lagrangeov Grassmannian. Eksplicitno opisujemo operatore u rezoluciji, te određujemo njihov red. Dokazujemo egzaktnost dobivenog kompleksa nad velikom afinom ćelijom.

BGG komplekse uveli su Joseph Bernstein, Israel Gelfand i Sergei Gelfand, u [BGG75]. Za bilo koju poluprostu Liejevu algebru $\mathfrak{g}$ (konačno-dimenzionalnu, nad $\mathbb{C}$), konstruirali su, za svaki konačno-dimenzionalan ireducibilan $\mathfrak{g}$-modul F, rezoluciju koja se sastoji od direktnih suma Vermaovih modula, koji su inducirani s Borelove podalgebre. Najveće težine Vermaovih modula koje se javljanju u rezoluciji od F podudaraju se s elementima orbite Weylove grupe (s afinim djelovanjem) najveće težine od F, a sama rezolucija ima istu strukturu usmjerenog grafa kao i Weylova grupa s Bruhatovim uređajem. Ta konstrukcija je generalizirana u radovima Jamesa Lepowskoga [Lep77] i Alvany Rocha-Caridi u [RC80], s Borelovog slučaja na slučaj proizvoljne paraboličke podalgebre. Oni su pokazali kako konstruirati BGG rezoluciju koja se sastoji od direktnih suma generaliziranih Vermaovih modula, koji su inducirani s proizvoljne paraboličke podalgebre $\mathfrak{p}$. Najveće težine generaliziranih Vermaovih modula koji se javljanju u rezoluciji odgovaraju $\mathfrak{p}$-dominantnim elementima u orbiti Weylove grupe, a oni se pak mogu parametrizirati određenim podskupom Weylove grupe koji se zove Hasseov dijagram paraboličke podalgebre $\mathfrak{p}$. Hasseov dijagram je neovisan o početnom $\mathfrak{g}$-modulu F, stoga se može zaključiti kako za fiksiranu Liejevu algebru $\mathfrak{g}$ i paraboličku podalgebru $\mathfrak{p}$, sve BGG rezolucije (u regularnom infinitezimalnom karakteru) imaju isti oblik.

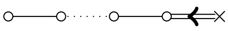
Poznato je da homomorfizmi generaliziranih Vermaovih modula korespondiraju (kontravarijantno) invarijatnim diferencijalnim operatorima koji djeluju među snopovima prereza homogenih vektorskih svežnjeva nad generaliziranom mnogostrukošću zastava G/P. Na ovoj geometrijskoj strani, BGG rezolucije su se prvo pojavile u radu Michaela Eastwooda [Eas85], no tada su bile konstruirane ad hoc, te je njihova veza s čisto algebarskim Vermaovim modulima uočena kasnije, [ER87] i Robert Bastonov članak [Bas91]. U posebnim slučajevima kada je parabolička podalgebra $|1|$-graduirana, što je ekvivalentno s činjenicom da G/P ima strukturu Hermitski simetričnog prostora, holomorfni tangencijalni svežanj nad G/P je ireducibilan, te se BGG rezolucija u trivijalnom infinitezimalnom karakteru podudara s holomorfnim de Rhamovim kompleksom. U višoj graduaciji, BGG rezolucija u trivijalnom infinitezimalnom karakteru je potkompleks holomorfnog de Rhamovog kompleksa – objekti su nižih dimenzija, no diferencijalni operatori mogu biti reda višeg od jedan. BGG kompleksi su sistematično proučavani u radu Andreasa Čapa, Jana Slováka i Vladimíra Součeka [ČSS01]. Oni su konstruirali BGG komplekse u općenitijoj teoriji zakrivljenih paraboličkih geometrija, [ČS09], za koje je naš G/P poseban slučaj – tzv. ravni model. U ravnome modelu, njihova konstrukcija daje, za svaki konačno-dimenzionalan G-modul F, rezoluciju konstantnog snopa definiranog s F, sastavljenu od direktnih suma homogenih vektorskih svežnjeva i invarijantnih diferencijalnih operatora. Ova rezolucija je u određenom smislu dual od one koju su konstruirali Lepowsky i Rocha-Caridi. Nadalje, ove BGG rezolucije su lokalno egzaktne, dakle, rezolucije su u kategoriji snopova. Konstrukcija takvih BGG rezolucija je dalje proširena i pojednostavljena u radovima Davida Calderbanka and Tammoa Diemera u [CD01].

Mnogi važni diferencijalni operatori djeluju među homogenim svežnjevima singularnog infinitezimalnog karaktera (na primjer skalarni valni operator na prostoru Minkowskog, Dirac-Weylovi operatori na konformnim mnogostrukostima, Dirac-Feuterovi operatori na kvaternionskim mnogostrukostima, ...), stoga se ne mogu pronaći u prije spomenutim BGG rezolucijama. Ne postoje općenite konstrukcije tog tipa rezolucija u singularnom infinitezimalnom karakteru. U singularnom infinitezimalnom karakteru pojavljuje se nekoliko problema, jedan od kojih je nedostatak standardnih diferencijalnih operatora. U BGG rezolucijama u regularnom infinitezimalnom karakteru, svi diferencijalni operatori su standardni, što po definiciji znači da se mogu dobiti kao direktne slike, s obzirom na fibraciju $G/B \to G/P$ s pune mnogostrukosti zastava. Takvi operatori su, barem u principu potpuno poznati. No, singularni infinitezimalni karakter ima netrivijalan stabilizator u Weylovoj grupi, te stoga navedena direktna slika poništava mnoge diferencijalne operatore. Stoga, da bi se konstruirala rezolucija iz singularne orbite, potrebno je naći mnogo nestandardnih diferencijalnih operatora. Pokazalo se da je Penroseova transformacija, u formi koju su definirali Robert Baston i Michael Eastwood u [BE16], izrazito korisno sredstvo u konstrukciji nestandardnih invarijantnih diferencijalnih operatora.

Robert Baston je započeo konstrukcije BGG rezolucija u singularnom infinitezimalnom karakteru pomoću Penroseove transformacije u [Bas92]. Radio je na kvaternionskim mnogostrukostima na kojima takozvani Cauchy-Riemann-Feuterov operator ima istu ulogu koju ima Cauchy-Riemannov operator na kompleksnim mnogostrukostima (jezgre tih operatora definiraju „dobre“ funkcije). U niskoj dimenziji, Cauchy-Riemann-Feuterov operator se podudara sa Dirac-Weylovim operatorom koji djeluje na spinorima, a taj je pak dobro poznat u teoriji twistora. Posebno, poznato je da jezgra tog operatora korespondira određenoj snopovskoj kohomološkoj grupi na prostoru twistora, a ta je korespondencija poznata kao klasična Penroseova transformacija. Baston je iskoristio Penroseovu transformaciju da konstruira analogon Dolbeaultovog kompleksa nad kvaternionskim mnogostrukostima – konačnu lokalno egzaktnu rezoluciju s homogenim svežnjevima i invarijantnim diferencijalnim operatorima, u kojoj je prvi operator točno Cauchy-Riemann-Feuterov. To je upravo BGG rezolucija u singularnom infinitezimalnom karakteru za tip A (kvaternionska mnogostrukost se modelira kao određen kvocijent od $\mathrm{GL}(2n+2,\mathbb{C})$, s Dynkinovom notacijom). Ponovo kao i u regularnom slučaju, najveće težine koje se javljaju u rezoluciji se podudaraju s Levi-dominantnim dijelom singularne orbite početne težine pod djelovanjem Weylove grupe. Iako se radi o $|1|$-graduiranoj situaciji, u singularnoj BGG rezoluciji se pojavljuju diferencijalni operatori reda dva, i točno ti su nestandardni.

Lukáš Krump i Vladimír Souček proučavali su više graduiranu situaciju u [KS06], gdje je $G = \mathrm{Spin}(2+2n,\mathbb{C})$ i njena realna forma $\mathrm{Spin}(2,2n;\mathbb{R})$. Rezultat je popriličnо drugačiji nego u kvaternionskom slučaju – ovdje se ne mogu sve točke u singularnoj orbiti pronaći u jednoj singularnoj BGG rezoluciji, te različite singularne BGG rezolucije mogu imati zajedničke točke. Vidi i [Sal17a], [Sal17b].

Nedavno su Pavle Pandžić i Vladimír Souček u [PS16] konstruirali singularne BGG rezolucije nad velikom afinom ćelijom, za tip A, te za sve $|1|$-graduirane paraboličke podalgebre, tj. sve kompleksne Grassmanniane. Iz konstrukcije je vidljivo da singularna BGG rezolucija pokriva cijelu singularnu orbitu, te nadalje, da singularna BGG rezolucija ima strukturu usmjerenog grafa istu kao i regularna u nižem rangu. Taj defekt u rangu jednak je dvostrukom broju zidova u kojima početna težina singularna leži. Također su opisani redovi diferencijalnih operatora koji se javljaju u rezoluciji.

Slični rezultati dobiveni su i za tip C, čime se ova disertacija bavi. Ovdje je G simplektička grupa $\mathrm{Sp}(2n,\mathbb{C})$. Tu postoji samo jedna standardna $|1|$-graduirana parabolička podalgebra $\mathfrak{p}$, te je G/P Lagrangeov Grassmannian (u Dynkinovoj notaciji: ). No, ovdje imamo dvije vrste singularnosti: singularnost prve vrste uključuje samo kratke proste korijene, dok singularnost druge vrste uključuje i dugi prosti korijen. U samoj kontrukciji pretpostavljamo da je infinitezimalni karakter semi-regularan, tj. ortogonalan na točno jedan prosti korijen.

Glavno sredstvo korišteno u konstrukciji invarijantnih diferencijalnih operatora jest

Penroseova transformacija, u formi koju su definirali Robert Baston i Michael Eastwood u [BE16]. Izabiremo takozvani prostor twistora G/R, te za zadani integralni semi-regularni infinitezimalni karakter λ izabiremo konjugat od λ s obzirom na Weylovu grupu od G koji je $\mathfrak{r}$-dominantan, pa stoga definira homogeni vektorski svežanj E nad G/R. Formiramo dvostruku fibraciju

$$\begin{tikzcd} & G/Q \arrow[dl] \arrow[dr] & \\ G/R & & G/P, \end{tikzcd}$$

gdje je $Q = R \cap P$. Izabravši „dobar“ otvoren podskup $X \subseteq G/P$ (najčešće afin ili Steinov), dobijemo restringiranu dvostruku fibraciju

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ Z & & X. \end{tikzcd}$$

Određeni tehnički uvjeti moraju biti zadovoljeni: vlakna lijeve restringirane fibracije moraju biti glatko kontraktibilna, dok vlakna desne fibracije moraju biti kompaktna. Penroseova transformacija se tipično provodi u dva koraka:

(a) Povlak (inverzna slika) snopa prereza od E sa Z na Y. Grubo govoreći, svaka kohomološka klasa na Z se može shvatiti kao kohomološka klasa na Y koja je konstantna na vlaknima od $Y \to Z$. Taj uvjet konstantnosti na vlaknima može se iskazati u terminima diferencijalnih jednadžbi. Ako su vlakna kontraktibilna, povlak je izomorfizam na kohomologiji, [Buc83].

(b) Potisak (direktna slika) na X. Inverznu sliku od E na Y rezolviramo takozvanom relativnom BGG rezolucijom $\Delta^\bullet$. To je zapravo kolekcija regularnih BGG rezolucija, po jedna na svakom vlaknu, koje su homogeno „spojene“. Detaljan opis relativnih BGG nizova dan je u [ČS16] i [ČS15]. Zatim uzimamo više (derivirane) direktne slike od $\Delta^\bullet$ duž fibracije $Y \to X$. One se lako računaju koristeći Bott-Borel-Weilov teorem (relativnu verziju). Ključni dio dolazi iz homološke algebre: takozvani spektralni niz hiperkohomologije (zajedno sa Lerayevim spektralnim nizom) ima na E_1 nivou globalne prereze viših direktnih slika od $\Delta^\bullet$, te konvergira u kohomologiju $H^\bullet(Z, E)$ na prostoru twistora. Diferencijali u tom sprektralnom nizu su upravo invarijantni diferencijalni operatori. Na nivou E_1 imamo standardne diferencijalne operatore, budući da su oni direktne slike diferencijala iz relativne BGG rezolucije. Ali na višim nivoima javljaju se nestandardni diferencijalni operatori. Penroseova transformacija koristi navedeni spektralni niz tako da interpretira $H^\bullet(Z, E)$ u terminima invarijantnih diferencijalnih jednadžbi na X.

Kao i u [PS16], nestandardni diferencijalni operatori potrebni za singularnu BGG rezoluciju i u našem slučaju pojavljuju se na zadnjem nivou tog spektralnog niza, prije nego se isti stabilizira. Stoga kohomološke grupe $H^\bullet(Z, E)$ mjere razinu ne-egzaktnosti našeg singularnog kompleksa, do na određen pomak u stupnju. Dakle, problem se svodi na dokaz iščezavanja kohomologije $H^\bullet(Z, E)$ u određenim stupnjevima. To je riješeno Lerayjevim teoremom: Z se može pokriti s precizno određenim brojem kohomološki trivijalnih otvorenih podskupova, što daje traženo iščezavanje kohomologije. U holomorfnoj kategoriji, takvi podskupovi poznati su kao Steinovi podskupovi, [GR12].

Ključne riječi: BGG; Rezolucija; Homogeni svežnjevi; Invarijantni diferencijali operatori; Generalizirani Vermaovi moduli; Simplektička grupa, Lagrangeov Grassmannian; Penroseova transformacija

Contents

Introduction

The BGG complexes were first introduced by Joseph Bernstein, Israel Gelfand and Sergei Gelfand, in [BGG75]. For a general semisimple Lie algebra $\mathfrak{g}$ (finite-dimensional, over $\mathbb{C}$), they constructed for each finite-dimensional irreducible $\mathfrak{g}$-module F a resolution consisting of direct sums of Verma modules, which are induced from the Borel subalgebra. Highest weights of the Verma modules in the BGG resolution of F are precisely the elements of the Weyl group orbit (by the affine action) of the highest weight of F, and the resolution has the same directed-graph structure as the Weyl group with Bruhat order. This construction was generalized by James Lepowsky in [Lep77] and Alvany Rocha-Caridi in [RC80], from the Borel case to the case of any parabolic subalgebra. They showed how to construct BGG resolutions consisting of direct sums of generalized Verma modules, that can be induced from any standard parabolic subalgebra $\mathfrak{p}$. The highest weights of generalized Verma modules appearing in the resolution correspond to $\mathfrak{p}$-dominant elements in the Weyl group orbit, and those can be parametrized by a certain subset of the Weyl group, called the Hasse diagram attached to the parabolic subalgebra $\mathfrak{p}$. The Hasse diagram is independent of the starting $\mathfrak{g}$-module F, so one can conclude that for a fixed Lie algebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{p}$, all BGG resolutions (in regular infinitesimal character) have the same shape.

It is well known that homomorphisms of generalized Verma modules correspond (contravariantly) to invariant differential operators acting between sheaves of sections of homogeneous vector bundles over the generalized flag manifold G/P. In this geometric setting, BGG resolutions appeared first in Michael Eastwood's work in [Eas85], although in this article they were created ad hoc and the connection to the purely algebraic Verma modules was realised only later, eg. [ER87] and Robert Baston's [Bas91]. In a special case when the parabolic subalgebra $\mathfrak{p}$ is $|1|$-graded, which is equivalent to G/P having structure of a Hermitian symmetric space, the holomorphic tangent bundle of G/P is irreducible, and the BGG resolution in trivial infinitesimal character coincides with the holomorphic de Rham complex. In higher grading, the BGG resolution in trivial infinitesimal character is a subcomplex of the holomorphic de Rham complex – objects are of smaller dimensions, but differential operators can be of order greater than one.

BGG complexes were systematically studied in the work of Andeas Čap, Jan Slovák and Vladimír Souček [ČSS01]. They constructed BGG complexes in the more general theory of curved parabolic geometries [ČS09], for which our G/P is a special case called the flat model. In the flat model, their construction yields, for each finite-dimensional G-module F, a resolution of the constant sheaf over G/P defined by F, by direct sums of homogeneous vector bundles and invariant differential operators. This BGG resolution is in a certain sense dual to those in the work of Lepowsky and Rocha-Caridi. Moreover, these BGG complexes are locally exact, so they are resolutions in the sheaf category. Construction of such BGG complexes was then extended and simplified by David Calderbank and Tammo Diemer in [CD01].

Many important invariant differential operators act between homogeneous bundles of singular infinitesimal character (for example, scalar wave operator on the Minkowski space, Dirac-Weyl operators on conformal manifolds, Dirac-Feuter operators on quaternionic manifolds, ...), hence they cannot be found in the above mentioned BGG resolutions. There are no such general constructions of resolutions in singular infinitesimal character. Several problems emerge in singular infinitesimal character, one of which is a lack of standard differential operators. In BGG resolutions in regular infinitesimal character, all differential operators are standard ones, which means by definition that they can be obtained by the direct image with respect to the fibration $G/B \to G/P$ from the full flag manifold. These operators are, at least in principle, completely known. But a singular infinitesimal character has a nontrivial stabilizer in the Weyl group, and consequently, the above direct image annihilates many differential operators. So, in order to make a resolution out of the singular orbit, one must invent many non-standard invariant differential operators. It turned out that the Penrose transform, in the form given by Robert Baston and Michael Eastwood in [BE16], is a particularly useful tool for the construction of non-standard invariant differential operators.

Robert Baston initiated the construction of BGG resolutions in singular infinitesimal character via Penrose transform in [Bas92]. He worked on quaternionic manifolds, on which the so called Cauchy-Riemann-Feuter operator has the same role as the Cauchy-Riemann operator on complex manifolds (kernels of these operators define the notion of a "good" function). In low dimension, the Cauchy-Riemann-Feuter operator coincides with the Dirac-Weyl operator acting on spinors, which is well known in twistor theory. In particular, it is known that the kernel of this operator corresponds to a certain holomorphic sheaf cohomology group on the twistor space, by a correspondence known as the classical Penrose transform. Baston used the Penrose transform to construct analogues of the Dolbeault complex over a quaternionic manifold – a finite locally exact resolution by homogeneous bundles and invariant differential operators, in which the first operator is precisely the Cauchy-Riemann-Feuter operator. This is a BGG resolution in singular infinitesimal character, in the type A (the quaternionic manifold is modeled as a certain quotient of $\mathrm{GL}(2n+2,\mathbb{C})$, having the Dynkin diagram notation $\circ\!-\!\!\times\!\!-\!\circ\!-\!\circ\cdots\circ$). Again, as in the regular case, highest weights in the resolution completely exhaust the Levi-dominant part of the singular Weyl group orbit of the starting weight. Even though this is a $|1|$-graded situation, some differential operators in the singular BGG resolution are of order two, and exactly those are non-standard.

Some observations have been done in higher grading by Lukáš Krump and Vladimír Souček in [KS06], where $G = \mathrm{Spin}(2+2n, \mathbb{C})$ and its real form $\mathrm{Spin}(2, 2n; \mathbb{R})$. The result is quite different from the quaternionic case – here not all points in a singular orbit can be found in one singular BGG resolution, and singular BGG resolutions can intersect. See also [Sal17a], [Sal17b].

Recently, in [PS16] Pavle Pandžić and Vladimír Souček constructed singular BGG resolutions over the big affine cell in type A, for all $|1|$-graded parabolic subalgebras, that is, all complex Grassmannians. From the construction, it is visible that singular BGG resolutions cover the whole singular orbit, and moreover, that the singular BGG resolution has the same directed-graph structure as a regular one, but of lower rank. The defect in rank is precisely twice the number of walls on which the starting singular weight sits. The orders of the differential operators in the resolution are also described.

Similar results are also obtained in type C, which is the topic of this dissertation. Here G is the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. There is only one standard $|1|$-graded parabolic subalgebra $\mathfrak{p}$ in this case, and G/P is the Lagrangian Grassmannian (in Dynkin notation: $\circ\!-\!\circ \cdots \circ\!-\!\circ\!\Leftarrow\!\times$). But here we have two types of singularities: singularity of the first kind, involving only short simple roots, and singularity of the second kind, involving also the long simple root. In the construction, we assume that the infinitesimal character is semi-regular, that is, orthogonal to exactly one simple root.

Our main tool for construction of invariant differential operators is the Penrose transform, in the form given by Robert Baston and Michael Eastwood in [BE16]. We choose a so called twistor space G/R, and for an integral semi-regular infinitesimal character λ we choose a Weyl group conjugate of λ which is $\mathfrak{r}$-dominant and therefore defines a homogeneous vector bundle E over G/R. We form the double fibration

$$\begin{tikzcd} & G/Q \arrow[dl] \arrow[dr] & \\ G/R & & G/P, \end{tikzcd}$$

where $Q = R \cap P$. By choosing a "nice" open subset $X \subseteq G/P$ (usually affine or Stein), we get a restricted double fibration

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ Z & & X. \end{tikzcd}$$

Certain technical conditions must be imposed here: fibers of the restricted left fibration should be smoothly contractible, and fibers of the right one should be compact. The

Penrose transform typically consists of two steps:

(a) Pull-back (inverse image) of the sheaf of sections of E from Z to Y. Roughly speaking, any cohomology class on Z is now regarded as a cohomology class on Y, constant on the fibers of $Y \to Z$. This constancy can be interpreted by means of differential equations. If the fibers are contractible, then this pull-back is an isomorphism on cohomology, [Buc83].

(b) Push-down (direct image) to X. Now we resolve the inverse image of E on Y by the so called relative BGG resolution $\Delta^\bullet$. This is just a collection of regular BGG resolutions, one on each fiber, that are glued together homogeneously. A detailed treatment of the relative BGG sequences in given in [ČS16] and [ČS15]. We take higher (derived) direct images of $\Delta^\bullet$ along the fibration $Y \to X$. These are easily calculated by (the relative version of) Bott-Borel-Weil theorem. The crucial part comes from homological algebra: the so called hypercohomology spectral sequence (together with Leray spectral sequence) has on the E_1 level global sections of the higher direct images of $\Delta^\bullet$, and converges to the cohomology $H^\bullet(Z, E)$ on the twistor space. Differentials in this spectral sequence turn out to be invariant differential operators. On the level E_1 we have standard differential operators, since they are direct images of differentials in the relative BGG resolution on Y. But higher levels give non-standard differential operators. The Penrose transform uses this spectral sequence to interpret $H^\bullet(Z, E)$ in terms of invariant differential equations on X.

As in [PS16], differential operators wanted for the singular BGG resolution appear in the last level of this spectral sequence before it stabilizes. Therefore, the cohomology groups $H^\bullet(Z, E)$ measure the extent of failure of exactness of our complex, up to a certain shift in degree. So the problem reduces to obtaining the vanishing of the cohomology $H^\bullet(Z, E)$ in the correct degrees. This is settled by Leray's theorem: Z can be covered with a precise number of cohomologically trivial open subsets, which implies required vanishing of the cohomology. In the holomorphic category, these are known as Stein subsets, [GR12].

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Chapter 1

Preliminaries

Let G be a semisimple complex Lie group, connected and simply connected, and $\mathfrak{g}$ its Lie algebra, also complex. Fix once and for all a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h}) \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$. Denote by $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Delta^+(\mathfrak{g}, \mathfrak{h})$ the set of simple positive roots. We have a non-degenerate, invariant, bilinear form on $\mathfrak{h}$ and $\mathfrak{h}^*$, the Killing form or some form positively proportional to it, written $\langle \cdot, \cdot \rangle$. It becomes a positive-definite scalar product when restricted to the real form $\mathfrak{h}^*_{\mathbb{R}}$, the real span of the roots. A hyperplane perpendicular to a root is called a ***wall***. For each root α we have the corresponding coroot $\check{\alpha} = \frac{2}{\langle \alpha, \alpha \rangle}\alpha$.

A weight λ, that is, an element of $\mathfrak{h}^*$, is ***dominant*** if $\langle \lambda, \check{\alpha}_i \rangle$ for $i = 1, \ldots, n$ are non-negative real numbers, and ***integral*** if those numbers are integers. Weights can be written in the Dynkin diagram notation: on the Dynkin diagram for $\mathfrak{g}$ write $\langle \lambda, \check{\alpha}_i \rangle$ over the node corresponding to the simple root α_i. A weight is ***singular*** if it lies in some wall, otherwise it is ***regular***. ***Fundamental weights*** $\omega_1, \ldots, \omega_n$ are dual elements to the simple coroots, with respect to the form $\langle \cdot, \cdot \rangle$, and they constitute a $\mathbb{Z}$-basis for the lattice of integral weights. The half sum of all positive roots will be denoted by ρ, and it is equal to the sum of all fundamental weights. It is the smallest integral, dominant and regular weight.

1.1 Weyl group

For a root α, denote by $\sigma_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$ the ***reflection*** $\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$. Such reflections generate the ***Weyl group***, denoted $W_{\mathfrak{g}}$, a finite subgroup of $\langle \cdot, \cdot \rangle$-orthogonal transformations which preserve roots. The Weyl group is in bijection with its orbit of any regular weight, and exactly one element of that orbit is dominant. If $\alpha = \alpha_i$ is a simple root, then the reflection σ_{α_i} is called a ***simple reflection***. Simple reflections generate $W_{\mathfrak{g}}$ as a group. Denote by $l(w)$ the ***length*** of w, that is, the minimal number of simple reflections required to obtain w. Such a minimal composition is called a ***reduced form*** of the given element. Denote also

$$\Phi_w := \left\{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h}) \ : \ w^{-1}\alpha < 0\right\} = w(-\Delta^+(\mathfrak{g},\mathfrak{h})) \cap \Delta^+(\mathfrak{g},\mathfrak{h}).$$

A subset $S \subseteq \Delta^+(\mathfrak{g},\mathfrak{h})$ is said to be ***saturated*** if for any $\alpha,\beta \in S$ such that $\alpha+\beta$ is a root, we have $\alpha+\beta \in S$. A subset $S \subseteq \Delta^+(\mathfrak{g},\mathfrak{h})$ is said to be ***admissible*** if S and $\Delta^+(\mathfrak{g},\mathfrak{h}) \setminus S$ are saturated. Admissible subsets can be characterized by the following two conditions:

(a) If $\alpha,\beta \in S$ and $\alpha+\beta \in \Delta^+(\mathfrak{g},\mathfrak{h})$, then $\alpha+\beta \in S$,

(b) If $\alpha,\beta \in \Delta^+(\mathfrak{g},\mathfrak{h})$ and $\alpha+\beta \in S$, then $\alpha \in S$ or $\beta \in S$.

Definition 1.1. *For $w, w' \in W_{\mathfrak{g}}$ we write $w \xrightarrow{\alpha} w'$ if $l(w') = l(w)+1$ and $w' = \sigma_\alpha \circ w$, for some $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})$ (not necessarily simple). We often omit α from the notation, and write $w \longrightarrow w'$.*

This way, $W_{\mathfrak{g}}$ becomes a directed graph. The following proposition gives a method for finding its graph structure:

Proposition 1.2. *Denote by $\langle S\rangle$ the sum of all elements in S, and by $|S|$ its cardinality.*

(a) The map $w \mapsto \Phi_w$ is a bijection from $W_{\mathfrak{g}}$ to the set of all admissible subsets of $\Delta^+(\mathfrak{g},\mathfrak{h})$,

(b) $l(w) = |\Phi_w|$,

(c) $w \xrightarrow{\alpha} w'$ if and only if $|\Phi_{w'}| = |\Phi_w| + 1$ and $\langle\Phi_{w'}\rangle = \langle\Phi_w\rangle + k\alpha$, for some $k \in \mathbb{Z}$.

Proof. See [ČS09, 3.2.]. □

Another algorithm (which we will not use) for finding the graph structure and all reduced forms of elements in $W_{\mathfrak{g}}$ by calculating the orbit of the weight ρ is described in [BE16, 4.1].

We can extend the relation '$\longrightarrow$' by transitivity to a partial order '$\leq$' on $W_{\mathfrak{g}}$, called the (left) ***Bruhat order***.

Besides the standard action of $W_{\mathfrak{g}}$ on $\mathfrak{h}^*$, we also use the "ρ-shifted" action:

Definition 1.3. *The **affine action** (or the **dot-action**) of $W_{\mathfrak{g}}$ on $\mathfrak{h}^*$ is given by the formula*

$$w \cdot \lambda = w(\lambda+\rho) - \rho.$$

*We say that a weight λ is **dot-regular**, if $\lambda+\rho$ is regular.*

For $\lambda \in \mathfrak{h}^*$ integral and dominant for $\mathfrak{g}$, we write $F(\lambda)$ for the complex, finite-dimensional, irreducible representation of $\mathfrak{g}$ with highest weight λ, and $E(\lambda)$ for the complex, finite-dimensional, irreducible representation of $\mathfrak{g}$ with lowest weight $-\lambda$, that is, highest weight $-w_0\lambda$, where $w_0 \in W_{\mathfrak{g}}$ is the longest element. Note that $E(\lambda)$ and $F(\lambda)$ are mutually contragredient. Since G is simply connected, $F(\lambda)$ and $E(\lambda)$ automatically integrate to group representations. We use the same notation for the group representations.

1.2 Parabolic subalgebras

Definition 1.4. *The **standard parabolic subalgebra** of $\mathfrak{g}$ attached to a subset $\Gamma \subseteq \Pi$ of the simple roots is a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ constructed in the following way:*

- *Root subspaces for the roots in Γ and their negatives generate the **Levi factor** $\mathfrak{l}$, a reductive subalgebra of $\mathfrak{g}$ having the same Cartan subalgebra.*
- *Positive root subspaces of $\mathfrak{g}$ not in the Levi factor constitute the **nilpotent radical** $\mathfrak{u}$, a nilpotent subalgebra of $\mathfrak{g}$.*
- *Then, $\mathfrak{p} := \mathfrak{l} \oplus \mathfrak{u}$, and this direct sum is called the **Levi decomposition** of $\mathfrak{p}$.*

We will denote standard parabolic subalgebras by the Dynkin diagram for $\mathfrak{g}$ on which we will put crosses over nodes corresponding to $\Pi \setminus \Gamma$.

Note that $\mathfrak{h} = \mathfrak{h}_{\mathfrak{l}_{ss}} \oplus Z_{\mathfrak{l}}$, where $\mathfrak{h}_{\mathfrak{l}_{ss}} = \mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}]$ is a Cartan subalgebra of the semisimple part $\mathfrak{l}_{ss} = [\mathfrak{l}, \mathfrak{l}]$, and $Z_{\mathfrak{l}}$ is the center of $\mathfrak{l}$. The dimension of $Z_{\mathfrak{l}}$ is equal to the number of crossed nodes. We can get the Dynkin diagram for $\mathfrak{l}_{ss}$ by removing the crossed nodes and the corresponding edges from the diagram for $\mathfrak{p}$. Weights for $\mathfrak{l}$ (which are again elements of $\mathfrak{h}^*$) can be again written in the Dynkin diagram notation, but with the diagram representing $\mathfrak{p}$. A weight is $\mathfrak{p}$-***dominant*** (resp. $\mathfrak{p}$-***integral***) if the numbers over the non-crossed nodes are non-negative (resp. integers), that is, if it is dominant (resp. integral) for $\mathfrak{l}_{\mathrm{ss}}$.

For a standard parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, denote by $\mathfrak{u}^-$ the subalgebra consisting of root subspaces opposite of those in $\mathfrak{u}$ (so we have $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$). We define the ***opposite*** parabolic subalgebra $\mathfrak{p}^- := \mathfrak{l} \oplus \mathfrak{u}^-$. It is standard for the opposite choice of positive roots.

The minimal standard parabolic subalgebra is the ***standard Borel subalgebra***, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Its Levi subalgebra is just the Cartan subalgebra $\mathfrak{h}$, and its nilpotent radical $\mathfrak{n}$ consists of all positive root subspaces. Its Dynkin diagram has all the nodes crossed.

We will be mostly interested in maximal standard parabolic subalgebras, which are obtained by crossing only one node (we do not consider $\mathfrak{p} = \mathfrak{g} \oplus 0$).

In general, a ***parabolic subalgebra*** is a subalgebra containing some ***Borel subalgebra*** (a maximal solvable subalgebra). It is a standard fact that any parabolic subalgebra is conjugate to a unique standard parabolic subalgebra. So we will work only with standard parabolic subalgebras, without loss of generality.

1.2.1 Hasse diagram of a parabolic subalgebra

Fix a standard parabolic subalgebra $\mathfrak{p} := \mathfrak{l} \oplus \mathfrak{u}$. We denote by $W_\mathfrak{p}$ the Weyl group $W_{\mathfrak{l}_{ss}}$, but considered as the subgroup of $W_\mathfrak{g}$ generated by reflections with respect to non-crossed nodes. For $w \in W_\mathfrak{g}$ the following holds: $w \in W_\mathfrak{p}$ if and only if $\Phi_w \subseteq \Delta^+(\mathfrak{l}, \mathfrak{h})$ (this follows from Proposition 1.2). This motivates the next definition:

Definition 1.5. *The* ***Hasse diagram*** *of a parabolic subalgebra* $\mathfrak{p}$ *is the full subgraph of* $W_\mathfrak{g}$ *with the following nodes:*

$$\begin{aligned} W^\mathfrak{p} &:= \{w \in W_\mathfrak{g} \;:\; \Phi_w \subseteq \Delta(\mathfrak{u})\} \\ &= \left\{w \in W_\mathfrak{g} \;:\; w^{-1}\left(\Delta^+(\mathfrak{l}, \mathfrak{h})\right) \subseteq \Delta^+(\mathfrak{g}, \mathfrak{h})\right\}. \end{aligned}$$

Sometimes we will call this the ***regular*** Hasse diagram, as opposed to singular ones, which will be defined in Subsection 1.4.5.

Proposition 1.6. $W^\mathfrak{p}$ *consists of all elements in* $W_\mathfrak{g}$ *which map* $\mathfrak{g}$*-dominant weights to* $\mathfrak{p}$*-dominant weights; or equivalently, at least one regular* $\mathfrak{g}$*-dominant weight (for example* ρ*) to a* $\mathfrak{p}$*-dominant one.*

Proof. Follows from the equality $\langle w\lambda, \alpha\rangle = \langle \lambda, w^{-1}\alpha\rangle$ for $w \in W_\mathfrak{g}$, and $\lambda, \alpha \in \mathfrak{h}^*$. □

Remark 1.7 Each element $w \in W_\mathfrak{g}$ has a unique decomposition $w = w_\mathfrak{p} w^\mathfrak{p}$ for $w_\mathfrak{p} \in W_\mathfrak{p}$ and $w^\mathfrak{p} \in W^\mathfrak{p}$, and moreover $l(w) = l(w_\mathfrak{p}) + l(w^\mathfrak{p})$. It follows that $W^\mathfrak{p}$ is the set of unique minimal length representatives of the right cosets $W_\mathfrak{p} \backslash W_\mathfrak{g}$. Furthermore, the map $w \mapsto w^{-1}\rho^\mathfrak{p}$ defines a bijection from $W^\mathfrak{p}$ to the orbit $W_\mathfrak{g}\rho^\mathfrak{p}$, where in the Dynkin diagram notation $\rho^\mathfrak{p}$ has 1's over crossed nodes, and 0's over uncrossed nodes (which is the same as the sum of fundamental weights corresponding to $\mathfrak{u}$). This gives a method for finding $W^\mathfrak{p}$, but we will use another approach. For details and proofs, see [ČS09, 3.2.]. ■

Note the two extreme cases. For minimal (Borel) case $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ we have $W^\mathfrak{b} = W_\mathfrak{g}$. Other (not so interesting) extreme is $\mathfrak{p} = \mathfrak{g} \oplus 0$, and here $W^\mathfrak{p} = \{\mathrm{Id}\}$.

1.2.2 |1|-graded parabolic subalgebras

We will mostly be interested in parabolic subalgebras $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ with abelian nilpotent radical $\mathfrak{u}$. Such parabolic subalgebras are called |1|-***graded*** (because $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ is a $\{-1, 0, 1\}$-grading), and also parabolic subalgebras of ***Hermitian type*** (because of Proposition 1.12). They are necessarily maximal. Here is the full classification of |1|-graded parabolic subalgebras:

Proposition 1.8. *For a maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ of a simple Lie algebra $\mathfrak{g}$, the following properties are equivalent:*

(a) $\mathfrak{u}$ *is abelian,*

(b) $\mathfrak{g}/\mathfrak{p}$ *is an irreducible* $\mathfrak{p}$*-module,*

(c) there exists E in the center of $\mathfrak{l}$ such that

$$\mathfrak{u} = \{X \in \mathfrak{g} \colon [E, X] = X\}.$$

*Such an element is unique, and it is called the **grading element**. Also,*

$$\mathfrak{u}^- = \{X \in \mathfrak{g} \colon [E, X] = -X\},$$

(d) the crossed simple root occurs with coefficient 1 in the highest root of $\mathfrak{g}$ (table 2 in [Hum72, p. 66]),

(e) $(\mathfrak{g}, \mathfrak{l}_{ss})$ *is one of the pairs in Figure 1.1.*

Proof. See [ČS09, 3.2.] and [EHP14]. □

Finding the graph structure of $W^{\mathfrak{p}}$ is much easier in the |1|-graded case. Every subset of $\Delta(\mathfrak{u})$ is saturated, so we have to care only about saturated complements in $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Moreover, Proposition 1.2 gets simplified:

Proposition 1.9. *In the |1|-graded case:*

(a) The map $w \mapsto \Phi_w$ is a bijection from $W^{\mathfrak{p}}$ to the set of all admissible subsets $S \subseteq \Delta(\mathfrak{u})$. A subset $S \subseteq \Delta(\mathfrak{u})$ is admissible if and only if the following condition holds:

$$\textit{If } \alpha \in \Delta(\mathfrak{u}) \textit{ and } \beta \in \Delta^+(\mathfrak{l}, \mathfrak{h}) \textit{ such that } \alpha + \beta \in S, \textit{ then } \alpha \in S. \tag{1.1}$$

(b) $w \xrightarrow{\alpha} w'$ in $W^{\mathfrak{p}}$ if and only if $|\Phi_{w'}| = |\Phi_w| + 1$ and $\Phi_{w'} = \Phi_w \cup \{\alpha\}$.

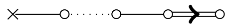
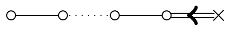
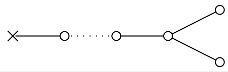
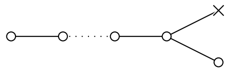
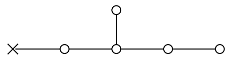
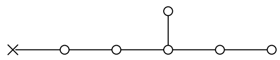
types	Lie algebras	Dynkin notation
$(A_n, A_{k-1} \times A_{n-k})$	$(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{sl}(k,\mathbb{C}) \times \mathfrak{sl}(n-k+1,\mathbb{C}))$	
(B_n, B_{n-1})	$(\mathfrak{so}(2n+1,\mathbb{C}), \mathfrak{so}(2n-1,\mathbb{C}))$	
(C_n, A_{n-1})	$(\mathfrak{sp}(2n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C}))$	
(D_n, D_{n-1})	$(\mathfrak{so}(2n,\mathbb{C}), \mathfrak{so}(2n-2,\mathbb{C}))$	
(D_n, A_{n-1})	$(\mathfrak{so}(2n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C}))$	
(E_6, D_5)	$(\mathfrak{e}6, \mathfrak{so}(10,\mathbb{C}))$	
(E_7, E_6)	$(\mathfrak{e}7, \mathfrak{e}6)$	

Figure 1.1: List of $|1|$-graded parabolic subalgebras

Proof. See [ČS09, 3.2.] and [KS03]. □

Remark 1.10 This is the simplest case of a more general notion: $|k|$-grading. For a definition of $|k|$-grading, of the corresponding grading element, and their relation to parabolic subalgebras, see [ČS09, 3.1.2., 3.2.]. ■

1.2.3 Representations of parabolic subalgebras

Given a parabolic subalgebra $\mathfrak{p}$, for $\lambda \in \mathfrak{h}^*$ integral and dominant for $\mathfrak{p}$, we write $F_{\mathfrak{p}}(\lambda)$ for the complex, finite-dimensional, irreducible representation of $\mathfrak{p}$ such that its restriction to $\mathfrak{l}$ is irreducible, $\mathfrak{h}$-semisimple, with highest weight λ, and $\mathfrak{u}$ acts by 0. This way we get all complex, finite-dimensional, irreducible, $\mathfrak{h}$-semisimple representations of $\mathfrak{p}$. Dually, we write $E_{\mathfrak{p}}(\lambda)$ for the complex, finite-dimensional, irreducible representation of $\mathfrak{p}$ such that its restriction to $\mathfrak{l}$ is irreducible, $\mathfrak{h}$-semisimple, with lowest weight $-\lambda$, and $\mathfrak{u}$ acts by 0. Again, we have $E_{\mathfrak{p}}(\lambda) = F_{\mathfrak{p}}(\lambda)^*$.

In the $|1|$-graded case, each of the representations $F_{\mathfrak{p}}(\lambda)$ is completely determined by its highest weight for the semisimple part $\mathfrak{l}_{ss}$, and by the scalar by which the one-dimensional center of $\mathfrak{l}$ acts. That scalar is called the ***generalized conformal weight***, and it is equal to

$$\lambda(E) = \frac{2\langle \lambda, \omega_i \rangle}{\langle \alpha_i, \alpha_i \rangle},$$

where E is the grading element, α_i is the crossed simple root, and ω_i is the corresponding fundamental weight.

1.3 Geometric setup

1.3.1 Generalized flag manifolds

Once we choose a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$, we get the corresponding ***parabolic subgroup*** $P \subseteq G$ as the normalizer of $\mathfrak{p}$ in G:

$$P := \{g \in G\colon \ \mathrm{Ad}(g)(\mathfrak{p}) = \mathfrak{p}\}.$$

The subgroup P is complex Lie subgroup, closed, connected, self-normalizing, and has the Lie algebra $\mathfrak{p}$. Moreover, $P = LU$ (the semi-direct product), where L and U are the analytic subgroups of G corresponding to $\mathfrak{l}$ and $\mathfrak{u}$, and are also called the ***Levi factor*** and the ***nilpotent radical***, respectively. Normalizer of the Borel subalgebra $\mathfrak{b}$ is called the ***Borel subgroup***, and denoted by B. We form the (complex) ***generalized flag manifold*** G/P. It is a complex, compact, connected, simply connected, homogeneous manifold (see [FHW06]). It is also naturally a non-singular projective variety, and therefore Kähler (see [ČS09, 3.2]). The Dynkin notation for $\mathfrak{p}$ will also denote P and G/P; which is meant should be clear from the context.

The representations $F_{\mathfrak{p}}(\lambda)$ and $E_{\mathfrak{p}}(\lambda)$ integrate to the representations of the group P if and only if λ is $\mathfrak{p}$-dominant and $\mathfrak{g}$-integral. In that case, we use the same notation for the group representations.

Let us describe the projective realization of G/P. Consider the irreducible finite-dimensional representation $F(\rho^{\mathfrak{p}})$ of G, where the weight $\rho^{\mathfrak{p}}$ is the sum of the fundamental weights corresponding to the simple roots in $\mathfrak{u}$ (that is, $\rho^{\mathfrak{p}}$ has 1's over the crossed nodes, and $0's$ over the uncrossed nodes in its Dynkin diagram notation). The group G acts on the projectivization $\mathbb{P}(F(\rho^{\mathfrak{p}}))$ of the vector space $F(\rho^{\mathfrak{p}})$. Let $v \in F(\rho^{\mathfrak{p}})$ be the highest weight vector. Then:

Proposition 1.11. *The stabilizer of the line $[v]$ in $\mathbb{P}(F(\rho^{\mathfrak{p}}))$ is exactly the parabolic subgroup P. Therefore, we have an isomorphism $G/P \cong G[v]$ onto the orbit $G[v]$, which is a closed complex submanifold of $\mathbb{P}(F(\rho^{\mathfrak{p}}))$, and therefore a smooth projective subvariety.*

Proof. See [ČS09, 3.2.]. ☐

Furthermore, we have:

Proposition 1.12. *G/P has a natural structure of a Hermitian symmetric space if and only if the parabolic subalgebra $\mathfrak{p}$ is $|1|$-graded.*

Proof. See [ČS09, 3.2] or [EHP14] (for the proof and for the definition of the Hermitian symmetric space). ☐

Let us recall the ***Bruhat decomposition*** of G/P. The quotient $N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ acts on $\mathfrak{h}^*$ via the coadjoint map

$$g \cdot Z_G(\mathfrak{h}) \mapsto \left(\lambda \mapsto \lambda \circ \operatorname{Ad}(g^{-1})\right),$$

and becomes naturally isomorphic to the Weyl group $W_{\mathfrak{g}}$. For each $w \in W_{\mathfrak{g}}$, denote by $\tilde{w}$ its representative in $N_G(\mathfrak{h})$.

Theorem 1.13 (Bruhat decomposition)**.** *The Hasse diagram W^p parametrizes the orbits of the B-action on G/P:*

$$G/P = \bigsqcup_{w \in W^{\mathfrak{p}}} B\tilde{w}^{-1}P.$$

Moreover, each B-orbit $X_w := B\tilde{w}^{-1}P$ is isomorphic to the affine space $\mathbb{C}^{l(w)}$. Furthermore, $X_{w'} \subseteq \bar{X}_w$ if and only if $w' \leq w$, and therefore the closure of X_w is

$$\bar{X}_w = \bigsqcup_{w' \in W^{\mathfrak{p}},\ w' \leq w} X_{w'}.$$

Proof. See [ČS09, 3.2.] and [Pro07, 10.6.] ☐

Each X_w is called a ***Schubert cell***, its closure $\bar{X}_w$ a ***Schubert variety***. Closure in both the analytic and the Zariski topology are the same in this case. Schubert varieties are often singular.

Denote by $N = \exp \mathfrak{n}$ the connected subgroup of G with the Lie algebra $\mathfrak{n}$. The subgroup N is simply connected, and the exponential map $\exp\colon \mathfrak{n} \to N$ is an isomorphism (see [Kna02, I.16., VI.4.]).

Corollary 1.14. *In the projective realization of G/P from Proposition 1.11, each Schubert cell X_w becomes the N-orbit $N[v_w]$, where v_w is the vector of the weight $w^{-1}\rho^{\mathfrak{p}}$.*

Denote by $U^- = \exp \mathfrak{u}^-$ the connected subgroup of G with the Lie algebra $\mathfrak{u}^-$.

Proposition 1.15. *The following composition defines the canonical affine chart around the line generated by the highest weight vector $[v] \in G[v] \cong G/P$:*

$$\mathfrak{u}^- \xrightarrow{\exp} U^- \xrightarrow{\cdot[v]} U^-[v]. \tag{1.2}$$

The set $U^-[v]$ is open and dense in G/P, of the dimension $l(w^0) = |\Delta(\mathfrak{u})|$.

See [Kem78]. The set $U^-[v]$ is called the ***big affine cell*** around $[v]$. Moreover, each $\tilde{w}^{-1}U^-[v]$ is also an affine set, around $[v_w]$, and also called a big affine cell. The biggest Schubert cell in 1.13 and 1.14 is $X_{w^0} = (\tilde{w}^0)^{-1}U^-[v]$.

For more detailed information, and a (co)homological interpretation of the Schubert cells, see [BGG73] and [Kos63].

The importance of the Hasse diagram is evident from the Bruhat decomposition, since it gives the CW-structure of G/P and its homology. Much more important role to us of the Hasse diagram will be presented in Subsection 1.3.5.

1.3.2 Fibrations

If we have two standard parabolic subgroups $Q \subseteq P$, then the natural map

$$G/Q \to G/P, \qquad gQ \mapsto gP$$

is a ***fibration***. It is locally a product, and therefore an open map. Its fibers are also generalized flag manifolds, of the form $P/Q \cong L_{\mathrm{ss}}/(L_{\mathrm{ss}} \cap Q)$, where L_{ss} is the semisimple part the of Levi factor of P. The relation $Q \subseteq P$ is seen on the Dynkin diagrams: every node that is crossed in the diagram for P must also be crossed in the diagram for Q. We can get the Dynkin diagram for the fiber by deleting all the crossed nodes in Q which are also crossed in P, and then deleting all the connected components without the crosses. The ***relative Hasse diagram*** of the fibration $G/Q \to G/P$, denoted $W_{\mathfrak{p}}^{\mathfrak{q}}$, is by definition the Hasse diagram of the parabolic subalgebra $\mathfrak{l}_{\mathrm{ss}} \cap \mathfrak{q}$ in $\mathfrak{l}_{\mathrm{ss}}$. It is also a subgraph of $W_{\mathfrak{g}}$.

1.3.3 Homogeneous vector bundles and homogeneous sheaves

For a parabolic subgroup P and its finite-dimensional holomorphic representation $\pi\colon P \to \mathrm{End}(V)$, we can form the ***homogeneous bundle***

$$\begin{array}{ll} G \times_P V = G \times V/_{\sim}, & \qquad \text{where } (gp, v) \sim (g, \pi(p)v). \\ \quad\downarrow & \\ G/P & \end{array}$$

It is a holomorphic vector bundle, whose fibers are copies of V. The fiber over eP recovers the original P-module V. Also, P-invariant maps lift to (G-invariant) morphisms of homogeneous vector bundles in a functorial way, and such construction defines an equivalence of the corresponding categories.

Denote by $\mathcal{O}(V)$ the ***sheaf of holomorphic sections*** of the homogeneous bundle induced from V. It is a locally free sheaf of $\mathcal{O}_{G/P}$-modules of the finite rank, and therefore coherent. Sheaves and their operations will be heavily used latter. For the introduction, see [Tay02, 7.], [WW90, 3] or [Mrđ11].

Sections of the sheaf $\mathcal{O}(V)$ correspond to the holomorphic functions defined on open subsets of G, to V, that are P-equivariant:

$$f(gp) = \pi(p^{-1})f(g), \quad g \in G,\ p \in P.$$

The equivalent differentiated condition is

$$X^l f = -d\pi(X)f, \tag{1.3}$$

where $X^l \in \mathfrak{p}$ is thought of as the left-invariant vector field on G defined by X:

$$(X^l f)(g) = \frac{d}{dt}\bigg|_0 f(g \cdot \exp(tX)),$$

and the action on the right-hand side of (1.3) is the negative of the differentiated representation π, applied pointwise on f.

The group G acts on the sections of $\mathcal{O}(V)$ (so we call it a ***homogeneous sheaf***) by the left translation:

$$(g \cdot f)(g') = f(g^{-1}g')). \tag{1.4}$$

Differentiating this action, we see that the Lie algebra $\mathfrak{g}$ acts on the sections as the negative of the action by the right-invariant vector fields:

$$(X \cdot f)(g) = \frac{d}{dt}\bigg|_0 f(\exp(-tX) \cdot g) = (-X^r f)(g).$$

There is an interesting relation between real forms of G and the complex structures on G/P. We will not use this however, so for details see [FHW06, 1.].

Example 1.16 The most basic homogeneous bundles/sheaves are the following:

- From the trivial representation $\mathbb{C}$, we get the trivial bundle of the rank one, and the holomorphic structure sheaf $\mathcal{O}(\mathbb{C}) = \mathcal{O}_{G/P}$ of G/P.
- From the adjoint action of P on $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{u}^- \cong \mathfrak{u}^*$, we get the holomorphic tangent bundle and the sheaf of holomorphic vector fields.
- From the action of P on $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{u}$, we get the holomorphic cotangent bundle and the sheaf of holomorphic differential 1-forms.

The identifications are made via the Killing form. The holomorphic tangent and cotangent bundles are irreducible if and only if the parabolic subalgebra is $|1|$-graded, that is, if and only if G/P is a Hermitian symmetric space. ■

Assume that $\lambda \in \mathfrak{h}^*$ is $\mathfrak{g}$-integral and $\mathfrak{p}$-dominant. Denote by $\mathcal{O}_{\mathfrak{p}}(\lambda)$ the homogeneous sheaf $\mathcal{O}(E_{\mathfrak{p}}(\lambda))$. The notations for λ will also denote $E_{\mathfrak{p}}(\lambda)$ and $\mathcal{O}_{\mathfrak{p}}(\lambda)$; which is meant should be clear from the context. We will write $E(\lambda)$ also for the locally constant sheaf on G/P whose space of sections over any open connected subset is the vector space $E(\lambda)$.

The ***sheaf cohomologies*** $H^k(G/P, \mathcal{O}_{\mathfrak{p}}(\lambda))$ carry G-module structures. They coincide with the classical Dolbeault cohomologies of the holomorphic vector bundle $G \times_P E_{\mathfrak{p}}(\lambda)$, see [WW90, 3.3] or [Tay02, 10.]. The strucures on the cohomologies are revealed in the following theorem.

Theorem 1.17 (Bott-Borel-Weil). *Let $\lambda \in \mathfrak{h}^*$ be $\mathfrak{g}$-integral and $\mathfrak{p}$-dominant weight. If $\lambda + \rho$ is singular for $\mathfrak{g}$, then*

$$H^k(G/P, \mathcal{O}_{\mathfrak{p}}(\lambda)) = 0, \quad \textit{for } k \geq 0.$$

Otherwise, there exists a unique $w \in W_{\mathfrak{g}}$ such that $w \cdot \lambda$ is dominant for $\mathfrak{g}$ (necessarily $w^{-1} \in W^{\mathfrak{p}}$). Then, as G-modules

$$H^k(G/P, \mathcal{O}_{\mathfrak{p}}(\lambda)) \cong \begin{cases} E(w \cdot \lambda) & : k = l(w) \\ 0 & : k \neq l(w). \end{cases}$$

Proof. See [Bot57]. There is a sketch of the proof in [BE16, 5.1]. □

We will use Bott-Borel-Weil theorem in a relative form. Let $\tau \colon G/Q \to G/P$ be the fibration corresponding to the inclusion $Q \subseteq P$ of the standard parabolic subgroups, and denote by τ_*^q the q-th ***right derived functor*** of the left exact ***direct image functor*** τ_*. The functors τ_*^q are also called the ***higher direct images*** of τ. For calculating τ_*^q, we use the fact that the higher direct images along a fibration are actually the sheaf cohomologies of the restrictions to the fibers (proved in [Bot57]), where we can apply Bott-Borel-Weil theorem:

Corollary 1.18. *Let $\tau \colon G/Q \to G/P$ and $\lambda \in \mathfrak{h}^*$ be $\mathfrak{g}$-integral and $\mathfrak{p}$-dominant weight. If $\lambda + \rho$ is $\mathfrak{p}$-singular, then all the higher direct images $\tau_*^q \mathcal{O}_{\mathfrak{q}}(\lambda)$ are 0. Otherwise, there exists a unique $w \in W_{\mathfrak{p}}$, such that $w \cdot \lambda$ is $\mathfrak{p}$-dominant (and necessarily $w^{-1} \in W_{\mathfrak{p}}^{\mathfrak{q}}$). Then,*

$$\tau_*^{l(w)} \mathcal{O}_{\mathfrak{q}}(\lambda) \cong \mathcal{O}_{\mathfrak{p}}(w \cdot \lambda),$$

and all other higher direct images are 0.

Remark 1.19 The algebraic version of Bott-Borel-Weil theorem is known as Kostant's theorem. See [Kos61], [ČS09, 3.3.] or [Vog81, 3.], [KV95, IV.9.-11.]. ■

1.3.4 Jet bundles and invariant differential operators

The goal of this subsection is to give a coordinate-free definition of an invariant differential operator. They are usually defined as maps of the sections of vector bundles that in local coordinates look like

$$D = \sum_{\substack{I=(i_1,\ldots,i_n) \\ i_1+\ldots+i_n\leq k}} f_I \frac{\partial^{i_1}}{\partial z^{i_1}} \cdots \frac{\partial^{i_n}}{\partial z^{i_n}} \tag{1.5}$$

for some $k \in \mathbb{N}$ and holomorphic functions f_I. We will have a benefit of a slightly more abstract approach, following [Spe69].

Suppose $E \to X$ is a holomorphic vector bundle of a finite rank. Denote by $\mathcal{O}_E$ its sheaf of the holomorphic sections, and $\mathcal{O}_{E,x}$ its stalk at $x \in X$. Define

$$j^k\mathcal{O}_{E,x} = \{s \in \mathcal{O}_{E,x} : \text{all partial derivatives of } s \text{ of order} < k \text{ vanish at } x$$
$$\text{(in any, thus all holomorphic charts)}\}\,.$$

Set

$$\mathcal{O}_{E,x} \xrightarrow{j^k_x} J^k_x E := \mathcal{O}_{E,x} \Big/ j^{k+1}\mathcal{O}_{E,x}.$$

The image $j^k_x(s)$ in $J^k_x E$ of a section s is called the ***k*-jet** of s around x, and can be represented by the Taylor polynomial approximation of the order k of s around x, in some holomorphic chart. By Taylor expansion theorem, $J^k_x E$ is a finite-dimensional vector space.

There is a unique holomorphic vector bundle $J^k E \to X$ with the fibers $J^k_x E$, called the ***k*-jet bundle** of E. There are obvious homogeneous bundle homomorphisms

$$E \cong J^0 E \leftarrow J^1 E \leftarrow \ldots \leftarrow J^k E \leftarrow J^{k+1} E \leftarrow \ldots, \tag{1.6}$$

which are holomorphic submersions. We can take the (inverse) limit of the diagram (1.6)

$$J^\infty E := \varprojlim J^k E \to E,$$

which is called the ***formal jet bundle***. It is still a vector bundle, but of infinite rank. The kernels of the homomorphims in (1.6) are easy to describe: there are canonical short exact sequences of homogeneous bundles

$$0 \to S^k\Omega^1_X \otimes E \to J^k E \to J^{k-1} E \to 0, \tag{1.7}$$

where $S^k\Omega^1_X$ is the k-th symmetric power of the holomorphic cotangent bundle of X, which are called the ***jet exact sequences***.

Definition 1.20. *The **universal differential operator** of order k is the following* *map of sheaves:*

$$\mathcal{O}_E \xrightarrow{j^k} \mathcal{O}_{J^kE}, \qquad s \mapsto \left(j^k(s) \colon x \mapsto j^k_x(s)\right). \tag{1.8}$$

It is $\mathbb{C}$-linear, but not $\mathcal{O}_X$-linear for $k > 0$ (in particular, it is not induced from some bundle map $E \to J^kE$). A ***differential operator*** *(always assumed linear) between holomorphic vector bundles E and F over X is a $\mathbb{C}$-linear morphism of sheaves $D \colon \mathcal{O}_E \to \mathcal{O}_F$ that factorizes over the sections of some jet bundle*

$$\begin{tikzcd} \mathcal{O}_E \arrow[r, "D"] \arrow[d, "j^k"'] & \mathcal{O}_F \\ \mathcal{O}_{J^kE}, \arrow[ur, "\tilde{D}"'] & \end{tikzcd}$$

in such way that $\tilde{D}$ is $\mathcal{O}_X$-linear, that is, induced from a vector bundle homomorphism $\tilde{D} \colon J^kE \to F$ (the same notation). The smallest such possible k is called the ***order*** *of D, and $\tilde{D}$ is then uniquely determined. The* ***(principal) symbol*** *of D is the composition*

$$\sigma(D) \;:\; S^k\Omega^1_X \otimes E \hookrightarrow J^kE \xrightarrow{\tilde{D}} F, \tag{1.9}$$

by means of (1.7).

Definition 1.20 says that a differential operator of the order k depends on the derivatives of the argument up to the order k in the $\mathcal{O}_X$-linear way; this is equivalent to (1.5). Two differential operators with the same symbol always differ by a lower order differental operator.

Remark 1.21 Peetre's theorem states that any local map between the sections (meaning that the support of a section is not increased) of vector bundles must be a differential operator. See [KMS93, V.19.]. ∎

Remark 1.22 For a more functional-analytic approach, see [Nac82]. ∎

Suppose now X is a generalized flag manifold G/P and E a homogeneous vector bundle on X. Then all the jet bundles J^kE are also homogeneous vector bundles, because of the chain rule; we will describe the corresponding P-representations in Proposition 1.45. All the maps in (1.6), (1.7) and (1.8) are G-invariant. The invariance in case of the homogeneous sheaves in considered with respect to the left translation (1.4), or its differentiated version. A differential operator $D \colon \mathcal{O}_E \to \mathcal{O}_F$ is invariant if and only if $\tilde{D} \colon J^kE \to F$ is invariant, and therefore corresponds to a homomorphism of the finite-dimensional P-modules. Hence, the classification of the invariant differential operators becomes a completely algebraic problem. Since the P-representation corresponding to J^kE tends to be complicated (for example, not completely reducible), it is easier to classify the possibilities for the symbols $\sigma(D) \;:\; S^k\Omega^1_X \otimes E \to F$, using the following ***symbol principle***: Suppose E and F are the irreducible homogeneous bundles defined by $E_{\mathfrak{p}}(\lambda)$ and $E_{\mathfrak{p}}(\mu)$. Because of the G-invariance and transitivity, a symbol $\sigma(D)$ is completely determined by its restriction to the fiber over eP, and this restriction is a P-invariant map $S^k\mathfrak{u}\otimes E_{\mathfrak{p}}(\lambda) \to E_{\mathfrak{p}}(\mu)$. We decompose $S^k\mathfrak{u}\otimes E_{\mathfrak{p}}(\lambda)$ into the direct sum of $\mathfrak{l}$-irreducibles. If $\sigma(D)$ is non-zero, then by Schur's lemma, it can only correspond to a projection from $S^k\mathfrak{u} \otimes E_{\mathfrak{p}}(\lambda)$ onto the one of its irreducible factors isomorphic to $E_{\mathfrak{p}}(\mu)$, if such even exists. It has to be checked that such projection is P-invariant.

In the $|1|$-graded situation, more information is available: The order of a non-zero invariant differential operator is determined by the highest weights of the induced representations. More precisely, it is equal to the difference between the generalized conformal weights in the domain and the codomain (Proposition 1.46). Such an operator is then unique up to a non-zero scalar. The symbol $\sigma(D)$ completely determines D. More information will be given in Section 1.4.

Consider the Borel subgroup $B \subseteq P$, where P is not necessarily $|1|$-graded, and take $\lambda, \mu \in \mathfrak{h}^*$ to be $\mathfrak{g}$-integral and $\mathfrak{p}$-dominant. The following fact will be clear in Subsection 1.4.3 and Section 1.5: if there exists a non-zero invariant differential operator $\mathcal{O}_{\mathfrak{b}}(\lambda) \to \mathcal{O}_{\mathfrak{b}}(\mu)$, then it is unique up to a non-zero scalar. The direct image of such map via the fibration $G/B \to G/P$ is again an invariant differential operator, which is called the ***standard*** differential operator $\mathcal{O}_{\mathfrak{p}}(\lambda) \to \mathcal{O}_{\mathfrak{p}}(\mu)$. It may happen that the standard differential operator is zero, and there may exist invariant differential operators which are ***non-standard***, for $P \neq B$. More information will be given in Section 1.4. Criteria for the existence of non-standard operators are currently unknown in general.

We will denote $J^k(G \times_P E_{\mathfrak{p}}(\lambda))$ by $J^k(\mathcal{O}_{\mathfrak{p}}(\lambda))$. The sheaf of differential operators $\mathcal{O}_E \to \mathcal{O}_F$ will be denoted by $\mathrm{Diff}(\mathcal{O}_E, \mathcal{O}_F)$, and the invariant ones by $\mathrm{Diff}_G(\mathcal{O}_E, \mathcal{O}_F)$.

1.3.5 Regular BGG resolution

Theorem 1.23 (Bernstein-Gelfand-Gelfand, Čap-Slovák-Souček)**.** *Let $\lambda \in \mathfrak{h}^*$ be a weight, integral and dominant for $\mathfrak{g}$. There is an exact sequence of sheaves on G/P and invariant differential operators, called the **(regular) BGG resolution**:*

$$0 \to E(\lambda) \to \Delta^\bullet(\lambda), \quad \text{where}^1 \quad \Delta^k(\lambda) = \bigoplus_{w \in W^{\mathfrak{p}},\ l(w)=k} \mathcal{O}_{\mathfrak{p}}(w \cdot \lambda). \tag{1.10}$$

The morphisms in the sequence are the direct sums of invariant differential operators $\mathcal{O}_{\mathfrak{p}}(w \cdot \lambda) \to \mathcal{O}_{\mathfrak{p}}(w' \cdot \lambda)$ for $w \to w'$ in $W^{\mathfrak{p}}$, which are all standard and non-zero.

Proof. See [ČSS01] for the proof in a much greater generality of parabolic geometries. In [BE16, 8.4] there is a construction of the resolution for a parabolic subgroup P, assuming the resolution for the Borel subgroup B. In Subsection 1.4.4 we will give references to the algebraic proofs. □

[1] The notation $0 \to E(\lambda) \to \Delta^\bullet(\lambda)$ is a shorthand for $0 \to E(\lambda) \to \Delta^0(\lambda) \to \Delta^1(\lambda) \to \Delta^2(\lambda) \to \ldots$.

Notice that every homogeneous sheaf in the resolution if well defined because of Proposition 1.6. Also, every $\mathfrak{p}$-dominant $W_{\mathfrak{g}}$-dot-conjugate of λ will appear at some point in the resolution. Moreover, the resolution has the same directed-graph structure as the Hasse diagram, which is independent of the starting weight λ.

In the $|1|$-graded situation, the resolution is sometimes called the generalized de Rham complex. For $\lambda = 0$ all the operators in the resolution are of order one, and one gets the usual holomorphic de Rham complex. These resolutions show up in suprisingly many different areas – see [Eas99].

1.4 Algebraic Setup

1.4.1 (Generalized) infinitesimal character

In this subsection we allow $\mathfrak{g}$ to be a reductive Lie algebra. The universal enveloping algebra of $\mathfrak{g}$ will be denoted by $U(\mathfrak{g})$, and the center of $U(\mathfrak{g})$ by $Z(\mathfrak{g})$. A $U(\mathfrak{g})$-module M has the ***generalized infinitesimal character*** χ (where $\chi\colon Z(\mathfrak{g}) \to \mathbb{C}$ is a homomorphism of algebras, always assumed unital) if $(z - \chi(z)1)^n v = 0$ for all $v \in M$, $z \in Z(\mathfrak{g})$ and some $n \in \mathbb{N}$. If $n = 1$ (that is, $Z(\mathfrak{g})$ acts with scalars), then χ is called the ***infinitesimal character***. Algebra homomorphisms $Z(\mathfrak{g}) \to \mathbb{C}$ are identified with the elements from $\mathfrak{h}^*$ by means of the ***Harish-Chandra isomorphism***, which we briefly recall here.

Using the Poincaré-Birkhoff-Witt's basis for $U(\mathfrak{g})$ corresponding to the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, one sees that

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \left(\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}\right).$$

Denote by p the projection $U(\mathfrak{g}) \to U(\mathfrak{h})$ corresponding to the above decomposition. Denote by S_ρ the ρ-shift: the algebra automorphism of $U(\mathfrak{h})$ defined on the generators $\mathfrak{h}$ by $h \mapsto h - \rho(h)1$. Under the identification $U(\mathfrak{h}) \cong \mathcal{P}(\mathfrak{h}^*)$, the ρ-shift corresponds to the map $(S_\rho P)(\lambda) = P(\lambda - \rho)$, for a polynomial function $P\colon \mathfrak{h}^* \to \mathbb{C}$.

The ***Harish-Chandra map*** is defined as

$$\gamma := S_\rho \circ p \;:\; U(\mathfrak{g}) \to U(\mathfrak{h}) = S(\mathfrak{h}) \cong \mathcal{P}(\mathfrak{h}^*).$$

Theorem 1.24 (Harish-Chandra)**.** *The map $\gamma|_{Z(\mathfrak{g})}$ is an algebra isomorphism from $Z(\mathfrak{g})$ onto the $W_{\mathfrak{g}}$-invariants in $S(\mathfrak{h})$, and does not depend on the choice of the positive roots.*

For any $\lambda \in \mathfrak{h}^*$ we can obtain a homomorphism $\chi_\lambda \colon Z(\mathfrak{g}) \to \mathbb{C}$ by the formula:

$$\chi_\lambda(z) := \lambda(\gamma(z)).$$

Theorem 1.25 (Harish-Chandra)**.** *Every homomorphism $\chi \colon Z(\mathfrak{g}) \to \mathbb{C}$ is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$. Furthermore, $\chi_\lambda = \chi_\mu$ if and only if λ and μ are in the same $W_\mathfrak{g}$-orbit.*

Proofs. See [KV95, IV.7-8.], or (with some different conventions) [Hum08, 1.7-10.]. □

Therefore, a generalized infinitesimal character $\chi_\lambda \colon Z(\mathfrak{g}) \to \mathbb{C}$ is identified with the element $\lambda \in \mathfrak{h}^*$, or more precisely, its $W_\mathfrak{g}$-orbit.

By Dixmier-Schur's lemma ([KV95, IV.7.]), $Z(\mathfrak{g})$ acts by scalars on any irreducible $U(\mathfrak{g})$-module. Therefore, any irreducible $U(\mathfrak{g})$-module has an infinitesimal character.

Relations between modules with generalized infinitesimal characters are severely restricted by the following proposition (see [KV95, VII.2.]):

Proposition 1.26 (Wigner)**.** *Let M, N be $U(\mathfrak{g})$-modules having respectively generalized infinitesimal characters $\lambda, \mu \in \mathfrak{h}^*$, such that $\chi_\lambda \neq \chi_\mu$. Then $\mathrm{Ext}^n_\mathfrak{g}(M, N) = 0$ for all $n \geq 0$. In particular, $\mathrm{Hom}_\mathfrak{g}(M, N) = 0$.*

1.4.2 Parabolic category $\mathcal{O}^\mathfrak{p}$

Fix a standard parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \subseteq \mathfrak{g}$.

Definition 1.27. *The **parabolic category** $\mathcal{O}^\mathfrak{p}$ is the full subcategory of $U(\mathfrak{g})$-modules M that satisfy:*

(a) M is a finitely generated $U(\mathfrak{g})$-module,

(b) as a $U(\mathfrak{l}_{ss})$-module, M is a direct sum of finite-dimensional simple $U(\mathfrak{l}_{ss})$-modules,

*(c) M is **locally $\mathfrak{u}$-finite**, meaning that each element in M has a finite-dimensional $U(\mathfrak{u})$-orbit.*

The morphisms are all the $\mathfrak{g}$-invariant linear maps.

The category $\mathcal{O}^\mathfrak{p}$ is closed under taking the finite direct sums, submodules, quotients, and tensoring with finite-dimensional $U(\mathfrak{g})$-modules. Moreover, each object of $\mathcal{O}^p$ can be decomposed as a direct sum of finitely many $U(\mathfrak{g})$-submodules in $\mathcal{O}^\mathfrak{p}$ with a generalized infinitesimal character. Therefore, there is a direct sum decomposition of categories

$$\mathcal{O}^\mathfrak{p} = \bigoplus_{\lambda \in \mathfrak{h}^*/W_\mathfrak{g}} \mathcal{O}^\mathfrak{p}_\lambda,$$

where $\mathcal{O}^{\mathfrak{p}}_{\lambda}$ denotes the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ consisting of the modules with the generalized infinitesimal character λ. These subcategories are called the ***(infinitesimal) blocks***, even though they may also be decomposable (as we will see later in Theorem 1.43). Details and proofs can be found in [Hum08, 9.].

Dangerous bend: the category $\mathcal{O}^{\mathfrak{p}}$ is not closed under the extensions in the category of all $U(\mathfrak{g})$-modules. This is important to keep in mind when working with the Ext-functors.

1.4.3 (Generalized) Verma modules

Definition 1.28. *For a weight λ, integral and dominant for $\mathfrak{p}$, we define the **generalized Verma module** as*

$$M_{\mathfrak{p}}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_{\mathfrak{p}}(\lambda).$$

Here $U(\mathfrak{g})$ acts with the multiplication from the left.

It is a cyclic module with the highest weight λ, and it belongs to the category $\mathcal{O}^{\mathfrak{p}}$. Using Poincaré-Birkhoff-Witt's theorem, one sees that

$$M_{\mathfrak{p}}(\lambda) \cong U(\mathfrak{u}^-) \otimes_{\mathbb{C}} F_{\mathfrak{p}}(\lambda),$$

as an $U(\mathfrak{u}^-)$-module. It has a unique maximal submodule, and therefore a unique irreducible quotient $L(\lambda)$ (which does not depend on $\mathfrak{p}$). All the simple modules in $\mathcal{O}^{\mathfrak{p}}$ are obtained in this way.

Proposition 1.29. *The $U(\mathfrak{g})$-module $L(\lambda)$ is finite-dimensional if and only if λ is integral and dominant for $\mathfrak{g}$, and then $L(\lambda) = F(\lambda)$.*

Proposition 1.30. *The modules $M_{\mathfrak{p}}(\lambda)$ and $L(\lambda)$ have the infinitesimal character $\lambda+\rho$.*

Proofs. See again [Hum08, 9.] for all the details and proofs. □

If we start with the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, then any weight is integral and dominant for $\mathfrak{b}$. We denote $\mathcal{O} = \mathcal{O}^{\mathfrak{b}}$ and $M(\lambda) = M_{\mathfrak{b}}(\lambda)$, which is called the ***Verma module***. The generalized Verma module $M_{\mathfrak{p}}(\lambda)$ is a quotient of $M(\lambda)$. The Verma module $M(\lambda)$ is simple, that is, $M(\lambda) = L(\lambda)$, if and only if λ is antidominant, meaning that $\langle \lambda + \rho, \hat{\alpha} \rangle \notin \mathbb{N}$ for all $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})$ (for proof, see [Hum08, 4.]). There is also a more general simplicity criterion for $M_{\mathfrak{p}}(\lambda)$ (see [Hum08, 9.12-13.]).

The morphisms between (ordinary) Verma modules are completely classified. We will describe the classification.

Definition 1.31. *For $\lambda, \mu \in \mathfrak{h}^*$, we write $\mu \uparrow \lambda$ if $\mu = \lambda$ or there is a root $\beta > 0$ such that $\mu = \sigma_\beta \cdot \lambda < \lambda$; in other words, $\langle \lambda + \rho, \check{\beta} \rangle > 0$. More generally, if $\mu = \lambda$ or there* *exist positive roots $\beta_1, \ldots, \beta_r$ such that*

$$\mu = (\sigma_{\beta_1} \ldots \sigma_{\beta_r}) \cdot \lambda \uparrow (\sigma_{\beta_2} \ldots \sigma_{\beta_r}) \cdot \lambda \uparrow \ldots \uparrow \sigma_{\beta_r} \cdot \lambda \uparrow \lambda,$$

*we say that μ is **strongly linked** to λ and write again $\mu \uparrow \lambda$.*

Theorem 1.32 (Verma-Bernstein-Gelfand-Gelfand)**.** *For weights $\lambda, \mu \in \mathfrak{h}^*$, the following conditions are equivalent:*

(a) There is a non-zero morphism of Verma modules $M(\mu) \to M(\lambda)$.

(b) $L(\mu)$ is a subquotient of $M(\lambda)$.

(c) μ is strongly linked to λ.

In that case, such morphism is unique up to a non-zero scalar, and injective.

Corollary 1.33. *Suppose λ is dominant and integral for $\mathfrak{g}$. There is a non-zero morphism $M(w \cdot \lambda) \to M(v \cdot \lambda)$ if and only if $v \leq w$ in $W_{\mathfrak{g}}$. In that case, this morphism is unique up to a non-zero scalar, and injective.*

Proofs. See [Hum08, 5.]. □

Every non-trivial morphism $M(\mu) \to M(\lambda)$ factors through $M_{\mathfrak{p}}(\mu)$, $M_{\mathfrak{p}}(\lambda)$ and the quotient projections, in the following way:

$$\begin{tikzcd} M(\mu) \arrow[r] \arrow[d, two heads] & M(\lambda) \arrow[d, two heads] \\ M_{\mathfrak{p}}(\mu) \arrow[r, dashed] & M_{\mathfrak{p}}(\lambda) \end{tikzcd}$$

The induced map $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ is called the ***standard morphism*** of the generalized Verma modules. It may not be injective, and may even be zero (but is unique up to a scalar, given $\mathfrak{p}$-dominant $\mu \uparrow \lambda$). We give a criteria for a standard morphism to be zero:

Proposition 1.34. *Suppose μ, λ are $\mathfrak{p}$-dominant, and that μ is strongly linked to λ. The standard morphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ is zero if and only if μ is strongly linked to $\sigma_\alpha \cdot \lambda$ for some $\alpha \in \Delta^+(\mathfrak{l}, \mathfrak{h})$.*

Corollary 1.35. *Suppose λ is dominant for $\mathfrak{g}$, and $w, v \in W^{\mathfrak{p}}$ such that $v \leq w$. The standard morphism $M_{\mathfrak{p}}(w \cdot \lambda) \to M_{\mathfrak{p}}(v \cdot \lambda)$ is zero if and only if there is a path from v to w in $W_{\mathfrak{g}}$ exiting $W^{\mathfrak{p}}$.*

A morphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ that is not standard is called ***non-standard***. There are no known criteria for their existence in general, but we give some necessary conditions for the existence.

Proposition 1.36. *If there exists a non-zero morphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$, then μ is strongly linked to λ, and $L(\mu)$ is a subquotient of $M_{\mathfrak{p}}(\lambda)$.*

Proofs. See [Bas85, 3.6]. □

In general, it is easy to see that a morphism $f\colon M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ is completely determined by $f(1 \otimes v_\mu)$, where $v_\mu \in F_{\mathfrak{p}}(\mu)$ is the highest weight vector. The vector $f(1 \otimes v_\mu)$ is a ***maximal vector*** (or sometimes called a ***singular vector***), meaning that it is annihilated by $\mathfrak{n}$. Morphisms $f\colon M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ are in a 1-1 correspondence with the maximal vectors in $M_{\mathfrak{p}}(\lambda)$ of the weight μ.

Remark 1.37 In the $|1|$-graded case however, it turns out that the dimensions of the Hom-spaces between generalized Verma modules are all less or equal to one. See [BC90a], [BC90b] and [BC86]. ■

Remark 1.38 Let us mention that both families $\{L(\lambda)\colon \lambda\, \mathfrak{p}\text{-dominant}\}$ of all the simple modules, and $\{M_{\mathfrak{p}}(\lambda)\colon \lambda\, \mathfrak{p}\text{-dominant}\}$ of all the generalized Verma modules are bases for the Grothendieck group of $\mathcal{O}^{\mathfrak{p}}$. The transition matrix between these bases is a subject of the Kazhdan-Lusztig conjectures, solved in the Borel case in 1981 by Beilinson-Bernstein, and independently by Brylinski-Kashiwara, using methods from the D-module theory and the intersection cohomology. The parabolic version was settled by Deodhar and Casian-Collinwood in 1987. See [Hum08, 8., 9.7.]. ■

1.4.4 Regular BGG resolution

Now we state the existence of the regular BGG resolution in the category $\mathcal{O}^{\mathfrak{p}}$, which is exactly dual of the resolution (1.10), in the sense described in Section 1.5.

Theorem 1.39 (Bernstein-Gelfand-Gelfand)**.** *Let $\lambda \in \mathfrak{h}^*$ be weight, integral and dominant for $\mathfrak{g}$. There is an exact sequence, called the **(regular) BGG resolution**, of the irreducible finite-dimensional $U(\mathfrak{g})$-module $F(\lambda) = L(\lambda)$ via the direct sums of the generalized Verma modules:*

$$\ldots \to \bigoplus_{w \in W^{\mathfrak{p}},\ l(w)=k} M_{\mathfrak{p}}(w \cdot \lambda) \to \ldots \to M_{\mathfrak{p}}(\lambda) \to L(\lambda) \to 0.$$

The morphisms in the above sequence are the direct sums of the standard morphisms $M_{\mathfrak{p}}(w \cdot \lambda) \to M_{\mathfrak{p}}(v \cdot \lambda)$ for $v \to w$ in $W^{\mathfrak{p}}$, which are all non-zero (see Corollary 1.35).

Proof. For the Borel case the proof can be found in [Hum08, 6.] or [BGG75], and for a general parabolic subalgebra in [RC80] or [Lep77]. □

1.4.5 Singular blocks

Any two blocks with regular generalized infinitesimal characters are mutually equivalent, by the theory of the Yantzen-Zuckerman translation functors (see [Hum08, 7.]), so we usually work only with $\mathcal{O}^{\mathfrak{p}}_{\rho}$, and denote it by $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$. It is usualy called the ***principal block***. The Hasse diagram $W^{\mathfrak{p}}$ parametrizes (almost by the definition) the $\mathfrak{p}$-dominant elements of the affine $W_{\mathfrak{g}}$-orbit of a dominant element. So, $W^{\mathfrak{p}}$ parametrizes the generalized Verma modules in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$, and also the simple modules in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$, by the affine action. We can do similarly in the singular blocks. Take an integral weight $\lambda \in \mathfrak{h}^*$ such that $\lambda + \rho$ is dominant, and denote by Σ the set of the ***simple singular roots*** for λ:

$$\Sigma = \{\alpha \in \Pi \colon \langle \lambda + \rho, \check{\alpha} \rangle = 0\}\,.$$

The subgroup of $W_{\mathfrak{g}}$ generated by $\{\sigma_\alpha \colon \alpha \in \Sigma\}$, denoted by W_Σ, is equal to the stabilizer $\{z \in W_{\mathfrak{g}} \colon z \cdot \lambda = \lambda\}$ (which follows from the lemma [Hum72, 10.3.B]). So, $\lambda + \rho$ is regular if and only if $\Sigma = \emptyset$.

Definition 1.40. *We define the* ***singular Hasse diagram*** *attached to the pair* $(\mathfrak{p}, \Sigma)$ *(or to the pair* $(\mathfrak{p}, \lambda)$*) to be*

$$W^{\mathfrak{p},\Sigma} := \{w \in W^{\mathfrak{p}} \colon w\sigma_\alpha \in W^{\mathfrak{p}} \textit{ and } w < w\sigma_\alpha, \textit{ for all } \alpha \in \Sigma\} \subseteq W^{\mathfrak{p}}.$$

The next two propositions are taken from [BN05].

Proposition 1.41. *The singular Hasse diagram* $W^{\mathfrak{p},\Sigma}$ *parametrizes the* $\mathfrak{p}$*-dominant elements of the affine orbit* $W_{\mathfrak{g}} \cdot \lambda$*.*

Proof. Let us first prove the following claim: For $w \in W_{\mathfrak{g}}$,

$$w \cdot \lambda \text{ is } \mathfrak{p}\text{-dominant if and only if } wW_\Sigma \subseteq W^{\mathfrak{p}}.$$

Assume $w \cdot \lambda$ is $\mathfrak{p}$-dominant, which implies $\langle \lambda + \rho, w^{-1}\alpha \rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{l}, \mathfrak{h})$. Since $\lambda + \rho$ is dominant, we must have $w^{-1}\alpha > 0$, that is, $w \in W^{\mathfrak{p}}$. If $z \in W_\Sigma$, then of course $wz \cdot \lambda = w \cdot \lambda$, and the same argument shows that $wz \in W^{\mathfrak{p}}$. Conversely, assume that $wW_\Sigma \subseteq W^{\mathfrak{p}}$ (in particular $w \in W^{\mathfrak{p}}$), and that $\langle \lambda + \rho, w^{-1}\alpha \rangle \leq 0$ for some $\alpha \in \Delta^+(\mathfrak{l}, \mathfrak{h})$. Then $w^{-1}\alpha > 0$ and $\langle \lambda + \rho, w^{-1}\alpha \rangle = 0$, therefore $\sigma_{w^{-1}\alpha} \in W_\Sigma$. By the assumption, we have $w\sigma_{w^{-1}\alpha} \in W^{\mathfrak{p}}$. But $(w\sigma_{w^{-1}\alpha})^{-1}(\alpha) = -w^{-1}\alpha < 0$, a contradiction.

To prove the proposition, it is enough now to prove that $W^{\mathfrak{p},\Sigma}$ is precisely the set of unique minimal length representatives of the left cosets wW_Σ that are contained in $W^{\mathfrak{p}}$. Any such representative is obviously in $W^{\mathfrak{p},\Sigma}$. Conversely, assume that $w \in W^{\mathfrak{p},\Sigma}$. If $\alpha \in \Sigma$, from $w < w\sigma_\alpha$ and the proposition [ČS09, 3.2.14.(4)] follows $w\alpha > 0$. If $w\alpha \in \Delta^+(\mathfrak{l}, \mathfrak{h})$, then $(w\sigma_\alpha)^{-1}(w\alpha) = -\alpha < 0$ would be a contradiction with $w\sigma_\alpha \in W^{\mathfrak{p}}$. We conclude that

$$w\Sigma \subseteq \Delta(\mathfrak{u}). \tag{1.11}$$

Now we will prove the following claim: If $z \in W_\Sigma$ and $\alpha \in \Sigma$ satisfy $wz \in W^{\mathfrak{p}}$ and

$z < z\sigma_\alpha$, then $wz < wz\sigma_\alpha$ and $wz\sigma_\alpha \in W^\mathfrak{p}$. From this, it follows by an induction on $l(z)$ that $wW_\Sigma \subseteq W^\mathfrak{p}$ and that w is a unique element of the minimal length in wW_Σ.

To prove the claim, take $z \in W_\Sigma$ and $\alpha \in \Sigma$ so that $wz \in W^\mathfrak{p}$ and $z < z\sigma_\alpha$. From $z < z\sigma_\alpha$ we see that $z\alpha > 0$, and from $z \in W_\Sigma$ and the dominance of $\lambda+\rho$, we conclude $z\alpha \in \operatorname{span}\Sigma$. Because of (1.11), it follows that $wz\alpha \in \Delta(\mathfrak{u})$, and in particular $wz\alpha > 0$, so $wz < wz\sigma_\alpha$. Now take $\beta \in \Delta^+(\mathfrak{l},\mathfrak{h})$. Since $wz \in W^\mathfrak{p}$, we have $z^{-1}w^{-1}\beta > 0$, and since $wz\alpha \in \Delta(\mathfrak{u})$, we have $z^{-1}w^{-1}\beta \neq \alpha$. From the lemma [Hum72, 10.2.B] we conclude that $\sigma_\alpha z^{-1}w^{-1}\beta > 0$, hence $zw\sigma_\alpha \in W^\mathfrak{p}$. □

So, $W^{\mathfrak{p},\Sigma}$ parametrizes the Verma modules in $\mathcal{O}^\mathfrak{p}_\lambda$, and also the simple modules in $\mathcal{O}^\mathfrak{p}_\lambda$, by the affine action.

Proposition 1.42. *Let $\mathfrak{p}$ be a $|1|$-graded parabolic subalgebra of a complex simple Lie algebra. If Σ contains two adjacent simple roots, then $\mathcal{O}^\mathfrak{p}_\lambda = \emptyset$.*

Proof. Assume that $w \in W^{\mathfrak{p},\Sigma}$ and that $\alpha, \beta \in \Sigma$ are adjacent. Then $\alpha+\beta$ is a root. From (1.11) we see that $w\alpha, w\beta \in \Delta(\mathfrak{u})$. But $w\alpha + w\beta = w(\alpha+\beta)$ is a root, again in $\Delta(\mathfrak{u})$, which is a contradiction with $\mathfrak{u}$ being abelian. □

The criteria for a singular block to be zero are given in [Pla11].

There is a certain equivalence of categories of a singular infinitesimal block and a regular infinitesimal block of some other parabolic subalgebra of some other Lie algebra (of the lower rank), called the ***Enright-Shelton equivalence***:

Theorem 1.43 (Enright-Shelton)**.** *Let $\mathfrak{p}$ be a $|1|$-graded parabolic subalgebra of a complex simple Lie algebra $\mathfrak{g}$, and let λ, Σ and $W^{\mathfrak{p},\Sigma}$ be as before.*

- *Assume that either Π has only one root length, or that Π has two root lengths and all the roots in Σ are short. Then there exists an equivalence of categories*

$$\mathcal{E}\colon \mathcal{O}^\mathfrak{p}_\lambda \xrightarrow{\sim} \mathcal{O}^{\mathfrak{p}'}_{reg},$$

  for some $|1|$-graded parabolic subalgebra $\mathfrak{p}'$ of a complex simple Lie algebra $\mathfrak{g}'$, of the lower rank. Furthermore, there exists a bijection $W^{\mathfrak{p},\Sigma} \xrightarrow{\sim} W^{\mathfrak{p}'}$, $w \mapsto w'$, such that

$$\begin{aligned} \mathcal{E}\colon\ & L(w\cdot\lambda) \mapsto L(w'\cdot 0), \\ & M_\mathfrak{p}(w\cdot\lambda) \mapsto M_{\mathfrak{p}'}(w'\cdot 0). \end{aligned}$$

- *Assume that Π has two root lengths and that Σ contains the long root. Then there exists an equivalence of categories*

$$\mathcal{E}\colon \mathcal{O}^\mathfrak{p}_\lambda \xrightarrow{\sim} \mathcal{O}^{\mathfrak{p}'}_{reg} \oplus \mathcal{O}^{\mathfrak{p}'}_{reg},$$

for some $|1|$*-graded parabolic subalgebra* $\mathfrak{p}'$ *of a complex simple Lie algebra* $\mathfrak{g}'$*, of the lower rank. Furthermore, we have a disjoint union*

$$W^{\mathfrak{p},\Sigma} = W^{\mathfrak{p},\Sigma}_{even} \cup W^{\mathfrak{p},\Sigma}_{odd}$$

that corresponds to a decomposition of $\mathcal{O}^{\mathfrak{p}}_{\lambda}$ *into a direct sum of categories*

$$\mathcal{O}^{\mathfrak{p}}_{\lambda} = \mathcal{O}^{\mathfrak{p}}_{\lambda,even} \oplus \mathcal{O}^{\mathfrak{p}}_{\lambda,odd},$$

such that all the extensions between modules in the different summands are zero. There exist bijections $W^{\mathfrak{p},\Sigma}_{even} \xrightarrow{\sim} W^{\mathfrak{p}'}$ *and* $W^{\mathfrak{p},\Sigma}_{odd} \xrightarrow{\sim} W^{\mathfrak{p}'}$*, both denoted by* $w \mapsto w'$*, and equivalences of categories*

$$\mathcal{E}_{even}\colon \mathcal{O}^{\mathfrak{p}}_{\lambda,even} \xrightarrow{\sim} \mathcal{O}^{\mathfrak{p}'}_{reg}, \quad \mathcal{E}_{odd}\colon \mathcal{O}^{\mathfrak{p}}_{\lambda,odd} \xrightarrow{\sim} \mathcal{O}^{\mathfrak{p}'}_{reg},$$

such that $\mathcal{E} = \mathcal{E}_{even} \oplus \mathcal{E}_{odd}$ *and*

$$\begin{aligned} \mathcal{E}_{even}, \mathcal{E}_{odd}\colon \; & L(w \cdot \lambda) \mapsto L(w' \cdot 0), \\ & M_{\mathfrak{p}}(w \cdot \lambda) \mapsto M_{\mathfrak{p}'}(w' \cdot 0). \end{aligned}$$

If the parabolic subalgebra $\mathfrak{p}$ *is one of* $(A_n, A_{i-1} \times A_{n-i})$*,* (C_n, A_{n-1}) *or* (D_n, A_{n-1})*, then* $\mathfrak{p}'$ *is* $(A_{n-2s}, A_{i-s-1} \times A_{n-s-i})$*,* (C_{n-2s}, A_{n-2s-1}) *or* (D_{n-2s}, A_{n-2s-1})*, respectively, where* $s = |\Sigma|$*. There are explicit descriptions of* $\mathfrak{p}'$ *in the other cases also.*

Proof. It is proved in [ES87] and [ES89], and explained in [EHP14, 5.5] and [BH09, 7.3.]. Even though the construction of the functors $\mathcal{E}$, $\mathcal{E}_{\text{even}}$ and $\mathcal{E}_{\text{odd}}$ is extremely complicated, the bijections between the singular Hasse diagrams and the regular ones are very simple and explicit. In the case when the long root is singular, the decomposition of the sigular Hasse diagram to the even and the odd part can be done by considering the Lascoux-Schützenberger notation. This will be explained in Chapter 4 for the type (C_n, A_{n-1}). □

1.5 Duality

For fixed G and P, there in a contravariant correspondence between the generalized Verma modules $M_{\mathfrak{p}}(\lambda)$ and the homogeneous sheaves $\mathcal{O}_{\mathfrak{p}}(\lambda)$ over G/P. More precisely,

Theorem 1.44. *If λ and μ are integral for $\mathfrak{g}$ and dominant for $\mathfrak{p}$, then*

$$\mathrm{Hom}_{\mathfrak{g}}(M_{\mathfrak{p}}(\mu), M_{\mathfrak{p}}(\lambda)) \cong \mathrm{Diff}_G(\mathcal{O}_{\mathfrak{p}}(\lambda), \mathcal{O}_{\mathfrak{p}}(\mu)). \tag{1.12}$$

There is a proof in [ČSS01, appendix of the preprint], and also brief sketches in [ČS09, 1.4.], [BE16, 11.1] and [Bas85, 3.3]. We describe it here.

If we think of the elements of $\mathfrak{g}$ as the left-invariant vector fields on G, then $U(\mathfrak{g})$ corresponds to the invariant differential operators $\mathcal{O}_G \to \mathcal{O}_G$, and this correspondence respects the order. For a pair $(X, v) \in U(\mathfrak{g}) \times F_{\mathfrak{p}}(\lambda)$ we can define an operator acting on the stalks over $eP \in G/P$ of the sheaves $\mathcal{O}_{\mathfrak{p}}(\lambda) \to \mathcal{O}_{G/P}$, by defining it on the fiber over $eP \in G/P$ of the bundles $J^k(\mathcal{O}_{\mathfrak{p}}(\lambda)) \to G/P \times \mathbb{C}$, where k is the order of X. The formula is

$$j^k_e(s) \mapsto v\Big((Xs)(e)\Big).$$

For $Y \in \mathfrak{p}$, by (1.3) it follows that the pairs (Y, v) and $(1, Yv)$ define the same operator. So the previous map descends to $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_{\mathfrak{p}}(\lambda) = M_{\mathfrak{p}}(\lambda)$, and it becomes a natural isomorphism of $U(\mathfrak{g})$-modules

$$M_{\mathfrak{p}}(\lambda) \cong \mathrm{Diff}(\mathcal{O}_{\mathfrak{p}}(\lambda), \mathcal{O}_{G/P})_{eP}. \tag{1.13}$$

Moreover, $M_{\mathfrak{p}}(\lambda)$ induces $\mathrm{Diff}(\mathcal{O}_{\mathfrak{p}}(\lambda), \mathcal{O}_{G/P})$ as a homogeneous sheaf. There is a natural filtration on $M_{\mathfrak{p}}(\lambda)$ induced by the filtration of $U(\mathfrak{u}^-)$ by the order:

$$M_{\mathfrak{p}}(\lambda)_k := U(\mathfrak{u}^-)_k \otimes_{\mathbb{C}} F_{\mathfrak{p}}(\lambda),$$

and these are $\mathfrak{p}$-submodules of $M_{\mathfrak{p}}(\lambda)$. The isomorphism (1.13) respects the filtrations, so we get

$$M_{\mathfrak{p}}(\lambda)_k \cong \mathrm{Hom}_{\mathbb{C}}(J^k(\mathcal{O}_{\mathfrak{p}}(\lambda))_{eP}, \mathbb{C}) = \Big(J^k(\mathcal{O}_{\mathfrak{p}}(\lambda))_{eP}\Big)^*.$$

Since those are finite-dimensional spaces, we can conclude:

Proposition 1.45. *As a homogeneous vector bundle, the jet bundle $J^k(\mathcal{O}_{\mathfrak{p}}(\lambda))$ is induced from the $\mathfrak{p}$-module $M_{\mathfrak{p}}(\lambda)^*_k$. Moreover, the sequence (1.6) is induced from the modules contragredient to the filtration*

$$M_{\mathfrak{p}}(\lambda)_0 \cong F_{\mathfrak{p}}(\lambda) \hookrightarrow M_{\mathfrak{p}}(\lambda)_1 \hookrightarrow M_{\mathfrak{p}}(\lambda)_2 \hookrightarrow \dots,$$

and the formal jet bundle $J^\infty(\mathcal{O}_{\mathfrak{p}}(\lambda))$ by

$$\varprojlim\, (M_{\mathfrak{p}}(\lambda)^*_k) = \Big(\varinjlim M_{\mathfrak{p}}(\lambda)_k\Big)^* = M_{\mathfrak{p}}(\lambda)^*.$$

The jet exact sequence (1.7) is induced from the modules contragredient to

$$0 \to M_{\mathfrak{p}}(\lambda)_{k-1} \hookrightarrow M_{\mathfrak{p}}(\lambda)_k \to {}^{M_{\mathfrak{p}}(\lambda)_k}\big/_{M_{\mathfrak{p}}(\lambda)_{k-1}} \cong S^k(\mathfrak{u}^-) \otimes_{\mathbb{C}} F_{\mathfrak{p}}(\lambda) \to 0.$$

Proof of 1.44. Now let us see how the invariant differential operators correspond to the homomorphisms of generalized Verma modules. Suppose $D\colon \mathcal{O}_{\mathfrak{p}}(\lambda) \to \mathcal{O}_{\mathfrak{p}}(\mu)$ is an invariant differential operator of order k. It is determined by a homogeneous bundle morphism

$$J^k(\mathcal{O}_{\mathfrak{p}}(\lambda)) \to G \times_P E_{\mathfrak{p}}(\mu),$$

which is (by the invariance and the transitivity of the G-action) determined by its restriction on the fiber over eP:

$$M_{\mathfrak{p}}(\lambda)_k^* \to E_{\mathfrak{p}}(\mu),$$

and this map is $\mathfrak{p}$-invariant. Dualizing, we get a $\mathfrak{p}$-invariant map

$$F_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)_k \hookrightarrow M_{\mathfrak{p}}(\lambda).$$

Such map (by the Frobenius reciprocity) defines a $\mathfrak{g}$-invariant morphism

$$M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda).$$

This process is clearly reversible, so we have proved (1.12). □

It is not hard to check that this correspondence is functorial. One can also see that standard morphisms of generalized Verma modules correspond to the standard invariant differential operators of homogeneous sheaves.

Let us see now what is the possible order of an invariant differential operator, in the $|1|$-graded case.

Proposition 1.46. *Suppose P is $|1|$-graded. If there exists a non-trivial invariant differential operator $D\colon \mathcal{O}_{\mathfrak{p}}(\lambda) \to \mathcal{O}_{\mathfrak{p}}(\mu)$, then its order is $(\lambda - \mu)(E)$, where $E \in \mathfrak{h}$ is the grading element for $\mathfrak{p}$.*

Proof. By the duality (1.13), D corresponds to a homomorphism $\varphi\colon M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$, and by the construction, its order is the smallest k such that $\varphi(1 \otimes v_\mu) \in M_{\mathfrak{p}}(\lambda)_k$, where $v_\mu \in F_{\mathfrak{p}}(\mu)$ is the highest weight vector. Therefore, $\varphi(1 \otimes v_\mu)$ is a finite sum of terms

$$u_1 u_2 \dots u_s \otimes v, \quad s \le k, \tag{1.14}$$

where we can assume that all $u_j \in \mathfrak{u}^-$ and $v \in F_{\mathfrak{p}}(\lambda)$ are weight vectors. So each term (1.14) is a weight vector. But $\varphi(1 \otimes v_\mu)$ is of the weight μ, so each term (1.14) must also be of the weight μ. Let us calculate the action of E on a such term:

$$E \cdot u_1 u_2 \dots u_s \otimes v = E u_1 u_2 \dots u_s \otimes v =$$
$$= [E, u_1] u_2 \dots u_s \otimes v + u_1 [E, u_2] \dots u_s \otimes v + \quad \dots \quad + u_1 u_2 \dots u_s \otimes E \cdot v =$$
$$= (-s + \lambda(E))\, u_1 u_2 \dots u_s \otimes v.$$

It follows that $\mu(E) = -s + \lambda(E)$, for each term in (1.14). Therefore, each term has $s = k$, and $k = \lambda(E) - \mu(E)$. □

It follows from the previous proof that an operator (if it exists) is uniquely determined by its symbol. In a higher grading, a similar argument shows that $(\lambda - \mu)(E)$ is an upper bound for the order of an invariant differential operator.

There are more information and examples in [Bas90] and [ER87].

1.6 Spectral sequences

In this section we explain the terminology related to the theoy of spectral sequences, which will be used in Chapter 2. We do not give here the standard constructions of the particular spectral sequences used later, for example the Leray spectral sequence, or the hypercohomology specral sequence (also called the spectral sequence of a differental sheaf), which can be found in [WW90, 3.6], [Bry07], [Wei95, 5] and [KV95, D.]. We work in a fixed abelian category.

Definition 1.47. *A* ***(cohomological) spectral sequence*** *is a family E of objects:*

$$\{E_r^{pq} \quad : \quad p, q \in \mathbb{Z}, r \in \mathbb{N}\},$$

and morphisms (also called the ***differentials*** *of the bi-degree $(r, -r+1)$):*

$$d_r^{pq} \colon E_r^{pq} \to E_r^{p+r, q-r+1},$$

such that all the consecutive compositions are 0, and the conditions:

$$E_{r+1}^{pq} = {}^{\operatorname{Ker} d_r^{pq}} \Big/ {}_{\operatorname{Im} d_r^{p-r, q+r-1}} \qquad =: H(E_r^{pq}, d_r).$$

If $E_r^{pq} = 0$ whenever $p < 0$ or $q < 0$, we say that the spectral sequence is a ***first quadrant*** *one. For a fixed r, we call the family $\{E_r^{pq}\}_{pq}$ the r**-th page**.*

See Figure 1.2. The point is that on each page r, the lines with the slope $(r, -r+1)$ are cochain complexes, whose cohomologies are written in the next page.

$$E_1^{pq} = \begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
E_1^{03} \arrow[r] & E_1^{13} \arrow[r] & E_1^{23} \arrow[r] & E_1^{33} \arrow[r] & \dots \\
E_1^{02} \arrow[r] & E_1^{12} \arrow[r] & E_1^{22} \arrow[r] & E_1^{32} \arrow[r] & \dots \\
E_1^{01} \arrow[r] & E_1^{11} \arrow[r] & E_1^{21} \arrow[r] & E_1^{31} \arrow[r] & \dots \\
E_1^{00} \arrow[r] & E_1^{10} \arrow[r] & E_1^{20} \arrow[r] & E_1^{30} \arrow[r] & \dots
\end{tikzcd}$$

$$E_2^{pq} = \begin{tikzcd}
\vdots \arrow[drr] & \vdots \arrow[drr] & \vdots & \vdots & & \\
E_2^{03} \arrow[drr] & E_2^{13} \arrow[drr] & E_2^{23} \arrow[drr] & E_2^{33} \arrow[drr] & \dots & \dots \\
E_2^{02} \arrow[drr] & E_2^{12} \arrow[drr] & E_2^{22} \arrow[drr] & E_2^{32} \arrow[drr] & \dots & \dots \\
E_2^{01} \arrow[drr] & E_2^{11} \arrow[drr] & E_2^{21} \arrow[drr] & E_2^{31} \arrow[drr] & \dots & \dots \\
E_2^{00} & E_2^{10} & E_2^{20} & E_2^{30} & \dots & \dots
\end{tikzcd}$$

$$E_3^{pq} = \begin{tikzcd}
\vdots \arrow[ddrrr] & \vdots & \vdots & \vdots & & & \\
\vdots \arrow[ddrrr] & \vdots & \vdots & \vdots & & & \\
E_3^{03} \arrow[ddrrr] & E_3^{13} \arrow[ddrrr] & E_3^{23} \arrow[ddrrr] & E_3^{33} \arrow[ddrrr] & \dots & \dots & \dots \\
E_3^{02} \arrow[ddrrr] & E_3^{12} \arrow[ddrrr] & E_3^{22} \arrow[ddrrr] & E_3^{32} \arrow[ddrrr] & \dots & \dots & \dots \\
E_3^{01} & E_3^{11} & E_3^{21} & E_3^{31} & \dots & \dots & \dots \\
E_3^{00} & E_3^{10} & E_3^{20} & E_3^{30} & \dots & \dots & \dots
\end{tikzcd}$$

Figure 1.2: First three pages of a first quadrant spectral sequence

We assume from now on that our spectral sequences are first quadrant ones. It will be so in all our applications, and otherwise we would have to deal with a many technical issues.

Definition 1.48. *Let E be a first quadrant spectral sequence.*

- *We say that E **converges**, if for all p, q there exists r depending on (p, q) so that*

$$E_r^{pq} = E_{r+1}^{pq} = E_{r+2}^{pq} = \dots =: E_\infty^{pq}.$$

*We call $\{E_\infty^{pq}\}_{pq}$ the **abutment** of the spectral sequence E.*

- *If that r is independent of (p,q), we say that our spectral sequence* ***stabilizes*** *at the page (or the level) r. This is equivalent to the condition $d_s^{pq}=0$ whenever $s\geq r$.*
- *The spectral sequence* ***degenerates*** *at the page r, if it stabilizes at the page r, and for all k there exists $p(k)$ so that $E_r^{p,k-p}=0$ whenever $p\neq p(k)$. The last condition simply means that each skew-diagonal on the page r contains at most one non-trivial object.*
- *The spectral sequence* ***collapses****, if it degenerates at the page* 2.

Obviously, any first quadrant spectral sequence converges, simply because the differentials from any fixed point eventually go out of the first quadrant. The usual notation for the convergence is

$$E_{r_0}^{pq}\Longrightarrow H^{p+q}.$$

E_{r_0} is the initial page that is given explicitly, usualy for $r_0=1$ or 2. The notation means that we have a family of objects H^n, each implicitly possessing a finite decreasing filtration, so that $E_\infty^{p,q}$ is the p-th graded piece of H^{p+q}. By summing over the skew-diagonals $p+q=n$ on the abutment, for each n we have a graded isomorphism

$$\bigoplus_{p+q=n}E_\infty^{pq}\cong\mathrm{Gr}(H^n). \tag{1.15}$$

This is the most useful in the case of a spectral sequence that degenerate, because then for each n there is at most one pair (p,q) so that $p+q=n$ and $E_\infty^{pq}\neq 0$. So, (1.15) becomes

$$E_\infty^{pq}\cong H^{p+q},\quad\text{for all } p,q\geq 0.$$

Chapter 2

The Penrose transform

A standard reference for this chapter is the book [BE16]. Choose standard parabolic subgroups $P, R \subseteq G$. Their intersection $Q = P \cap R$ is also a standard parabolic subgroup, with the crosses on the Dynkin diagram obtained as the union of the crosses for P and R. Choose an open submanifold $X \subseteq G/P$, and define $Y := \tau^{-1}(X)$ and $Z := \eta(Y)$. The subsets Y and Z are open submanifolds of G/Q and G/R, respectively. Then we have the ***double fibration***, and the ***restricted double fibration***, respectively:

$$\begin{tikzcd} & G/Q \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ G/R & & G/P, \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z & & X. \end{tikzcd}$$

The left side of each double fibration is called the ***twistor space***. We will often restrict sheaves and bundles from the double fibration to the restricted double fibration without changing the notation, since the sheaf operations commute with the restrictions to the open submanifolds. The full double fibration is more comfortable for performing sheaf operations because of the homogeneity, but we are really interested in the restricted double fibration.

Start with a $\mathfrak{g}$-integral and $\mathfrak{r}$-dominant weight λ, and form the sheaf $\mathcal{O}_{\mathfrak{r}}(\lambda)$ on $Z \subseteq G/R$.

2.1 Pull-back stage

Consider the ***topological inverse image sheaf*** $\eta^{-1}\mathcal{O}_r(\lambda)$ on Y. Since η is an open map, the sheaf $\eta^{-1}\mathcal{O}_r(\lambda)$ can be identified with the sheaf of sections of the pull-back bundle $\eta^*(G \times_R E_{\mathfrak{r}}(\lambda)) \to Y$ that are constant along the fibers of η. So there is a canonical map on the global sections

$$\Gamma(Z, \mathcal{O}_{\mathfrak{r}}(\lambda)) \to \Gamma(Y, \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda)), \tag{2.1}$$

given by the precomposition with η. This map exists for any locally free sheaf on Z, the homogeneity of $\mathcal{O}_{\mathfrak{r}}(\lambda)$ was not important. We claim that (2.1) induces isomorphisms on all the cohomologies. To see this, choose a locally free acyclic resolution $0 \to \mathcal{O}_{\mathfrak{r}}(\lambda) \to \mathcal{E}^\bullet$ on Z (the Dolbeault resolution for example, [Tay02, 10.1]), and an injective resolution $0 \to \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda) \to \mathcal{I}^\bullet$ on Y. Recall that η^{-1} is an exact functor, so the sequence $0 \to \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda) \to \eta^{-1}\mathcal{E}^\bullet$ is exact. Observe the following composition of the canonical maps:

$$\begin{tikzcd}
H^r(Z, \mathcal{O}_{\mathfrak{r}}(\lambda)) \arrow[d, equal] & & H^r(Y, \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda)) \arrow[d, equal] \\
H^r(\Gamma(Z, \mathcal{E}^\bullet)) \arrow[r, "(2.1)"] & H^r(\Gamma(Y, \eta^{-1}\mathcal{E}^\bullet)) \arrow[r] & H^r(\Gamma(Y, \mathcal{I}^\bullet)).
\end{tikzcd} \tag{2.2}$$

When the fibers of η are connected, the map (2.1) is obviously an isomorphism. Moreover, from [Buc83] follows that if the fibers of η are smoothly contractible, then $H^r(Y, \eta^{-1}\mathcal{E}^k) = 0$ for $r > 0$, $k \geq 0$. Consequently, $0 \to \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda) \to \eta^{-1}\mathcal{E}^\bullet$ is an acyclic resolution, and so the unlabeled arrow in (2.2) is also an isomorphism. In conclusion:

Theorem 2.1. *In the restricted double fibration, suppose that the fibers of η are contractible by a smooth homotopy which preserves η. Then we have canonical isomorphims*

$$H^r(Z, \mathcal{O}_{\mathfrak{r}}(\lambda)) \cong H^r(Y, \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda)), \qquad \textit{for all } r \geq 0. \tag{2.3}$$

2.2 Push-down stage

The weight λ remains dominant on the fibers of η, which themselves are generalized flag manifolds. We have already seen that the restrictions of $\eta^{-1}\mathcal{O}_r(\lambda)$ to the fibers of η are locally constant sheaves. Therefore, on each fiber $\eta^{-1}(x) \subseteq G/Q$ we can construct the regular BGG resolution of the corresponding restriction $\eta^{-1}\mathcal{O}_r(\lambda)|_{\eta^{-1}(x)}$. One can see that these resolutions vary in a holomorphic homogeneous way as we vary $x \in G/R$. In other words:

Theorem 2.2. *In the above setting, there is an exact sequence of sheaves on G/Q and* *standard invariant differential operators, called the* ***relative BGG resolution****:*

$$0 \to \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda) \to \Delta_\eta^\bullet(\lambda), \quad \textit{where} \;\; \Delta_\eta^k(\lambda) = \bigoplus_{w \in W_{\mathfrak{r}}^{\mathfrak{q}},\; l(w)=k} \mathcal{O}_{\mathfrak{q}}(w \cdot \lambda). \tag{2.4}$$

Here $W_{\mathfrak{r}}^{\mathfrak{q}}$ is the relative Hasse diagram corresponding to the fibration $\eta\colon G/Q \to G/R$.

Technical details about the relative BGG sequences (in much greater generality of the parabolic geometries) can be found in [ČS16] and [ČS15].

Now we will consider the higher direct images along τ, denoted by τ_*^q, of the sequence (2.4). A standard tool for manipulating with the higher direct images is the ***Leray spectral sequence***:

$$E_2^{pq} = H^p(X, \tau_*^q \Delta_\eta^k(\lambda)) \implies H^{p+q}(Y, \Delta_\eta^k(\lambda)), \qquad k \geq 0. \tag{2.5}$$

Let us assume that $X \subseteq G/P$ is an open ***Stein subset*** (for the definition one can see [GR12] or [Tay02]), for example the big affine cell, or an open ball or a polydisc inside the big affine cell. One of the cornerstones of the theory of several complex variables is:

Theorem 2.3 (Cartan's theorem B)**.** *If $\mathcal{F}$ is a coherent sheaf on a Stein manifold X, then*

$$H^k(X, \mathcal{F}) = 0, \qquad k \geq 1.$$

By (the corollary of) Bott-Borel-Weil theorem, the sheaves $\tau_*^q \Delta_\eta^k(\lambda)$ are locally free, and therefore coherent. Cartan's theorem B then implies that for each $k \geq 0$ the Leray spectral sequence (2.5) collapses, and gives an isomorphism

$$\Gamma(X, \tau_*^q \Delta_\eta^k(\lambda)) \cong H^q(Y, \Delta_\eta^k(\lambda)), \qquad k \geq 0. \tag{2.6}$$

Lastly, the ***hypercohomology spectral sequence*** applied to the exact sequence (2.4) has the form

$$E_1^{pq} = H^p(Y, \Delta_\eta^q(\lambda)) \implies H^{p+q}(Y, \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda)). \tag{2.7}$$

Plugging (2.6) and (2.3) into (2.7), we get:

Theorem 2.4 (Baston-Eastwood)**.** *In the above setting, there is a first quadrant spectral sequence:*

$$E_1^{pq} = \Gamma(X, \tau_*^q \Delta_\eta^p(\lambda)) \implies H^{p+q}(Z, \mathcal{O}_{\mathfrak{r}}(\lambda)). \tag{2.8}$$

The differentials in the spectral sequence are of the bi-degree $(r, -r+1)$, and moreover, they are all invariant differential operators. On the first page E_1 they are all induced from the relative BGG sequence, and are the standard ones, but on the other pages the induced differentials are non-standard invariant differential operators.

Chapter 3

The symplectic group and the Lagrangian Grassmannian

Let V be a finite-dimensional complex ***symplectic vector space***, that is, a complex vector space equipped with a non-degenerate bilinear form B on V which is skew-symmetric (meaning that $B(v, w) = -B(w, v)$, for all $v, w \in V$). All such forms are equivalent on a complex vector space. We will use the standard notations $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ for the general linear group and the special linear group, and their Lie algebras $\mathfrak{gl}(V)$ and $\mathfrak{sl}(V)$, respectively.

3.1 The symplectic group

Definition 3.1. *The **complex symplectic group** $\mathrm{Sp}(V)$ is the group of all invertible linear operators $g\colon V \to V$ that preserve B, that is:*

$$B(gv, gw) = B(v, w), \quad v, w \in V.$$

It is obviously closed in $\mathrm{GL}(V)$, so it is a Lie group. By deriving the defining equation for $\mathrm{Sp}(V)$, we get:

Proposition 3.2. *The Lie algebra $\mathfrak{sp}(V)$ of $\mathrm{Sp}(V)$ (called the **complex symplectic Lie algebra**) consists of all linear operators $X\colon V \to V$ that satisfy*

$$B(Xv, w) + B(v, Xw) = 0, \quad v, w \in V.$$

Using an analogue of the Gram-Schmidt process, one can show the following:

Proposition 3.3. *The dimension of a symplectic vector space is necessarily an even* *number, say $2n$, and there exist a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ for V such that*

$$B(e_i, e_j) = B(f_i, f_j) = 0, \quad B(e_i, f_j) = \delta_{ij}, \quad i, j = 1, \dots, n. \tag{3.1}$$

*Such a basis is called a **symplectic basis**, or a **Darboux basis**, and the matrix of the symplectic form B in this basis is $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n is the $n \times n$ identity matrix.*

Sometimes it will be more convenient to write e_{n+i} instead of f_i, for $i = 1, \dots, n$.

Fix a symplectic basis once and for all, and from now on we work in the coordinates. It is easy to work out the following:

Proposition 3.4. *By fixing a symplectic basis for V, we have:*

- *The group $\mathrm{Sp}(V)$ is isomorphic to*

$$G = \mathrm{Sp}(2n, \mathbb{C}) := \{g \in \mathrm{GL}(2n, \mathbb{C}) \ : \ g^T J g = J\}.$$

It consists of matrices of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D \in M(n, \mathbb{C})$ satisfy

$$A^T C = C^T A, \ B^T D = D^T B, \ \text{and } A^T D - C^T B = I_n.$$

Moreover, it is a complex, simple, connected, simply connected, non-compact linear Lie group of the dimension $2n^2 + n$.

- *Its Lie algebra is*

$$\begin{aligned} \mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}) &= \{X \in M(2n, \mathbb{C}) \ : \ X^T J + J X = 0\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \ : \ A, B, C \in M(n, \mathbb{C}), \ B = B^T, \ C = C^T \right\}, \end{aligned} \tag{3.2}$$

a complex simple Lie algebra of the dimension $2n^2 + n$.

For a proof of simple connectedness of $\mathrm{Sp}(2n, \mathbb{C})$, see [FH04, §23.1.]. Since obviously $\mathfrak{sp}(2n, \mathbb{C}) \subseteq \mathfrak{sl}(2n, \mathbb{C})$ and since $\mathrm{Sp}(2n, \mathbb{C})$ is connected, it follows by exponentiating that $\mathrm{Sp}(2n, \mathbb{C}) \subseteq \mathrm{SL}(2n, \mathbb{C})$.

3.1.1 Roots, weights and the Weyl group

We choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consisting of the diagonal matrices. Elements of $\mathfrak{h}$ and $\mathfrak{h}^*$ will be written as n-tuples in the obvious way. The roots will be written in the round brackets, and weights in the square bracket, even though formally there is no difference. We use the standard euclidean product on $\mathfrak{h}$ and $\mathfrak{h}^*$, which is proportional to the Killing form. Using (3.2), it is easy but tedious to find the root space decomposition of $\mathfrak{g}$. It is presented in Figure 3.1, where the positive roots are:

$$\left(\begin{array}{ccccc|ccccc}
* & a_{12} & a_{13} & \cdots & a_{1n} & b_1 & c_{12} & c_{13} & \cdots & c_{1n} \\
-a_{12} & * & a_{23} & \cdots & a_{2n} & c_{12} & b_2 & c_{23} & \cdots & c_{2n} \\
-a_{13} & -a_{23} & * & \cdots & a_{3n} & c_{13} & c_{23} & b_3 & \cdots & c_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{1n} & -a_{2n} & -a_{3n} & \cdots & * & c_{1n} & c_{2n} & c_{3n} & \cdots & b_n \\
\hline
-b_1 & -c_{12} & -c_{13} & \cdots & -c_{1n} & * & -a_{12} & -a_{13} & \cdots & -a_{1n} \\
-c_{12} & -b_2 & -c_{23} & \cdots & -c_{2n} & a_{12} & * & -a_{23} & \cdots & -a_{2n} \\
-c_{13} & -c_{23} & -b_3 & \cdots & -c_{3n} & a_{13} & a_{23} & * & \cdots & -a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-c_{1n} & -c_{2n} & -c_{3n} & \cdots & -b_n & a_{1n} & a_{2n} & a_{3n} & \cdots & *
\end{array}\right)$$

Figure 3.1: Root space decomposition of $\mathfrak{sp}(2n, \mathbb{C})$

$$\begin{aligned}
\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{a_{ij} = \epsilon_i - \epsilon_j &= (\underbrace{\overbrace{0, \ldots, 0, 1}^{i}, 0, \ldots, 0, -1}_{j}, 0, \ldots, 0), && 1 \le i < j \le n, \\
b_i = 2\epsilon_i &= (\overbrace{0, \ldots, 0, 2}^{i}, 0, \ldots, 0), && 1 \le i \le n, \\
c_{ij} = \epsilon_i + \epsilon_j &= (\underbrace{\overbrace{0, \ldots, 0, 1}^{i}, 0, \ldots, 0, 1}_{j}, 0, \ldots, 0), && 1 \le i < j \le n\},
\end{aligned}$$

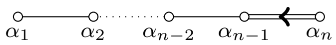
Here $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{h}^*$ are the projections to the coordinates. The simple roots are

$$\alpha_1 = a_{12}, \quad \alpha_2 = a_{23}, \quad \ldots, \quad \alpha_{n-1} = a_{n-1,n}, \quad \alpha_n = b_n,$$

and they constitute a basis for $\mathfrak{h}^*$. The decompositions of the positive roots with respect to the simple roots are the following:

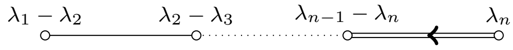
$$\begin{aligned}
a_{ij} &= \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}, \\
b_i &= 2\alpha_i + 2\alpha_{i+1} + \ldots + 2\alpha_{n-1} + \alpha_n, \\
c_{ij} &= \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} + 2\alpha_j + 2\alpha_{j+1} + \ldots + 2\alpha_{n-1} + \alpha_n.
\end{aligned}$$

The Dynkin diagram of $\mathfrak{g}$ is of the type C_n:



.

A weight $\lambda \in \mathfrak{h}^*$ may be written as an n-tuple $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$, or in the Dynkin diagram notation as:

$\lambda =$ 

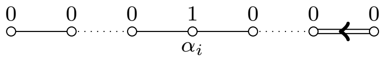
.

A weight λ is integral if all $\lambda_i \in \mathbb{Z}$, and dominant if $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n \geq 0$. A weight is regular if and only if it does not have two coordinates with the same absolute value, and all the coordinates are non-zero. The half sum of all positive roots is

$\rho = [n, n-1, \dots, 1] =$ 

.

The fundamental weights are

$\omega_i = [\underbrace{1, 1, \dots, 1}_{i}, 0, 0, \dots, 0] =$ 

, $\qquad 1 \leq i \leq n.$

The Weyl group $W_\mathfrak{g}$ acts on $\mathfrak{h}^*$ by the following formulas:

$$\begin{aligned}
\sigma_{a_{ij}}[\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_n] &= [\lambda_1, \dots, \lambda_j, \dots, \lambda_i, \dots, \lambda_n], \\
\sigma_{b_i}\ [\lambda_1, \dots, \lambda_i, \dots, \lambda_n] &= [\lambda_1, \dots, -\lambda_i, \dots, \lambda_n], \\
\sigma_{c_{ij}}[\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_n] &= [\lambda_1, \dots, -\lambda_j, \dots, -\lambda_i, \dots, \lambda_n].
\end{aligned}$$

Thus $W_\mathfrak{g}$ is isomorphic to a semidirect product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, where the first factor is normal, and therefore $|W_\mathfrak{g}| = 2^n n!$.

3.1.2 Fundamental representations

Let us now describe the ***fundamental representations*** of $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. Those are, by definition, the irreducible finite-dimensional representations whose highest weights are fundamental weights ω_i. Fix a symplectic basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ for V.

The vector space V is a representation for $\mathfrak{g}$ in the obvious way; it is called the ***standard representation***. The vectors e_i are of the weight ϵ_i, and f_i of the weight $-\epsilon_i$. Therefore, V is irreducible, with the highest weight $\epsilon_1 = \omega_1$. So, $V = F(\omega_1)$.

Now let us take $k \in \{2, \dots, n\}$, and observe the action of $\mathfrak{g}$ on the k-th exterior power $\bigwedge^k V$. The vector $e_1 \wedge \dots \wedge e_k$ is of the weight $\epsilon_1 + \dots + \epsilon_k = \omega_k$, and therefore generates $F(\omega_k)$ as an irreducible subrepresentation of $\bigwedge^k V$. To be more precise, there is a non-zero $\mathfrak{g}$-invariant map

$$Q\colon \bigwedge^k V \to \bigwedge^{k-2} V,$$

$$Q(v_1 \wedge \ldots \wedge v_k) = \sum_{i<j} B(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge \hat{v}_j \wedge \ldots \wedge v_k.$$

This is actually just the contraction with our skew-symmetric form $B \in \bigwedge^2 V^*$. So, $\operatorname{Ker} Q$ is a subrepresentation, and obviously $e_1 \wedge \ldots \wedge e_k \in \operatorname{Ker} Q$. It follows that $F(\omega_k) \subseteq \operatorname{Ker} Q$. In [FH04, 17.2.], the following is proved:

Proposition 3.5. $\operatorname{Ker} Q$ *is irreducible, so* $F(\omega_k) = \operatorname{Ker} Q$.

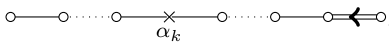
The kernel $\operatorname{Ker} Q$ is sometimes denoted by $\bigwedge_0^k V$, and its elements called ***traceless*** or ***trace-free*** exterior vectors.

For the record, let us mention the other extremal weight vectors in $F(\omega_k)$. An extremal weight is a $W_{\mathfrak{g}}$-conjugate of the highest weight. Any $W_{\mathfrak{g}}$-conjugate of ω_k consists of some 1's, some (-1)'s and other 0's, such that there are exactly k non-zero coordinates. Suppose 1's are on the positions $\{i_1 < \ldots < i_m\}$ and that (-1)'s are on $\{j_1 < \ldots < i_l\}$. Then these two sets are disjoint (some of them possibly empty), and $m + l = k$. The corresponding weight vector is $e_{i_1} \wedge \ldots \wedge e_{i_m} \wedge f_{j_1} \wedge \ldots \wedge f_{j_l} \in \operatorname{Ker} Q$.

Remark 3.6 Also for future use, note that $\bigwedge^k V$ is the fundamental representation of $\mathrm{SL}(2n, \mathbb{C})$ with the highest weight vector $e_1 \wedge \ldots \wedge e_k$ and the highest weight $\epsilon_1 + \ldots + \epsilon_k$. ■

3.1.3 Parabolic subalgebras

Choose $k \in \{1, \ldots, n\}$, and consider the following maximal standard parabolic subalgebra of $\mathfrak{g}$:

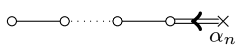
$\mathfrak{p}_k = \mathfrak{l}_k \oplus \mathfrak{u}_k :=$  .

The positive roots are arranged in the following way:

$$\begin{aligned} \Delta^+(\mathfrak{l}_k, \mathfrak{h}) = \{ & a_{ij}, && i < j \le k \text{ or } k < i < j, \\ & b_i, && k < i, \\ & c_{ij}, && k < i < j \}, \\ \Delta(\mathfrak{u}_k) = \{ & a_{ij}, && i \le k < j, \\ & b_i, && i \le k, \\ & c_{ij}, && i \le k \}. \end{aligned} \tag{3.3}$$

Thus, the parabolic subalgebra $\mathfrak{p}_k$ consists of the matrices in $\mathfrak{sp}(2n, \mathbb{C})$ that have the block structure presented in Figure 3.2.

We will mostly be interested in the case when $k = n$, that is, in the parabolic subalgebra

$\mathfrak{p}_n = \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} =$  ,

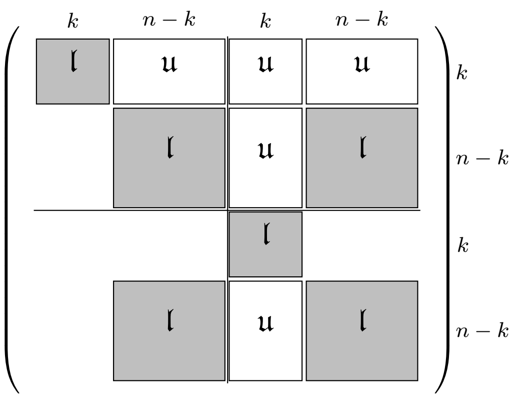


Figure 3.2: Parabolic subalgebras of $\mathfrak{sp}(2n, \mathbb{C})$

which has

$$
\begin{aligned}
\Delta^+(\mathfrak{l}, \mathfrak{h}) &= \{a_{ij} = \epsilon_i - \epsilon_j \ : \ 1 \le i < j \le n\}, \\
\Delta(\mathfrak{u}) &= \{b_i = 2\epsilon_i \ : \ 1 \le i \le n\} \cup \{c_{ij} = \epsilon_i + \epsilon_j \ : \ 1 \le i < j \le n\}.
\end{aligned} \tag{3.4}
$$

Its Levi factor, nilpotent radical, and the opposite nilpotent radical are respectively:

$$
\begin{aligned}
\mathfrak{l} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \ : \ A \in M(n, \mathbb{C}) \right\} \cong \mathfrak{gl}(n, \mathbb{C}) \cong \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}(1, \dots, 1), \\
\mathfrak{u} &= \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \ : \ B \in M(n, \mathbb{C}), \ B = B^T \right\}, \\
\mathfrak{u}^- &= \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \ : \ C \in M(n, \mathbb{C}), \ C = C^T \right\}.
\end{aligned}
$$

The nilpotent radical $\mathfrak{u}$ is abelian in this case, so the decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ is a $|1|$-grading. This is the only standard parabolic subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$ that is $|1|$-graded. The grading element is

$$
E = \operatorname{diag}\Big(\underbrace{\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}}_{n}, -\frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2} \Big). \tag{3.5}
$$

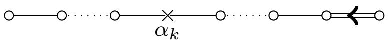
Other standard parabolic subalgebras are intersections of the maximal ones. Weights for the Levi factor can be written as n-tuples again, but for every crossed node α_i in the Dynkin diagram for the parabolic subalgebra, we will put a bar after the i-th coordinate of the weight. For example, $[3, 2 \,|\, 1, 0 \,|\,]$ is just a weight $[3, 2, 1, 0] \in \mathfrak{h}^*$, but considered as a weight for the Levi factor of the parabolic subalgebra $\mathfrak{q} = \circ\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\circ\!\Leftarrow\!\times$. Note that the bars also indicate the condition for the $\mathfrak{q}$-dominance. For example, $[\lambda_1, \lambda_2 \,|\, \lambda_3, \lambda_4 \,|\,]$

is $\mathfrak{q}$-dominant if and only if $\lambda_1 \geq \lambda_2$ and $\lambda_3 \geq \lambda_4$ (there is no condition on the order among λ_2 and λ_3, nor the non-negativity of λ_4).

3.2 Realization of G/P

3.2.1 (Isotropic) Grassmannian

Let (V, B) be a $2n$-dimensional symplectic vector space as before. A subspace $W \leq V$ is said to be ***isotropic***, if $B(v, w) = 0$ for all $v, w \in W$. A maximal isotropic subspace is called a ***Lagrangian subspace***, and it necessarily n-dimensional. Fix a symplectic basis $\{e_1, \ldots, e_n, f_1 = e_{n+1}, \ldots, f_n = e_{2n}\}$ satisfying (3.1), and fix the standard maximal parabolic subgroup

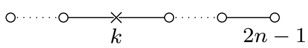
$P_k =$  $\subseteq \mathrm{Sp}(2n, \mathbb{C})$, for some $1 \leq k \leq n$.

Definition 3.7. *Note the following two sets:*

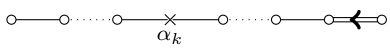
$$\begin{aligned} \mathrm{Gr}(k, V) = \mathrm{Gr}(k, 2n) &:= \{W \leq V :\ \dim W = k\}, \\ \mathrm{iGr}(k, V) = \mathrm{iGr}(k, 2n) &:= \{W \leq V :\ \dim W = k,\ W \text{ isotropic}\} \subseteq \mathrm{Gr}(k, V). \end{aligned}$$

The former one is called the ***(complex) Grassmannian****, and the latter one the* ***(complex) isotropic Grassmannian*** *(of k-planes in V). The isotropic Grassmannian of n-planes,* $\mathrm{iGr}(n, 2n)$*, is called the* ***Lagrangian Grassmannian****.*

The group $\mathrm{SL}(2n, \mathbb{C})$ acts transitively on $\mathrm{Gr}(k, V)$, and the stabilizer of the subspace $\mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_k\}$ is the parabolic subgroup consisting of matrices in the block-upper triangular form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A is of the type $k \times k$ and C of the type $(2n-k) \times (2n-k)$. Therefore, the Grassmannian $\mathrm{Gr}(k, V)$ is a generalized flag manifold of $\mathrm{SL}(2n, \mathbb{C})$, namely:

Proposition 3.8. $\mathrm{Gr}(k, V) =$  .

The group $\mathrm{Sp}(2n, \mathbb{C})$ obviously acts on $\mathrm{iGr}(k, V)$. Moreover, the action is transitive, which follows from Witt's extension theorem. The stabilizer of the isotropic subspace $\mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_k\}$ is precisely the parabolic subgroup P_k. Therefore:

Proposition 3.9. $\mathrm{iGr}(k, V) \cong \mathrm{Sp}(2n, \mathbb{C})/P_k =$ .

3.2.2 (Isotropic) flags

More generally, suppose now we have a not necessarily maximal standard parabolic subgroup $Q \subseteq \mathrm{Sp}(2n, \mathbb{C})$, and say it has crosses on the positions $k_1, \ldots, k_s$ in its Dynkin diagram. Writing it as an intersection of maximal parabolic subalgebras,

$$Q = \bigcap_{i=1}^{s} P_{k_i},$$

it is easy to see that

$$\begin{aligned}\mathrm{Sp}(2n, \mathbb{C})/Q \cong \Big\{(W_1, W_2, \ldots, W_s) \ : \ & W_1 \leq \ldots \leq W_s \leq V, \\ & \dim W_i = k_i \text{ for } i = 1, \ldots, s, \ W_s \text{ isotropic}\Big\},\end{aligned} \tag{3.6}$$

which is called the space of ***isotropic partial flags*** in V, of the type $(k_1, \ldots, k_s)$.

For the SL-case, the formula is completely the same, except that we do not require the spaces in the partial flags to be isotropic.

The (isotropic) partial flag of type $(1, \ldots, n)$ is called the ***full*** (isotropic) flag, and the space of those corresponds to the standard Borel subgroup.

3.2.3 Plücker embedding

Definition 3.10. *The* ***Plücker embedding*** *is a map*

$$p\colon \ \mathrm{Gr}(k, V) \to \mathbb{P}\left(\bigwedge^k V\right),$$

defined as follows: for a k-dimensional subspace $W \leq V$, choose its basis $\{w_1, \ldots, w_k\}$, and define

$$p(W) := [w_1 \wedge \ldots \wedge w_k] \in \mathbb{P}\left(\bigwedge^k V\right).$$

This is well defined, because a change of basis in W has the effect of multiplying $p(W)$ by a non-zero scalar (the determinant of the transition matrix), so it does not change the point in the projective space. Moreover, it is injective, and realizes $\mathrm{Gr}(k, V)$ as a smooth projective subvariety of $\mathbb{P}\left(\bigwedge^k V\right)$. Its image consists of the lines generated by the decomposable exterior vectors, that is, exterior vectors of the form $w_1 \wedge \ldots \wedge w_k$ for some linearly independent vectors $w_1, \ldots, w_k \in V$. For proofs and details, see [Pro07, 13.3].

It is easy to see that the condition on W to be isotropic, is equivalent to its image being in $\mathbb{P}\left(\bigwedge_0^k V\right)$ (see 3.1.2). So, the Plücker embedding restricts to the isotropic Grassmannian

$$p\colon \ \mathrm{iGr}(k, V) \hookrightarrow \mathbb{P}\left(\bigwedge_0^k V\right) \subseteq \mathbb{P}\left(\bigwedge^k V\right),$$

and moreover,

$$p\left(\mathrm{iGr}(k,V)\right) = p\left(\mathrm{Gr}(k,V)\right) \bigcap \mathbb{P}\left(\bigwedge\nolimits_0^k V\right),$$

which consists of the lines generated by the traceless decomposable exterior vector, and is a closed subset of $p\left(\mathrm{Gr}(k,V)\right)$. The Plücker embeddings of $\mathrm{Gr}(k,V)$ and $\mathrm{iGr}(k,V)$ are precisely the projective embeddings from Proposition 1.11. This follows from the description of the fundamental representation $F(\omega_k)$ in Subsection 3.1.2.

Let us describe this in coordinates. We represent a k-dimensional subspace $W \leq V$ by a $2n \times k$-matrix of the rank k, by choosing a basis $\{w_1, \ldots, w_k\}$ for W and then putting the coefficients of each w_i (with respect to the fixed symplectic basis) as the i-th column of the matrix. We will often identify a subspace with its matrix representation. Of course, such a matrix is not uniquely determined, but only up to the multiplication with the elements from $\mathrm{GL}(k,\mathbb{C})$ from the right[1], which corresponds to a change of basis in W. From the basis $\{e_1, \ldots, e_{2n}\}$ for V, we make a basis for $\bigwedge^k V$:

$$e_I := e_{i_1} \wedge \ldots \wedge e_{i_k}, \text{ for all } I = \{i_1 < \ldots < i_k\} \subseteq \{1, \ldots, 2n\}.$$

The coefficients in this basis are declared to be the homogeneous coordinates for $\mathbb{P}\left(\bigwedge^k V\right)$. For a $2n \times k$-matrix X of rank k and $I = \{i_1 < \ldots < i_k\} \subseteq \{1, \ldots, 2n\}$, denote by X_I the $k \times k$-minor of X consisting of the rows $i_1, \ldots, i_k$, and define $d_I(X) = \det(X_I)$. Because the rank is k, at least one of the $k \times k$-minors is regular. With these identifications, the Plücker embedding becomes

$$\begin{array}{ccc} \mathrm{Gr}(k,2n) & \hookrightarrow & \mathbb{P}^{\binom{2n}{k}-1} \\ \cup & & \\ \mathrm{iGr}(k,2n) & & \\ & & \\ X \cdot \mathrm{GL}(k,\mathbb{C}) & \longmapsto & [d_I(X)]_I, \end{array}$$

where I varies over all the possible strictly increasing k-tuples in $\{1, \ldots, 2n\}$. The functions d_I on $\mathrm{Gr}(k,2n)$ or $\mathrm{iGr}(k,2n)$ are called the ***Plücker coordinates***. They satisfy certain quadratic equations, called the Plücker relations, which define the Grassmannian as a projective subvariety. We will not use those relations, see [Pro07, 13].

[1]More precisely, the Stiefel manifold, which consists of all the full rank $2n \times k$-matrices, is a principal $\mathrm{GL}(k,\mathbb{C})$-bundle over the Grassmannian $\mathrm{Gr}(k,V)$.

3.2.4 Big affine cell

We will describe the canonical coordinates on the big affine cell in the Lagrangian Grassmannian $\mathrm{iGr}(n,2n) = \circ\!\!-\!\!\circ \cdots \circ\!\!\Leftarrow\!\!\underset{\alpha_n}{\times}$. Observe that the maps in (1.2) here become:

$$\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \overset{\exp}{\longmapsto} \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \xmapsto{\cdot e_1 \wedge \ldots \wedge e_n} \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \cdot \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} I \\ C \end{pmatrix}, \quad C \in M(n, \mathbb{C}) \text{ symmetric.}$$

We have the following conclusions:

Proposition 3.11. *The subset*

$$X = \{W \subseteq V \ : \ W \ \textit{Lagrangian}, \ d_{I_0}(W) \neq 0\} \subseteq \mathrm{iGr}(n, 2n)$$

where $I_0 = \{1, 2, \ldots, n\}$, is the big affine cell around $\mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_n\}$*. Moreover, for each $W \in X$ there is a unique $n \times n$ matrix C such that*

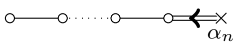
$$W \equiv \begin{pmatrix} I \\ C \end{pmatrix} \quad \big(\textit{modulo } \mathrm{GL}(n, \mathbb{C}) \textit{ from the right}\big).$$

The matrix C is symmetric, and $W \mapsto C$ defines the canonical affine coordinates on the big affine cell $X \cong \mathrm{Sym}_n(\mathbb{C}) \cong \mathbb{C}^{\frac{n^2+n}{2}}$.

Chapter 4

Structure of the regular Hasse diagram

The goal of this chapter is to explain the structure of the regular Hasse diagram $W^{\mathfrak{p}}$ for the $|1|$-graded parabolic subalgebra

$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} =$  .

Elements of $W^{\mathfrak{p}}$ can be be represented in two ways:

(a) By the bijection $w \mapsto w\rho$, $W^{\mathfrak{p}}$ corresponds to the orbit $W^{\mathfrak{p}}\rho$. The latter set consists of all $\mathfrak{p}$-dominant Weyl group conjugates of ρ. Recall from Subsection 3.1.1 that $\rho = [n, n-1, \ldots, 1]$, and that $W_{\mathfrak{g}}$ acts as permutations and sign changes of the coordinates. It follows that

$$W^{\mathfrak{p}} \cong \Big\{[\lambda_1, \ldots, \lambda_n] \,:\, \lambda_1 > \ldots > \lambda_n \text{ and } \{|\lambda_1|, \ldots, |\lambda_n|\} = \{1, \ldots, n\}\Big\}.$$

This description is more convenient for reconstructing the modules appearing in the BGG complex, but it is hard to visualize the shape of the complex, and the length of the objects.

(b) Using the bijection from Proposition 1.9,

$$w \mapsto \Phi_w := \Big\{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}) \,:\, w^{-1}\alpha < 0\Big\} \subseteq \Delta(\mathfrak{u}),$$

elements of $W^{\mathfrak{p}}$ are identified with the admissible subsets of $\Delta(\mathfrak{u})$. The main point of this identification is that the same proposition also provides a very simple criterion for the arrow relation.

We want to find an easy way to denote the admissible subsets of $\Delta(\mathfrak{u})$, and an easy switch between (a) and (b) above. Both will be given by the so called Lascoux-Schützenberger notation. The idea was taken from [EHP14], and it turned out to be the simplest way of describing both the regular and the singular BGG complexes of this type.

4.1 Admissible subsets of $\Delta(\mathfrak{u})$

4.1.1 Generalized Young diagrams

Recall the root arrangement (3.4). Notice that we can write $\Delta(\mathfrak{u})$ in the form that makes the additive structure transparent: Figure 4.1.

$$\begin{tikzcd}
\{c_{1n} \arrow[r, dotted, "+\alpha_{n-1}"] & c_{1,n-1} \arrow[r, dotted, "+\alpha_{n-2}"] & c_{1,n-2} \arrow[r, dotted] & \dots \arrow[r, dotted] & c_{13} \arrow[r, dotted, "+\alpha_2"] & c_{12} \arrow[r, dotted, "+\alpha_1"] & b_1 \\
c_{2n} \arrow[u, dotted, "+\alpha_1"] \arrow[r, dotted] & c_{2,n-1} \arrow[u, dotted] \arrow[r, dotted] & c_{2,n-2} \arrow[u, dotted] \arrow[r, dotted] & \dots \arrow[r, dotted] & c_{23} \arrow[u, dotted] \arrow[r, dotted] & b_2 \arrow[u, dotted] & \\
c_{3n} \arrow[u, dotted, "+\alpha_2"] \arrow[r, dotted] & c_{3,n-1} \arrow[u, dotted] \arrow[r, dotted] & c_{3,n-2} \arrow[u, dotted] \arrow[r, dotted] & \dots \arrow[r, dotted] & b_3 \arrow[u, dotted] & & \\
\vdots \arrow[u, dotted] & \vdots \arrow[u, dotted] & \vdots \arrow[u, dotted] & \iddots & & & \\
c_{n-2,n} \arrow[u, dotted] \arrow[r, dotted] & c_{n-2,n-1} \arrow[u, dotted] \arrow[r, dotted] & b_{n-2} \arrow[u, dotted] & & & & \\
c_{n-1,n} \arrow[u, dotted, "+\alpha_{n-2}"] \arrow[r, dotted] & b_{n-1} \arrow[u, dotted] & & & & & \\
b_n = \alpha_n\} \arrow[u, dotted, "+\alpha_{n-1}"] & & & & & &
\end{tikzcd}$$

Figure 4.1: $\Delta(\mathfrak{u})$ for $\mathfrak{p} = \circ\!-\!\circ\cdots\circ\!-\!\circ\Leftarrow\times$

Proposition 4.1. *Denote $b_i = c_{ii}$ for $i = 1, \dots, n$. A subset $S \subseteq \Delta(\mathfrak{u})$ is admissible if and only if*

$$c_{ij} \in S,\ i \le j, \quad \Longrightarrow \quad c_{kl} \in S \quad \text{for all} \quad k \ge i,\ l \ge j,\ k \le l \le n. \tag{4.1}$$

Proof. Note that sums of the labels of the consecutive arrows in Figure 4.1 are elements of $\Delta^+(\mathfrak{l}, \mathfrak{h})$. Then the condition (4.1) is equivalent to (1.1). $\square$

An admissible subset S will be represented in the following way: for each element $c_{ij} \in S$, we put a box $\square$ on the position c_{ij} in Figure 4.1. The diagram obtained this way will be called the ***generalized Young diagram*** of the corresponding Hasse diagram element.

For example, the maximal admissible subset is $\Delta(\mathfrak{u})$ itself, and we denote it by

$$\begin{array}{|c|c|c|c|c|}
\hline
c_{1n} & c_{1,n-1} & \cdots & c_{12} & b_1 \\
\hline
c_{2n} & c_{2,n-1} & \cdots & b_2 & \\
\cline{1-4}
\vdots & \vdots & \iddots & & \\
\hline
c_{n-1,n} & b_{n-1} & & & \\
\cline{1-2}
b_n & & & & \\
\cline{1-1}
\end{array}$$

This is also the maximal generalized Young diagram – others will be special subsets of this one. The labels inside the boxes will be omitted. For another example, the admissible subset $\{c_{13}, c_{23}, b_2, b_3\}$ in the rank $n = 3$ is denoted by $\ydiagram{1,2,1}$.

The condition (4.1) translated into the generalized Young diagram setting is: for each box in S, all the possible boxes bellow, and left of it are again contained in S.

For a non-example, take $S = \ydiagram{1+1,2,1}$. Then S is not a generalized Young diagram, because left of the top box there is a place for another box, which is not in S.

The notion of the length and the arrow relation transfer very nicely to the generalized Young diagram setting. Namely, the length of an element in $W^{\mathfrak{p}}$ is equal to the number of the boxes in the generalized Young diagram (follows from Proposition 1.2). Furthermore, an arrow between elements in $W^{\mathfrak{p}}$ corresponds to the ***"adding one box" operation*** on the generalized Young diagrams, and the label of that arrow is the same as the label of the added box (follows from Proposition 1.9). For example (in rank 3):

$$\sigma_{b_2}\sigma_{c_{23}}\sigma_{b_3} \xrightarrow{c_{13}} \sigma_{c_{13}}\sigma_{b_2}\sigma_{c_{23}}\sigma_{b_3} \qquad \text{corresponds to} \quad \ydiagram{2,1} \to \ydiagram{1,2,1}. \tag{4.2}$$

Remark 4.2 Let us mention that there is an easy way to find reduced forms of the elements in $W^{\mathfrak{p}}$ from the generalized Young diagram notation. See [EHP14, 3.7]. ■

4.1.2 Lascoux-Schützenberger notation

Notice that a generalized Young diagram is completely determined by a zig-zag line from the top left point to the diagonal. For each move to the right, we write 1, and for each move down, we write 0. This way we get a binary sequence of the length n, written with overline, called the ***Lascoux-Schützenberger notation*** (***LS*** for short) of the generalized Young diagram. For example,

$$\ydiagram{1,1,1} = \overline{100}, \ \ydiagram{1,2,1} = \overline{101}, \ \ydiagram{2,2,1} = \overline{110}, \ \ydiagram{3,2,1} = \overline{111}.$$

The example (4.2) in the LS notation is $\overline{011} \to \overline{101}$.

The following two propositions give a combinatorial description of the regular Hasse diagram.

Proposition 4.3. *As directed graphs,*

$$W^{\mathfrak{p}} \cong \left\{ \overline{d_1 d_2 \dots d_n} \ : \ d_i = 0 \ \textit{or} \ 1 \right\}, \tag{4.3}$$

with the following arrows on the right-hand side:

$$\overline{d_1 \dots 01 \dots d_n} \to \overline{d_1 \dots 10 \dots d_n} \quad \textit{and} \quad \overline{d_1 \dots d_{n-1} 0} \to \overline{d_1 \dots d_{n-1} 1}. \tag{4.4}$$

Moreover, if w has the digit 1 on the positions $i_1 < i_2 < \dots < i_k$, then

$$l(w) = (n+1) \cdot k - \sum_{j=1}^{k} i_j. \tag{4.5}$$

Proof. The bijection (4.3) follows from the definition of the LS notation. Obviously, the "adding one box" operation has the effect of switching a pair of consecutive digits $\overline{01}$ to $\overline{01}$ (if the added box is not the last possible in a row), or changing the last digit $\overline{0}$ to $\overline{1}$ (if the added box is the last possible in a row).

It remains to prove the formula (4.5). Recall that the length of an element is equal to the number of the boxes in the corresponding generalized Young diagram. The formula is obviously true for $n = 1$. Assume it is true for $n-1$, and take an LS word w of the length n as above. Let v be the subword of w consisting of the first $n-1$ digits of w. We will use the inductive assumption on v. There are two possible cases:

(a) $w = \overline{v0}$:

$$l(w) = l(v) + k = n \cdot k - \sum_{j=1}^{k} i_j + k = (n+1) \cdot k - \sum_{j=1}^{k} i_j.$$

(b) $w = \overline{v1}$:

$$l(w) = l(v) + k = n \cdot (k-1) - \sum_{j=1}^{k-1} i_j + k \overset{j_k = n}{=} (n+1) \cdot k - \sum_{j=1}^{k} i_j.$$

□

Proposition 4.4. *Let $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n] \in \mathfrak{h}^*$ and $w \in W^{\mathfrak{p}}$. Let $i_1 < i_2 < \dots < i_k$ denote the positions of the digit 1 in the LS notation for w. Then*

$$w\lambda = [\lambda_1, \lambda_2, \dots, \hat{\lambda}_{i_1}, \dots, \hat{\lambda}_{i_2}, \dots, \hat{\lambda}_{i_k}, \dots, \lambda_n, -\lambda_{i_k}, -\lambda_{i_{k-1}}, \dots, -\lambda_{i_1} \,|\,]. \tag{4.6}$$

Proof. Suppose first that $k = 1$. Then $w = \overline{0 \dots 010 \dots 0}$, and the corresponding generalized Young diagram has just one column of $n+1-i_1$ boxes. It follows that

$$w = \sigma_{c_{i_1,n}} \circ \ldots \circ \sigma_{c_{n-1,n}} \circ \sigma_{b_n}.$$

Applying this composition to λ gives $[\lambda_1, \ldots, \hat{\lambda_{i_1}}, \ldots, \lambda_n, -\lambda_{i_1}]$.

In general, the same principle is applicable. One can decompose w into the "columns", and calculate the action of each column from the left to the right. More precisely,

$$w = C_k \circ C_{k-1} \circ \ldots \circ C_1, \quad \text{where each } C_j = \sigma c_{i_j+1-j,n+1-j} \circ \ldots \circ \sigma c_{n-j,n+1-j} \circ \sigma b_{n+1-j}.$$

Applying this to λ gives (4.6). □

The last proposition gives an easy switch between (a) and (b) from Page 49. Namely, the positions of the digit 1 are precisely the positions of the coordinates of ρ that become negative in $w\rho$. For example,

$$\overline{0\mathbf{1}00\mathbf{1}000\mathbf{1}}\ [9, \mathbf{8}, 7, 6, \mathbf{5}, 4, 3, 2, \mathbf{1}] = [9, 7, 6, 4, 3, 2, -\mathbf{1}, -\mathbf{5}, -\mathbf{8}\,|\,].$$

Moreover, it is not hard to see that

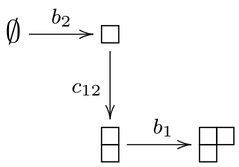
$$\overline{d_1 \ldots d_n} \leq \overline{e_1 \ldots e_n} \quad \Leftrightarrow \quad \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} e_i, \quad \text{for all } k = 1, \ldots, n.$$

On the figures 4.2, 4.3, 4.4 and 4.5 there are low-rank examples of $W^{\mathfrak{p}}$ described as generalized Young diagrams, in the LS notation, and also as the $W^{\mathfrak{p}}$-orbits of the weight ρ.

Remark 4.5 Note that, in the orbit $W^{\mathfrak{p}}\rho$, it is actually enough to keep track of the positive coordinates only. Then the arrow between two elements correspond to decrementing one positive coordinate of the first element. See [BC86] for such parametrization of $W^{\mathfrak{p}}$. ■

Definition 4.6. *We say that an element $w \in W^{\mathfrak{p}}$ is* ***even****, if there is an even number of the digits* 1 *in its LS notation; otherwise we say it is* ***odd****. The number of the digits* 1 *is equal to the number of the columns of the corresponding generalized Young diagram, and also to the number of the negative coordinates in $w\rho$.*

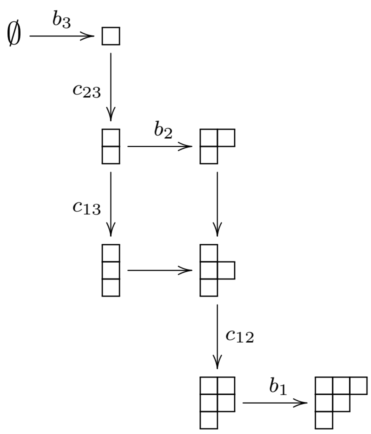
The distinction between even and odd elements of the (singular) Hasse diagram was mentioned in Enright-Shelton's theorem 1.43, in the long root singular case.



$$\begin{tikzcd} \overline{00} \arrow[r] & \overline{01} \arrow[d] & \\ & \overline{10} \arrow[r] & \overline{11} \end{tikzcd}$$

$$\begin{tikzcd} {[2,1\,|\,]} \arrow[r] & {[2,-1\,|\,]} \arrow[d] & \\ & {[1,-2\,|\,]} \arrow[r] & {[-1,-2\,|\,]} \end{tikzcd}$$

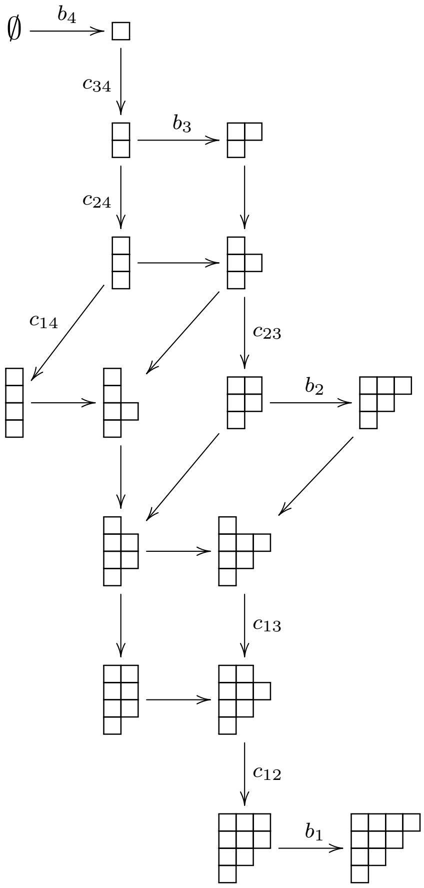
Figure 4.2: $W^{\mathfrak{p}}$ and $W^{\mathfrak{p}}\rho$ for $\mathfrak{p} =$



$$\begin{tikzcd} \overline{000} \arrow[r] & \overline{001} \arrow[d] & & \\ & \overline{010} \arrow[r] \arrow[d] & \overline{011} \arrow[d] & \\ & \overline{100} \arrow[r] & \overline{101} \arrow[d] & \\ & & \overline{110} \arrow[r] & \overline{111} \end{tikzcd}$$

$$\begin{tikzcd} {[3,2,1\,|\,]} \arrow[r] & {[3,2,-1\,|\,]} \arrow[d] & & \\ & {[3,1,-2\,|\,]} \arrow[r] \arrow[d] & {[3,-1,-2\,|\,]} \arrow[d] & \\ & {[2,1,-3\,|\,]} \arrow[r] & {[2,-1,-3\,|\,]} \arrow[d] & \\ & & {[1,-2,-3\,|\,]} \arrow[r] & {[-1,-2,-3\,|\,]} \end{tikzcd}$$

Figure 4.3: $W^{\mathfrak{p}}$ and $W^{\mathfrak{p}}\rho$ for $\mathfrak{p} =$



$$
\begin{tikzcd}
\overline{0000} \arrow[r] & \overline{0001} \arrow[d] & & \\
& \overline{0010} \arrow[r] \arrow[d] & \overline{0011} \arrow[d] & \\
& \overline{0100} \arrow[r] \arrow[dl] & \overline{0101} \arrow[dl] \arrow[d] & \\
\overline{1000} \arrow[r] & \overline{1001} \arrow[d] & \overline{0110} \arrow[r] \arrow[dl] & \overline{0111} \arrow[dl] \\
& \overline{1010} \arrow[r] \arrow[d] & \overline{1011} \arrow[d] & \\
& \overline{1100} \arrow[r] & \overline{1101} \arrow[d] & \\
& & \overline{1110} \arrow[r] & \overline{1111}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[4,3,2,1\,|\,]} \arrow[r] & {[4,3,2,-1\,|\,]} \arrow[d] & & \\
& {[4,3,1,-2\,|\,]} \arrow[r] \arrow[d] & {[4,3,-1,-2\,|\,]} \arrow[d] & \\
& {[4,2,1,-3\,|\,]} \arrow[r] \arrow[dl] & {[4,2,-1,-3\,|\,]} \arrow[dl] \arrow[d] & \\
{[3,2,1,-4\,|\,]} \arrow[r] & {[3,2,-1,-4\,|\,]} \arrow[d] & {[4,1,-2,-3\,|\,]} \arrow[r] \arrow[dl] & {[4,-1,-2,-3\,|\,]} \arrow[dl] \\
& {[3,1,-2,-4\,|\,]} \arrow[r] \arrow[d] & {[3,-1,-2,-4\,|\,]} \arrow[d] & \\
& {[2,1,-3,-4\,|\,]} \arrow[r] & {[2,-1,-3,-4\,|\,]} \arrow[d] & \\
& & {[1,-2,-3,-4\,|\,]} \arrow[r] & {[-1,-2,-3,-4\,|\,]}
\end{tikzcd}
$$

Figure 4.4: $W^{\mathfrak{p}}$ and $W^{\mathfrak{p}}\rho$ for $\mathfrak{p} = \circ\!-\!\!\!-\!\circ\!-\!\!\!-\!\circ\!\Leftarrow\!\times$

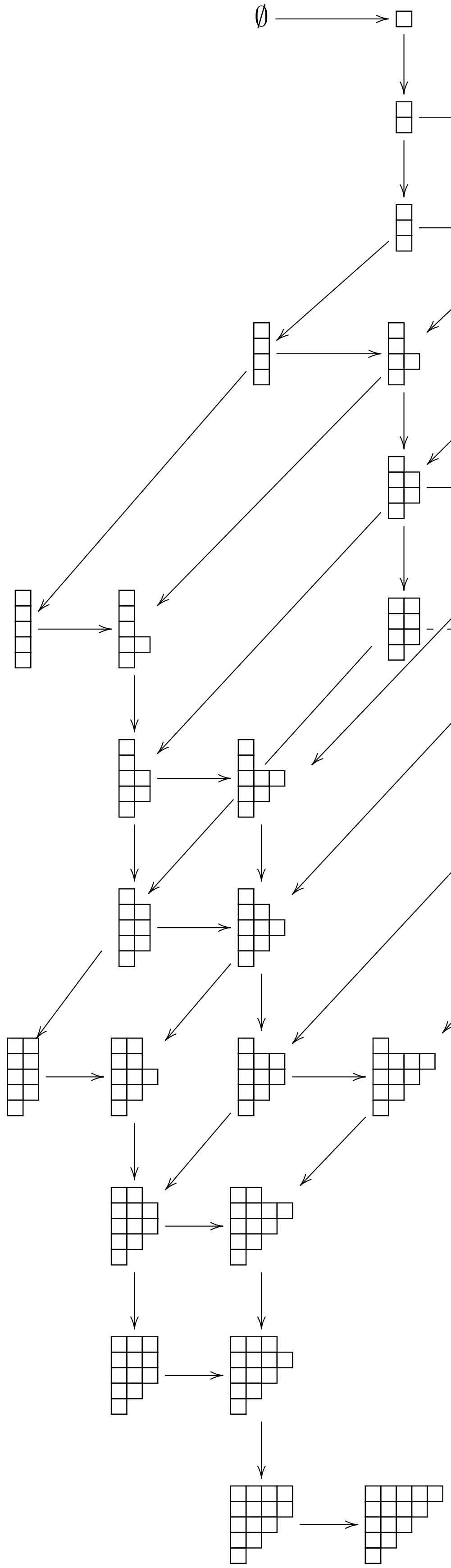


Figure 4.5: $W^{\mathfrak{p}}$ for $\mathfrak{p} =$ ○—○—○—○⇐×

4.2 Inductive structure of the Hasse diagram

Denote by W^n our Hasse diagram in the rank n, and note that $|W^n| = \frac{2^n \cdot n!}{n!} = 2^n$. The vertices of the graph W^n can be divided into two disjoint sets: those that do not have first column maximal (denoted W_0^n), and those that have (denoted W_1^n). Equivalently, W_0^n consists of the elements that have the first LS digit 0, and W_1^n consists of the elements that have the first digit 1. The former ones are naturally elements of W^{n-1}, and the latter ones become elements of W^{n-1} after we remove from them their maximal first column. Therefore, W^n consists of the two copies of W^{n-1},

$$W^n = W_0^n \bigsqcup W_1^n, \tag{4.7}$$

and some arrows between these copies. We want to describe the arrows between W_0^n and W_1^n. In the same induction principle, both W_0^n and W_1^n again decompose further into

$$W_0^n = W_{00}^n \bigsqcup W_{01}^n, \quad W_1^n = W_{10}^n \bigsqcup W_{11}^n,$$

where each $W_{ij}^n \cong W^{n-2}$ as a directed graph. Note that the index ij specifies the first two LS digits of the elements of W_{ij}^n. For any element of W_{01}^n (that is, a generalized Young diagram of the rank n with exactly $n-1$ boxes in the first column), there is an arrow to the element of W_{10}^n (that is, a generalized Young diagram of the rank n with the first column maximal, but the second column is not maximal) obtained by adding the top box in the first column.

In conclusion, W^n consists of the two pieces W^{n-1} that "glue" from the second copy of W^{n-2} in the first piece to the first copy of W^{n-2} in the second piece. What we get is a fractal-like shape, described in Figure 4.6. The position of $w \in W^{\mathfrak{p}}$ with respect to this inductive structure is encoded in its LS notation.

Having described $W^{\mathfrak{p}}$, we have also described the shape of the regular BGG resolutions (Theorem 1.23) over $G/P = \circ\!-\!\circ \cdots \circ\!-\!\circ\!\Leftarrow\!\times$. Some previous work on the regular BGG complexes for the orthogonal cases can be seen in [Krý04] and [Šmí04].

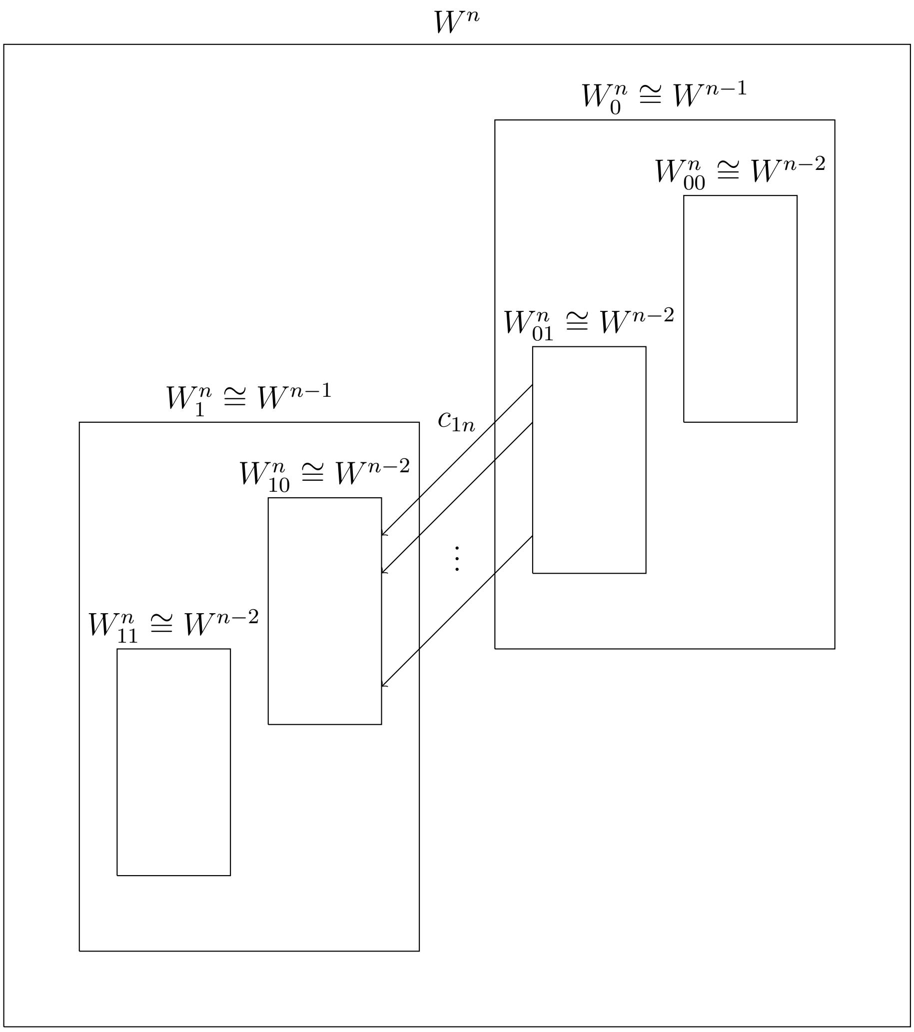


Figure 4.6: Inductive structure of $W^{\mathfrak{p}}$ for $\mathfrak{p} =$

4.3 Hasse diagrams for some other parabolic subalgebras

We will demonstrate here the simplest cases of the Hasse diagrams in a higher grading.

4.3.1 Subalgebra $\mathfrak{p}_1$

Let $\mathfrak{p}_1 =$ ×—○⋯○⇐○, and recall the root arrangement (3.3). The nilpotent radical is non-abelian, so the parabolic subalgebra is not $|1|$-graded. It is nevertheless very simple to find the Hasse diagram in this case. It is shown in Figure 4.7.

$$\emptyset \xrightarrow{a_{12}} \bullet \xrightarrow{a_{13}} \bullet \xrightarrow{a_{14}} \dots \xrightarrow{a_{1n}} \bullet \xrightarrow{b_1} \bullet \xrightarrow{c_{1n}} \bullet \xrightarrow{c_{1,n-1}} \dots \xrightarrow{c_{12}} \bullet$$

$$[n\,|\,n-1,\dots,1] \to [n-1\,|\,n,n-2,\dots,1] \to [n-2\,|\,n,n-1,n-3,\dots,1] \to \dots$$

$$\dots \to [1\,|\,n,n-1,\dots,2] \to [-1\,|\,n,\dots,2] \to [-2\,|\,n,\dots,3,1] \to \dots$$

$$\dots \to [-(n-1)\,|\,n,n-2,\dots,1] \to [-n\,|\,n-1,\dots,1]$$

Figure 4.7: $W^{\mathfrak{p}_1}$ and $W^{\mathfrak{p}_1}\rho$ for $\mathfrak{p}_1 =$ ×—○⋯○⇐○

We know from Section 3.2 that G/P_1 consists of all the isotropic complex lines in $\mathbb{C}^{2n}$. Since the form is skew-symmetric, all the lines are isotropic, and our space is homeomorpic to the complex projective space $\mathbb{P}^{2n-1}$. However, it is not isomorphic to it as a homogeneous complex manifold.

Remark 4.7 The parabolic subalgebra $\mathfrak{p}_1$ has so called complex contact grading. This property is equivalent to G/P_1 being a quaternionic symmetric space. For more information, see [ČS09, 3.2.4., 3.2.7.]. ■

4.3.2 Subalgebra $\mathfrak{p}_2$

For $\mathfrak{p}_2 = \mathfrak{l}_2 \oplus \mathfrak{u}_2 =$ ○—×—○⋯○⇐○, the Hasse diagram is more complicated. Its construction is a tedious exercise in using Proposition 1.2.

Proposition 4.8. *The Hasse diagram $W^{\mathfrak{p}_2}$ attached to the maximal parabolic subalgebra $\mathfrak{p}_2$ is given in Figure 4.8. The parallel arrows have the same label. Each bullet represents a subset of $\Delta(\mathfrak{u}_2)$ consisting of all the labels over some path from $\emptyset$ to that bullet, avoiding the diagonals of the hexagons.*

Proof. There are $2n(n-1)$ elements in the diagram, which is correct because in our

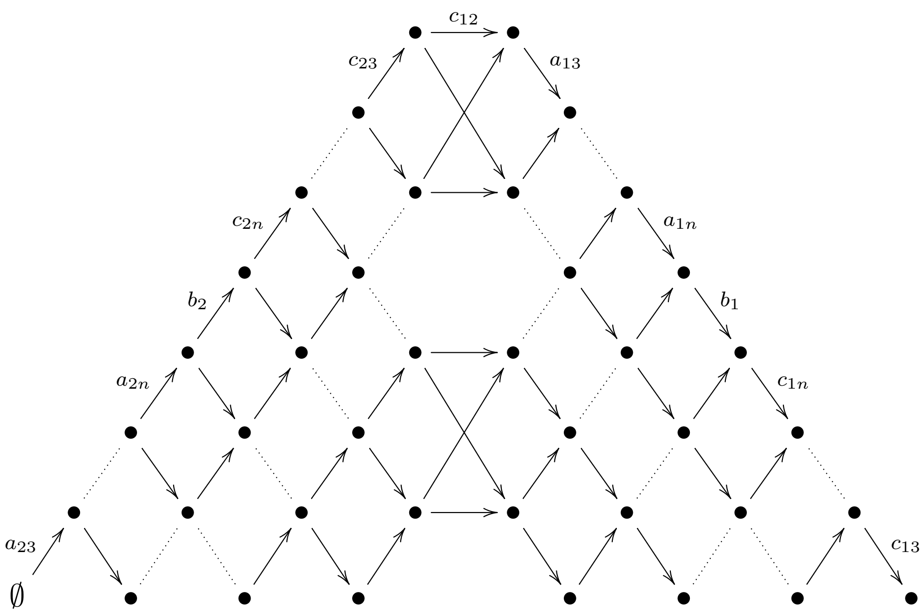


Figure 4.8: $W^{\mathfrak{p}_2}$ for $\mathfrak{p}_2 =$

case $|W^{\mathfrak{p}_2}| = \frac{2^n \cdot n!}{2 \cdot 2^{n-2} \cdot (n-2)!}$. Recall (3.3):

$$\begin{aligned}\Delta(\mathfrak{u}_2) = \{&a_{13}, \dots, a_{1n}, a_{23}, \dots, a_{2n},\\ &b_1, b_2,\\ &c_{12}, c_{13}, \dots, c_{1n}, c_{23}, \dots, c_{2n}\},\end{aligned}$$

$$\begin{aligned}\Delta^+(\mathfrak{l}_2, \mathfrak{h}) = \{&a_{12},\ a_{ij}, && 3 \le i < j \le n,\\ &b_3, \dots, b_n,\\ &c_{ij}, && 3 \le i < j \le n\}.\end{aligned}$$

We list all the subsets represented by the bullets in the diagram[1]:

$$\begin{aligned}
A_{st} &= \{a_{23}, \dots, a_{2s}\} \cup \{a_{13}, \dots, a_{1t}\}, && 2 \le t \le s \le n,\\
B_{st} &= \{a_{23}, \dots, a_{2n}\} \cup \{a_{13}, \dots, a_{1t}\} \cup \{b_2\} \cup \{c_{2,s+1}, \dots, c_{2n}\}, && 2 \le t \le s \le n,\\
C_{st} &= \{a_{23}, \dots, a_{2n}\} \cup \{a_{13}, \dots, a_{1t}\} \cup \{b_2\} \cup\\
&\qquad \cup \{c_{2,s+1}, \dots, c_{2n}\} \cup \{c_{12}\}, && 2 \le s \le t \le n,\\
D_{st} &= \{a_{23}, \dots, a_{2n}\} \cup \{a_{13}, \dots, a_{1n}\} \cup \{b_1, b_2\} \cup\\
&\qquad \cup \{c_{2,s+1}, \dots, c_{2n}\} \cup \{c_{1,t+1}, \dots, c_{1n}\} \cup \{c_{12}\}, && 2 \le s \le t \le n,
\end{aligned}$$

[1]We declare $\{a_{13}, \dots, a_{1t}\}$ to be empty for $t = 2$.

$$\begin{tikzcd}
& & & B_{22} \arrow[r] \arrow[dl, dotted, no head] & C_{22} \arrow[dr, dotted, no head] & & & \\
& & B_{n2} \arrow[dr, dotted, no head] & & & C_{2n} \arrow[dl, dotted, no head] \arrow[dr] & & \\
& A_{n2} \arrow[ur] \arrow[dl, dotted, no head] \arrow[dr, dotted, no head] & & B_{nn} \arrow[r] & C_{nn} \arrow[dr] & & D_{2n} \arrow[dl, dotted, no head] \arrow[dr, dotted, no head] & \\
A_{22} & & A_{nn} \arrow[ur] & & & D_{nn} & & D_{22}.
\end{tikzcd}$$

We check that they are all admissible subsets of $\Delta(\mathfrak{u}_2)$, so that they indeed represent the elements of $W^{\mathfrak{p}_2}$. For this, it is useful to list all the possible sums of two positive roots that give a root:

$$\begin{aligned}
a_{ij} + a_{jk} &= a_{ik}, \quad 1 \leq i < j < k \leq n, \\
a_{ij} + b_j &= c_{ij}, \quad 1 \leq i < j \leq n, \\
a_{ij} + c_{kj} &= c_{ki}, \quad 1 \leq k < i < j \leq n, \\
a_{ij} + c_{ij} &= b_i, \quad\ \ 1 \leq i < j \leq n, \\
a_{ij} + c_{kj} &= c_{ik}, \quad 1 \leq i < k < j \leq n, \\
a_{ij} + c_{jk} &= c_{ik}, \quad 1 \leq i < j < k \leq n.
\end{aligned} \tag{4.8}$$

For example, let us see that C_{st} is saturated. Only pairs of roots in C_{st} whose sum is a root again are $a_{2j} + c_{2j} = b_2$ $(s < j)$ and $a_{1j} + c_{2j} = c_{12}$ $(s < j \leq t)$. These sums are contained in C_{st}, which means that C_{st} is saturated. For the complement, we have to decompose each element in C_{st} as a sum of two positive roots in all the possible ways, and see that in each decomposition at least one summand is in C_{st}. Decompositions of a_{2j}, a_{1j} and b_2 are easier, so we omit them. Now take c_{2j}, $s < j \leq n$. According to (4.8), all the possible decompositions are:

$$\begin{aligned}
c_{2j} &= a_{2j} + b_j, & & \\
&= c_{2k} + a_{jk}, & & j < k, \\
&= a_{2k} + c_{jk}, & & j < k, \\
&= a_{2k} + c_{kj}, & & k < j.
\end{aligned}$$

The first summand in each sum belongs to C_{st}. Now consider c_{12}. According to (4.8), all the possible decompositions are:

$$\begin{aligned}
c_{12} &= b_2 + a_{12}, & & \\
&= a_{2k} + c_{1k}, & & 2 < k, \\
&= a_{1k} + c_{2k}, & & 2 < k.
\end{aligned}$$

Here, $b_2, a_{2k} \in C_{st}$. For $k \leq t$ we have $a_{1k} \in C_{st}$, and for $s < k$ we have $c_{2k} \in C_{st}$. One of these two cases must happen because $s \leq t$ by the definition of C_{st}. This shows that $\Delta^+(\mathfrak{g}, \mathfrak{h}) \setminus C_{st}$ is saturated. The proof is very similar for the sets A_{st}, B_{st} and D_{st}.

So far, we have constructed all the elements in the Hasse diagram. For the arrows, recall Proposition 1.2 (c): $w \xrightarrow{\alpha} w'$ if and only if $|\Phi_{w'}| = |\Phi_w|+1$ and $\langle \Phi_{w'} \rangle = \langle \Phi_w \rangle + k\alpha$, for some $k \in \mathbb{Z}$. This immediately gives us all the arrows and their labels in the diagram, except for the diagonals of the hexangons. We can prove their existence by the following calculation (for $2 \leq s < n$):

$$
\begin{tikzcd}
 & B_{ss} \arrow[r, "c_{12}"] & C_{ss} \arrow[dr, "a_{1,s+1}"] & \\
B_{s+1,s} \arrow[ur, "c_{2,s+1}"] \arrow[dr, "a_{1,s+1}"'] & & & C_{s,s+1} \\
 & B_{s+1,s+1} \arrow[r, "c_{12}"] & C_{s+1,s+1} \arrow[ur, "c_{2,s+1}"'] &
\end{tikzcd}
$$

$$
\begin{aligned}
\langle C_{s+1,s+1} \rangle - \langle B_{ss} \rangle &= a_{1,s+1} + c_{12} - c_{2,s+1} = 2 \cdot a_{1,s+1}, \\
\langle C_{ss} \rangle - \langle B_{s+1,s+1} \rangle &= c_{2,s+1} + c_{12} - a_{1,s+1} = 2 \cdot c_{2,s+1}.
\end{aligned}
$$

It remains to show that there are no other arrows. We can count and calculate in the coordinates:

$$
\begin{aligned}
|A_{st}| &= s + t - 4, & |B_{st}| &= 2n + t - s - 3, \\
|C_{st}| &= 2n + t - s - 2, & |D_{st}| &= 4n - s - t - 1,
\end{aligned}
$$

$$
\begin{aligned}
\langle A_{st} \rangle &= \left(t-2, s-2, -2, \ldots, \underbrace{-2}_{t}, -1, \ldots, \underbrace{-1}_{s}, 0, \ldots, 0 \right), \\
\langle B_{st} \rangle &= \left(t-2, 2n-s, -2, \ldots, \underbrace{-2}_{t}, -1, \ldots, \underbrace{-1}_{s}, 0, \ldots, 0 \right), \\
\langle C_{st} \rangle &= \left(t-1, 2n+1-s, -2, \ldots, \underbrace{-2}_{s}, -1, \ldots, \underbrace{-1}_{t}, 0, \ldots, 0 \right), \\
\langle D_{st} \rangle &= \left(2n+1-t, 2n+1-s, -2, \ldots, \underbrace{-2}_{s}, -1, \ldots, \underbrace{-1}_{t}, 0, \ldots, 0 \right).
\end{aligned}
$$

Let us consider the possible arrows between different A's. If we take $s \neq s'$ and $t \neq t'$, then $\langle A_{s't'} \rangle - \langle A_{st} \rangle$ would have at least three non-zero coordinates, and therefore cannot be a multiple of a root, so there is no arrow $A_{st} \to A_{s't'}$. If $s = s'$, then because of the cardinality condition, it is necessarily $t' = t + 1$, and analogously if $t = t'$. But these arrows were constructed before.

Now let us consider the possible arrows $A_{st} \to B_{s't'}$. The cardinality condition gives us $2n - s' - s = t - t'$. We have

$$\langle B_{s't'} \rangle - \langle A_{st} \rangle = (t' - t, 2n - s' - s + 2, \ldots) = (t' - t, t - t' + 2, \ldots)\,.$$

This can not be a multiple of a root if $s \neq s'$ and $t \neq t'$, as before. If $t = t'$, then $s + s' = 2n$, so $s = s' = n$, but this arrow was already constructed. If $s = s'$ and $t \neq t'$, then we must have second coordinate $t - t' + 2 = 0$. But $t' = t + 2$ implies

$$\langle B_{s't'} \rangle - \langle A_{st} \rangle = (2, 0, 0, \ldots, 0, -1, -1, 0, \ldots, 0)\,,$$

which is not a multiple of a root. One can check similarly for the possible arrows $B_{st} \to A_{s't'}$, and between different B's.

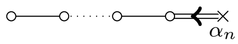
The only possible candidates for arrows between B's and C's are $B_{ss} \to C_{tt}$, because of the cardinality condition. Writing $\langle C_{tt} \rangle - \langle B_{ss} \rangle$ in coordinates, we see that s and t can only be equal or differ by one. Such arrows have been constructed before.

The other possible arrows on the right-hand side of the diagram, among C's and D's are eliminated similarly. $\square$

Chapter 5

Singular orbits and non-standard operators

From now on, the $|1|$-graded parabolic subalgebra

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} =$$


is fixed. Choose a subset of the simple singular roots $\Sigma \subseteq \Pi = \{\alpha_1, \dots, \alpha_n\}$, and take the minimal integral weight λ such that $\lambda + \rho$ is dominant and

$$\Sigma = \{\alpha \in \Pi \colon \langle \lambda + \rho, \check{\alpha} \rangle = 0\}\,.$$

We take the $W^{\mathfrak{p}}$-orbit of $\lambda + \rho$ (which is the same as the affine $W^{\mathfrak{p}}$-orbit of λ, up to the shift of the coordinates by ρ), and look for the elements that are strictly $\mathfrak{p}$-dominant, that is, whose coordinates are strictly decreasing sequences. The results that are not strictly $\mathfrak{p}$-dominant do not correspond to a homogeneous vector bundle over G/P or a generalized Verma module induced from $\mathfrak{p}$, so we will write $\times$ instead of them. The remaining part is what is called the ***singular orbit*** attached to the pair $(\mathfrak{p}, \Sigma)$ or $(\mathfrak{p}, \lambda)$. One can get the same result by calculating the $W^{\mathfrak{p},\Sigma}$-orbit of $\lambda + \rho$, which is 1-1 to the singular orbit, by Proposition 1.41. Some standard morphisms in the singular orbit are zero, which can be detected using Proposition 1.34, and we will denote them by the dotted line. Moreover, some maps are obviously identities. Because of Proposition 1.42, we can assume that Σ does not contain two adjacent simple roots.

5.1 Description of the singular orbits

Example 5.1 On the figures 5.1 to 5.7 we present all the singular orbits in the rank 4. By consulting the Enright-Shelton equivalence (Theorem 1.43), we can conclude the following:

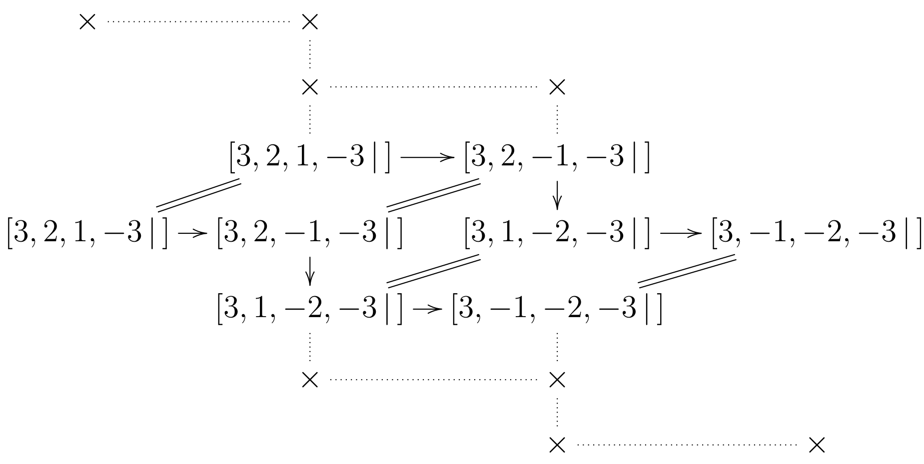


Figure 5.1: Singular orbit for $\Sigma = \{\alpha_1\}$, $\lambda + \rho = [3, 3, 2, 1]$

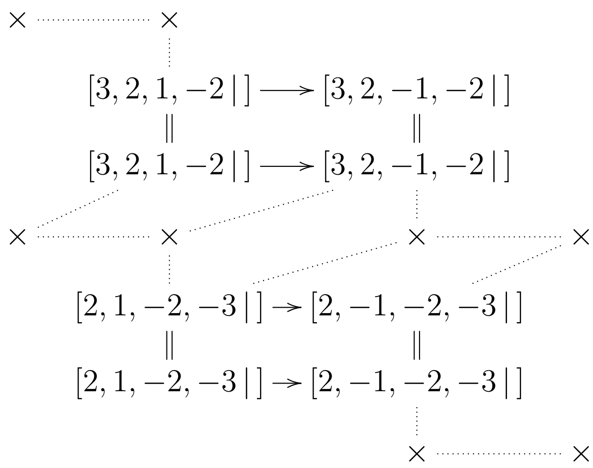


Figure 5.2: Singular orbit for $\Sigma = \{\alpha_2\}$, $\lambda + \rho = [3, 2, 2, 1]$

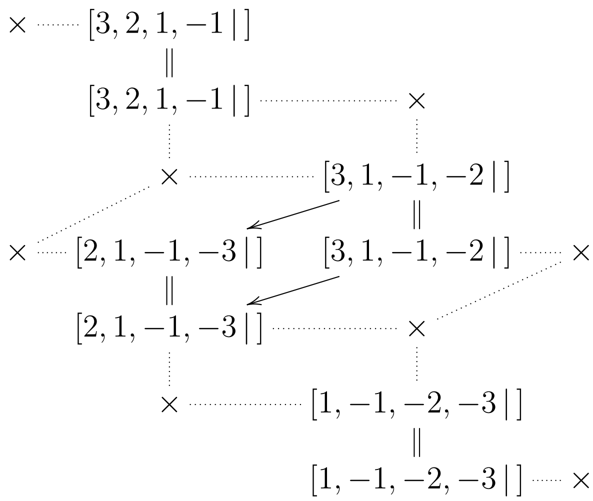


Figure 5.3: Singular orbit for $\Sigma = \{\alpha_3\}$, $\lambda + \rho = [3, 2, 1, 1]$

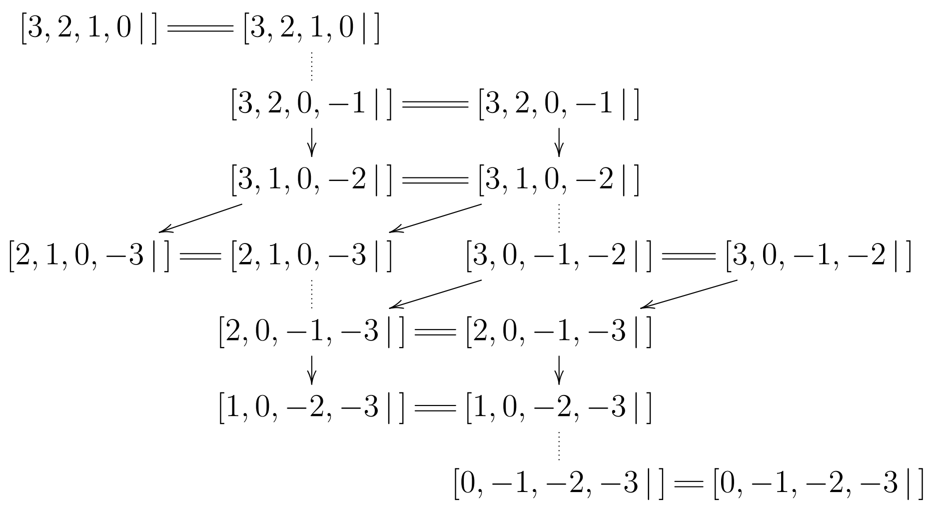


Figure 5.4: Singular orbit for $\Sigma = \{\alpha_4\}$, $\lambda + \rho = [3, 2, 1, 0]$

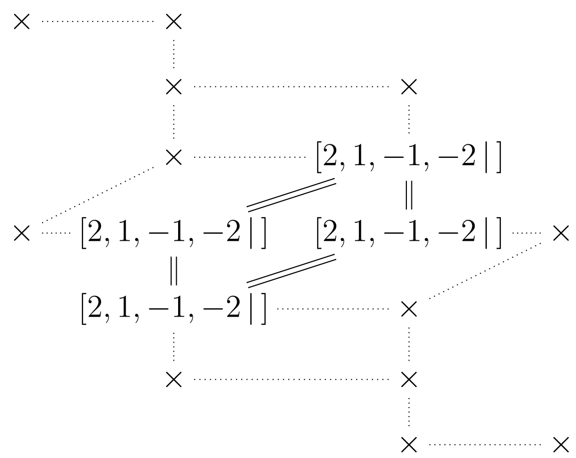


Figure 5.5: Singular orbit for $\Sigma = \{\alpha_1, \alpha_3\}$, $\lambda + \rho = [2, 2, 1, 1]$

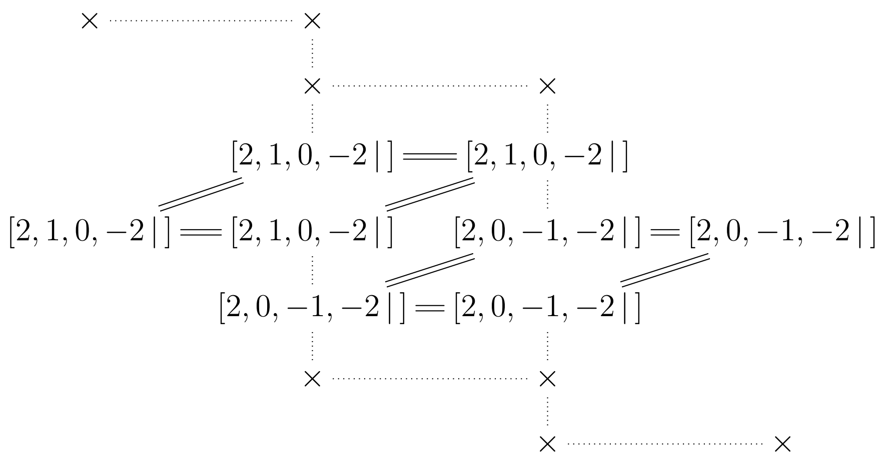


Figure 5.6: Singular orbit for $\Sigma = \{\alpha_1, \alpha_4\}$, $\lambda + \rho = [2, 2, 1, 0]$

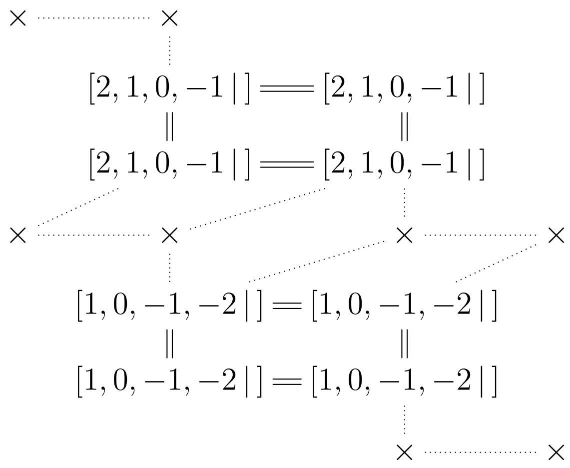


Figure 5.7: Singular orbit for $\Sigma = \{\alpha_2, \alpha_4\}$, $\lambda + \rho = [2, 1, 1, 0]$

- On Figure 5.2, there is a non-standard morphism $[3, 2, -1, -2\,|\,|\,] \to [2, 1, -2, -3\,|\,|\,]$. It is the only way that the singular orbit could be equivalent to the regular one in the rank 2, see Figure 4.2.

- For the same reason, we see that on Figure 5.3 there are now two non-standard morphisms:

$$[3, 2, 1, -1\,|\,|\,] \to [3, 1, -1, -2\,|\,|\,] \text{ and } [2, 1, -1, -3\,|\,|\,] \to [1, -1, -2, -3\,|\,|\,].$$

- On Figure 5.4, there are also two non-standard morphisms:

$$[3, 2, 1, 0\,|\,|\,] \to [3, 0, -1, -2] \text{ and } [2, 1, 0, -3\,|\,|\,] \to [0, -1, -2, -3\,|\,|\,],$$

  because this orbit should be equivalent to the two disjoint copies of the regular one in the rank 2. Each copy consists of the weights with the fixed parity of the negative coordinates, recall Definition 4.6.

These missing operators will be constructed using the Penrose transform from an appropriately chosen twistor space. Moreover, they will be (together with the standard operators) the differentials in the singular BGG complex. ∎

In the identification (4.3) of the elements of the regular Hasse diagram with the LS words, we can recognize those LS words that belong to the singular Hasse diagram $W^{\mathfrak{p},\Sigma}$:

Proposition 5.2. *Suppose* $\Sigma = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$.

- *If the long simple root* $\alpha_n \notin \Sigma$, *then:*

$$W^{\mathfrak{p},\Sigma} = \left\{\overline{d_1 d_2 \ldots d_n} \;:\; \overline{d_{i_k} d_{i_k+1}} = \overline{01} \quad \textit{for } k = 1, \ldots, s\right\}.$$

In particular, $|W^{\mathfrak{p},\Sigma}| = 2^{n-2s}$.

- *If* $\alpha_n = \alpha_{i_s} \in \Sigma$, *then:*

$$W^{\mathfrak{p},\Sigma} = \left\{\overline{d_1 d_2 \dots d_{n-1} 0} \ : \ \overline{d_{i_k} d_{i_k+1}} = \overline{01} \ \ \textit{for } k = 1, \dots, s-1\right\}.$$

In particular, $|W^{\mathfrak{p},\Sigma}| = 2^{n-2s+1}$.

Proof. Assume first that $\alpha_n \notin \Sigma$. It is easy to see that the coordinates of $\lambda + \rho$ are strictly decreasing, except on the positions $(i_k, i_k + 1)$, where it has an equal value depending on k, for $k = 1, \dots, s$. The necessary and sufficient condition for $w = \overline{d_1 \dots d_n} \in W^{\mathfrak{p}}$ to make $\lambda + \rho$ strictly $\mathfrak{p}$-dominant is that for each pair of the adjacent coordinates $(i_k, i_k + 1)$, exactly one of them becomes negative. By the formula (4.6), this is equivalent to $\overline{d_{i_k} d_{i_k+1}} = \overline{01}$ or $\overline{10}$. From the proof of Proposition 1.41 we know that $W^{\mathfrak{p},\Sigma}$ consists of unique minimal length representatives of the left cosets $vW_\Sigma \subseteq W^{\mathfrak{p}}$, where $W_\Sigma = \mathrm{Stab}_{W_{\mathfrak{g}}}(\lambda)$ and $v \in W^{\mathfrak{p}}$. Therefore, we must have $\overline{d_{i_k} d_{i_k+1}} = \overline{01}$ for all $k = 1, \dots, s$.

If $\alpha_n = \alpha_{i_s} \in \Sigma$, then also in addition to the previous conditions, the last coordinate of $\lambda + \rho$ is 0. So, both $\overline{d_1 d_2 \dots d_{n-1} 0}$ and $\overline{d_1 d_2 \dots d_{n-1} 1}$ are in the same left coset of W_Σ, and for $W^{\mathfrak{p},\Sigma}$ we choose the shorter one, which is $\overline{d_1 d_2 \dots d_{n-1} 0}$. □

Two different cases, depending on whether $\alpha_n \notin \Sigma$ or $\alpha_n \in \Sigma$, will be referred to as the ***singularity of the first kind***, and the ***singularity of the second kind***, respectively. The construction of non-standard operators will be more complicated for the singularity of the second kind.

In the singularity of the second kind, it is not hard to see that that there are no standard operators between the even and the odd part of the singular orbit (recall Definition 4.6). This fact also follows from Lemma 5.18, but here we give a more direct proof. The only possible situations for an arrow when the parity changes are the following:

$$\overline{\dots 00} \cong \overline{\dots 01} \to \overline{\dots 10},$$
$$\mu_1 = [\dots, 1, 0, \dots \mid] \to \mu_2 = [\dots, 0, -1, \dots \mid].$$

But we have $\sigma_{\alpha_i}\mu_1 = [\dots, 0, 1, \dots]$ for some $\alpha_i \in \Delta^+(\mathfrak{l}, \mathfrak{h})$, $\sigma_{b_{i+1}}(\sigma_{\alpha_i}\mu_1) = \mu_2$, and moreover, $\langle b_{i+1}, \sigma_{\alpha_i}\mu_1 \rangle > 0$. Now, it follows from Proposition 1.34 that the standard operator $\overline{\dots 00} \to \overline{\dots 10}$ in the singular orbit is trivial.

5.2 Double fibration

From now on, we will work with a weight λ such that $\lambda+\rho$ is orthogonal to only one simple root. In that case, we say that $\lambda+\rho$ is ***semi-regular***. So, $\Sigma=\{\alpha_k\}$ for some $k\leq n$. Minimal such $\lambda+\rho$ is

$$\lambda+\rho=[n-1,n-2,\ldots,n-k+1,\underbrace{\mathbf{n-k}}_{k},\underbrace{\mathbf{n-k}}_{k+1},n-k-1,\ldots,2,1] \tag{5.1}$$

for the singularity of the first kind ($k<n$), or

$$\lambda+\rho=[n-1,n-2,\ldots,2,1,\underbrace{\mathbf{0}}_{n}] \tag{5.2}$$

for the singularity of the second kind ($k=n$). We will work with this minimal $\lambda+\rho$, but we want to note that in the construction of the non-standard operators that follows, the minimality is not important, only the order among the coordinates of $\lambda+\rho$ plays a role. Of course, for a non-minimal $\lambda+\rho$, the orders of the constructed operators will increase (see Proposition 1.46).

Form the following double fibration:

$$\begin{tikzcd}[column sep=small] & G/Q = \times\!\!-\!\!\circ\cdots\circ\!\!-\!\!\circ\!\Leftarrow\!\times \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ G/R = \times\!\!-\!\!\circ\cdots\circ\!\!-\!\!\circ\!\Leftarrow\!\circ & & G/P = \circ\!\!-\!\!\circ\cdots\circ\!\!-\!\!\circ\!\Leftarrow\!\times, \end{tikzcd} \tag{5.3}$$

and start with the homogeneous sheaf $\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ on G/R, where

$$\tilde{\lambda}+\rho=[n-k\,|\,n-1,n-2,\ldots,2,1],\quad 1\leq k\leq n.$$

The weight $\tilde{\lambda}=[-k\,|\,0,0,\ldots,0,0]$ is obviously $\mathfrak{r}$-dominant, so the sheaf $\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ is indeed well defined.

5.2.1 Double fibration in coordinates

In Section 3.2 we saw that G/P can be realized as the Lagrangian Grassmannian $\mathrm{iGr}(n,2n)$, $G/R=G/P_1$ as the isotropic Grassmannian $\mathrm{iGr}(1,2n)$, and G/Q as the space of isotropic flags of the type $(1,n)$. More precisely, the double fibration (5.3) becomes:

$$\begin{tikzcd} & \{(L,W)\colon\ \dim L = 1,\ \dim W = n,\ L \leq W \text{ isotropic}\} \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \{L\colon\ \dim L = 1\} & & \{W\colon\ \dim W = n,\ W \text{ isotropic}\}, \end{tikzcd}$$

where η and τ are projections to the first and the second component, respectively.

Take $X \subseteq G/P$ to be the big affine cell, coordinatized by $\mathrm{Sym}_n(\mathbb{C})$, the symmetric $n \times n$ matrices (Proposition 3.11). For a symmetric matrix C, the Lagrangian subspace W that corresponds to C is the one spanned by the columns of the matrix $\begin{pmatrix} I \\ C \end{pmatrix}$. As before, put $Y := \tau^{-1}(X)$ and $Z := \eta(Y)$ to get the restricted double fibration

$$\begin{tikzcd} & Y \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z & & X. \end{tikzcd} \tag{5.4}$$

A general element in the fiber $\tau^{-1}(W)$, $W \in X$, is a pair (L, W), where any non-zero vector in L is a linear combination of the columns of $\begin{pmatrix} I \\ C \end{pmatrix}$. The coefficients in this linear combination are uniquely determined by L up to a non-zero scalar, so they define a point in the projective space $\mathbb{P}^{n-1}$. It follows that we have a biholomoprhic bijection

$$\mathrm{Sym}_n(\mathbb{C}) \times \mathbb{P}^{n-1} \cong Y,$$

$$(C, y) \mapsto \left(\begin{pmatrix} y \\ C \cdot y \end{pmatrix}, \begin{pmatrix} I \\ C \end{pmatrix} \right),$$

where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{P}^{n-1}$ are the projective (homogeneous) coordinates. Recall that $G/R \cong \mathbb{P}^{2n-1}$, so we put homogeneous coordinates on it with respect to the fixed symplectic basis. With these coordinates on X, Y and the twistor space, maps in (5.4) become:

$$\begin{tikzcd} & (C, y) \in Y \arrow[dl, mapsto, "\eta"'] \arrow[dr, mapsto, "\tau"] & \\ \begin{pmatrix} y \\ C \cdot y \end{pmatrix} & & C. \end{tikzcd}$$

Proposition 5.3. *For $Z = \eta(Y)$ we have:*

$$Z = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{P}^{2n-1} \quad : \quad \textit{at least one } y_i \neq 0 \right\}.$$

Proof. The condition in the curly brackets is necessary because y are projective coordinates. For the converse, assume $y_1 = 1$, and observe:

$$\left(\begin{array}{c|ccc} z_1 - \sum_{i=2}^n z_i \cdot y_i & z_2 & \dots & z_n \\ \hline z_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ z_n & 0 & \dots & 0 \end{array}\right) \begin{pmatrix} 1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

The proof is analogous if some other $y_i = 1$. □

The following proposition is needed for the pull-back stage of the Penrose transform, see Theorem 2.2.

Proposition 5.4. *The fibers of $\eta\colon Y \to Z$ are smoothly contractible.*

Proof. Given $\begin{pmatrix} y \\ z \end{pmatrix} \in Z$, the condition $C \cdot y = z$ is given by linear equations on the entries of the matrix C. So, the fiber $\eta^{-1}\begin{pmatrix} y \\ z \end{pmatrix} \subseteq \mathrm{Sym}_n(\mathbb{C}) \times y$ is a certain affine subspace of Y, and therefore smoothly contractible. □

Suppose $X' \subseteq X$ is a convex open subset, and denote $Y' := \tau^{-1}(X')$ and $Z' := \eta(Y')$. In this new restricted double fibration

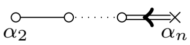
$$\begin{tikzcd} & Y' \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z' & & X', \end{tikzcd} \tag{5.5}$$

the fibers of $\eta\colon Y' \to Z'$ are equal to the intersection of an affine set (the fibers of η in Y) and a convex set (a copy of X' in Y). Therefore, these fibers are also smoothly contractible. Even though we are working with the fixed big affine cell, by a change of a symplectic basis in V (the action of $\mathrm{Sp}(2n, \mathbb{C})$ on $\mathrm{iGr}(n, 2n)$), we can make the big affine cell appear around any point in the Lagrangian Grassmannian. Moreover, we can take X' to be any open ball, or a polydisc around that point. The conclusion is that we have a valid setup for working with the Penrose transform locally, around any point in the Lagrangian Grassmannian.

5.2.2 Relative Hasse diagrams

Let us list other ingredients needed for the Penrose transform over the double fibration (5.3) and the restricted versions:

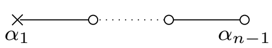
- To calculate the relative BGG resolution (2.4) of the inverse image $\eta^{-1}\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ on G/Q, we need the relative Hasse diagram $W_{\mathfrak{r}}^{\mathfrak{q}}$. It is easy to see that the fiber of η is

  $$R/Q \cong$$
  
  .

  So the relative Hasse diagram $W_{\mathfrak{r}}^{\mathfrak{q}}$ (and so the relative BGG resolution) has the same shape as the regular Hasse diagram in rank $n-1$. As a subset of $W_{\mathfrak{g}}$, $W_{\mathfrak{r}}^{\mathfrak{q}}$ operates on the last $n-1$ coordinates and completely ignores the first coordinate of a weight. Moreover, $W_{\mathfrak{r}}^{\mathfrak{q}}$ can be identified with the following subgraph of $W^{\mathfrak{p}}$:

  $$W_{\mathfrak{r}}^{\mathfrak{q}} = \left\{ w \in W^{\mathfrak{p}} \ : \ w = \overline{0\, d_1 d_2 \ldots d_{n-1}} \right\}. \tag{5.6}$$

- To apply Bott-Borel-Weil theorem for calculating the higher direct images along τ (Corollary 1.18), it is convenient to understand the relative Hasse diagram $W_{\mathfrak{p}}^{\mathfrak{q}}$. It is easy to see that the fiber of τ is

  $$P/Q \cong$$
  
  $$\cong \mathbb{P}^{n-1}.$$

  This is also $|1|$-graded, so an easy exercise with Proposition 1.9 gives:

  $$W_{\mathfrak{p}}^{\mathfrak{q}} = \left\{ \emptyset \xrightarrow{a_{12}} \bullet \xrightarrow{a_{13}} \bullet \xrightarrow{a_{14}} \ldots \xrightarrow{a_{1n}} \bullet \right\} \subseteq W_{\mathfrak{g}}. \tag{5.7}$$

  Actually, it is the inverses of the elements in $W_{\mathfrak{p}}^{\mathfrak{q}}$ that appear in the calculations, so we state them explicitly:

  $$\left(W_{\mathfrak{p}}^{\mathfrak{q}}\right)^{-1} = \{\mathrm{Id}\} \cup \{\sigma_{a_{12}} \circ \sigma_{a_{13}} \circ \ldots \circ \sigma_{a_{1t}} \ : \ t = 2, \ldots, n\} \tag{5.8}$$

  Moreover, from (5.7) it also follows that $l(\sigma_{a_{12}} \circ \sigma_{a_{13}} \circ \ldots \circ \sigma_{a_{1t}}) = t-1$. This can also be seen directly, by writing down its reduced form: $\sigma_{\alpha_{t-1}} \circ \ldots \circ \sigma_{\alpha_1}$. These are just compositions of the transpositions of adjacent coordinates, starting with (12), then (23), and so on until $(t-1, t)$.

5.2.3 Examples in rank 3

Example 5.5 $\Sigma = \{\alpha_1\}$, $\lambda + \rho = [2, 2, 1] \implies \tilde{\lambda} + \rho = [2 \,|\, 2, 1]$.

The relative BGG resolution of the sheaf $\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ on G/Q is obtained by applying (5.6) (with $n = 3$) to the weight $[2 \,|\, 2, 1 \,|\,]$. The first coordinate remains fixed, while the remaining part looks like in Figure 4.2:

$$0 \to \eta^{-1}[2 \,|\, 2, 1] \to [2 \,|\, 2, 1 \,|\,] \to [2 \,|\, 2, -1 \,|\,] \to [2 \,|\, 1, -2 \,|\,] \to [2 \,|\, -1, -2 \,|\,] \to 0. \quad (5.9)$$

Let us describe in detail how to calculate the higher direct images along τ of (5.9), using Corollary 1.18. In each object of the resolving part, remove the first bar:

$$[2, 2, 1 \,|\,] \to [2, 2, -1 \,|\,] \to [2, 1, -2 \,|\,] \to [2, -1, -2 \,|\,]. \quad (5.10)$$

If a weight has two coordinates equal, it is $\mathfrak{p}$-singular, and so all its higher direct images are 0. We denote them by $\times$ in the diagram below. Each of the remaining weights in this case has the strictly decreasing coordinates, so it is $\mathfrak{p}$-dominant. Therefore, they appear as higher direct images in the degree 0. So, the degrees of the surviving higher direct images of (5.10) are

$$\times \cdots\cdots \times \cdots\cdots 0 \to 0.$$

We organize this information in the first page of the spectral sequence (2.8), which in this example is the following:

$$E_1^{pq} = \begin{array}{|ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [2, 1, -2 \,|\,] \to & [2, -1, -2 \,|\,] & 0. \\ \hline \end{array}$$

The horizontal component p is the degree in the relative BGG resolution, and the vertical one q is the degree of the higher direct image; both indices start from 0. Note that the objects in the spectral sequence (2.8) are not really homogeneous sheaves, but rather its sections over X. We will omit $\Gamma(X, -)$ from the notation, and write only sheaves, or the defining (ρ-shifted) weights.

Also note that a standard operator between two adjacent objects in the relative BGG survives the higher direct image and appear in the spectral sequence as a standard operator, only if both these adjacent objects survive in the same degree. This follows from the functoriality of the direct images. ■

Example 5.6 $\Sigma = \{\alpha_2\}$, $\lambda + \rho = [2, 1, 1] \implies \tilde{\lambda} + \rho = [1 \,|\, 2, 1]$.

The relative BGG resolution is:

$$0 \to \eta^{-1}[1 \,|\, 2, 1] \to [1 \,|\, 2, 1 \,|\,] \to [1 \,|\, 2, -1 \,|\,] \to [1 \,|\, 1, -2 \,|\,] \to [1 \,|\, -1, -2 \,|\,] \to 0.$$

The weights $[1 \,|\, 2, 1 \,|\,]$ and $[1 \,|\, 1, -2 \,|\,]$ do not survive the higher direct images. For each of the remaining weights μ, there is a unique element w in (5.8) (a permutation of the coordinates) such that $w\mu$ is $\mathfrak{p}$-dominant (that is, having strictly decreasing coordinates). Then, $w\mu$ will appear in the degree $l(w)$. For the weight $[1\,|\,2,-1\,|\,]$, w is $\sigma_{a_{12}}$, the transposition of first two coordinates. For the last weight, $w = \mathrm{Id}$. We get:

$$\times \cdots 1 \cdots \times \cdots 0,$$

$$E_1^{pq} = \begin{array}{|ccccc} 0 & [2,1,-1\,|\,] & 0 & 0 & 0 \\ 0 & 0 & 0 & [1,-1,-2\,|\,] & 0. \\ \hline \end{array}$$

There are no standard operators in this spectral sequence. In Section 5.3 we will construct a non-standard one: $[2,1,-1\,|\,] \to [1,-1,-2\,|\,]$. ■

Example 5.7 $\Sigma = \{\alpha_3\}$, $\lambda + \rho = [2,1,0] \implies \tilde{\lambda} + \rho = [0\,|\,2,1]$.

This is the singularity of the second kind. The relative BGG resolution is:

$$0 \to \eta^{-1}[0\,|\,2,1] \to [0\,|\,2,1\,|\,] \to [0\,|\,2,-1\,|\,] \to [0\,|\,1,-2\,|\,] \to [0\,|\,-1,-2\,|\,] \to 0.$$

Note that now every weight survives a higher direct image, since there are no two coordinates in $\lambda + \rho$ with the same absolute value. This is typical for the singularity of the second kind. The degrees of the surviving higher direct images are:

$$2 \cdots 1 \to 1 \cdots 0.$$

The first page of the spectral sequence is:

$$E_1^{pq} = \begin{array}{|ccccc} [2,1,0\,|\,] & 0 & 0 & 0 & 0 \\ 0 & [2,0,-1\,|\,] & \to [1,0,-2\,|\,] & 0 & 0 \\ 0 & 0 & 0 & [0,-1,-2\,|\,] & 0. \\ \hline \end{array} \tag{5.11}$$

In Section 5.4 we will construct a non-standard operator $[2,1,0\,|\,] \to [0,-1,-2\,|\,]$. In Section 5.6 we will find the explicit formula as in (1.5), for this particular operator. ■

5.2.4 Examples in rank 4

Example 5.8 $\Sigma = \{\alpha_1\}$, $\lambda + \rho = [3,3,2,1] \implies \tilde{\lambda} + \rho = [3\,|\,3,2,1]$.

The relative BGG resolution of the sheaf $\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ on G/Q is obtained by applying (5.6) ($n = 4$) to the weight $[3\,|\,3,2,1\,|\,]$. The first coordinate remains fixed, while the remaining part looks like in Figure 4.3:

$$
\begin{tikzcd}
{[3 \mid 3,2,1 \mid \mid]} \arrow[r] & {[3 \mid 3,2,-1 \mid \mid]} \arrow[d] & & \\
& {[3 \mid 3,1,-2 \mid \mid]} \arrow[r] \arrow[d] & {[3 \mid 3,-1,-2 \mid \mid]} \arrow[d] & \\
& {[3 \mid 2,1,-3 \mid \mid]} \arrow[r] & {[3 \mid 2,-1,-3 \mid \mid]} \arrow[d] & \\
& & {[3 \mid 1,-2,-3 \mid \mid]} \arrow[r] & {[3 \mid -1,-2,-3 \mid \mid]},
\end{tikzcd}
\qquad
\begin{tikzcd}
\times \arrow[r, dash, dotted] & \times \arrow[d, dash, dotted] & \\
& \times \arrow[r, dash, dotted] \arrow[d, dash, dotted] & \times \arrow[d, dash, dotted] \\
& 0 \arrow[r] & 0 \arrow[d] \\
& & 0 \arrow[r] & 0.
\end{tikzcd}
$$

The right-hand side gives the degrees of the surviving higher direct images. The first page of the spectral sequence is:

$$
\begin{array}{|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & [3,2,1,-3 \mid \mid] \to & [3,2,-1,-3 \mid \mid] \to & [3,1,-2,-3 \mid \mid] \to & [3,-1,-2,-3 \mid \mid]. \\
\hline
\end{array}
$$

Notice that this agrees with the singular orbit on Figure 5.1. No new non-standard operators are missing in this case. This is true in any rank, for $\Sigma = \{\alpha_1\}$. ∎

Example 5.9 $\Sigma = \{\alpha_2\}$, $\lambda + \rho = [3,2,2,1] \implies \tilde{\lambda} + \rho = [2 \mid 3,2,1]$.

The relative BGG resolution is:

$$
\begin{tikzcd}
{[2 \mid 3,2,1 \mid \mid]} \arrow[r] & {[2 \mid 3,2,-1 \mid \mid]} \arrow[d] & & \\
& {[2 \mid 3,1,-2 \mid \mid]} \arrow[r] \arrow[d] & {[2 \mid 3,-1,-2 \mid \mid]} \arrow[d] & \\
& {[2 \mid 2,1,-3 \mid \mid]} \arrow[r] & {[2 \mid 2,-1,-3 \mid \mid]} \arrow[d] & \\
& & {[2 \mid 1,-2,-3 \mid \mid]} \arrow[r] & {[2 \mid -1,-2,-3 \mid \mid]},
\end{tikzcd}
\qquad
\begin{tikzcd}
\times \arrow[r, dash, dotted] & \times \arrow[d, dash, dotted] & \\
& 1 \arrow[r] \arrow[d, dash, dotted] & 1 \arrow[d, dash, dotted] \\
& \times \arrow[r, dash, dotted] & \times \arrow[d, dash, dotted] \\
& & 0 \arrow[r] & 0.
\end{tikzcd}
$$

The first page of the spectral sequence is:

$$
\begin{array}{|ccccccc}
0 & 0 & [3,2,1,-2 \mid \mid] \to & [3,2,-1,-2 \mid \mid] & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & [2,1,-2,-3 \mid \mid] \to & [2,-1,-2,-3 \mid \mid]. \\
\hline
\end{array}
$$

This also agrees with the corresponding singular orbit, on Figure 5.2. There is a non-standard operator $[3,2,-1,-2 \mid \mid] \to [2,1,-2,-3 \mid \mid]$ missing in the spectral sequence, which is to be constructed. ∎

Example 5.10 $\Sigma = \{\alpha_3\}$, $\lambda + \rho = [3,2,1,1] \implies \tilde{\lambda} + \rho = [1 \mid 3,2,1]$.

The relative BGG resolution is:

$$\begin{tikzcd}[column sep=small]
{[1\,|\,3,2,1\,|\,|\,]} \arrow[r] & {[1\,|\,3,2,-1\,|\,|\,]} \arrow[d] & & \\
& {[1\,|\,3,1,-2\,|\,|\,]} \arrow[r] \arrow[d] & {[1\,|\,3,-1,-2\,|\,|\,]} \arrow[d] & \\
& {[1\,|\,2,1,-3\,|\,|\,]} \arrow[r] & {[1\,|\,2,-1,-3\,|\,|\,]} \arrow[d] & \\
& & {[1\,|\,1,-2,-3\,|\,|\,]} \arrow[r] & {[1\,|\,-1,-2,-3\,|\,|\,]},
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
\times \arrow[r, dash, dotted] & 2 \arrow[d, dash, dotted] & & \\
& \times \arrow[r, dash, dotted] \arrow[d, dash, dotted] & 1 \arrow[d] & \\
& \times \arrow[r, dash, dotted] & 1 \arrow[d, dash, dotted] & \\
& & \times \arrow[r, dash, dotted] & 0.
\end{tikzcd}$$

Note that the degree of the surviving higher direct image is equal to the number of the transpositions of adjacent coordinates needed to move the first coordinate to its correct position. The first page of the spectral sequence is:

$$\begin{array}{|ccccccc}
0 & [3,2,1,-1\,|\,|\,] & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & [3,1,-1,-2\,|\,]\to & [2,1,-1,-3\,|\,|\,] & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & [1,-1,-2,-3\,|\,|\,]. \\
\hline
\end{array}$$

Compare this to Figure 5.3. Two non-standard operators are missing in this case: $[3,2,1,-1\,|\,|\,] \to [3,1,-1,-2\,|\,|\,]$ and $[2,1,-1,-3\,|\,|\,] \to [1,-1,-2,-3\,|\,|\,]$. ■

Example 5.11 $\Sigma = \{\alpha_4\}$, $\lambda+\rho = [3,2,1,0] \implies \tilde{\lambda}+\rho = [0\,|\,3,2,1]$.

Note that this is the singularity of the second kind. The relative BGG resolution is:

$$\begin{tikzcd}[column sep=small]
{[0\,|\,3,2,1\,|\,|\,]} \arrow[r] & {[0\,|\,3,2,-1\,|\,|\,]} \arrow[d] & & \\
& {[0\,|\,3,1,-2\,|\,|\,]} \arrow[r] \arrow[d] & {[0\,|\,3,-1,-2\,|\,|\,]} \arrow[d] & \\
& {[0\,|\,2,1,-3\,|\,|\,]} \arrow[r] & {[0\,|\,2,-1,-3\,|\,|\,]} \arrow[d] & \\
& & {[0\,|\,1,-2,-3\,|\,|\,]} \arrow[r] & {[0\,|\,-1,-2,-3\,|\,|\,]},
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
3 \arrow[r, dash, dotted] & 2 \arrow[d] & & \\
& 2 \arrow[r, dash, dotted] \arrow[d] & 1 \arrow[d] & \\
& 2 \arrow[r, dash, dotted] & 1 \arrow[d] & \\
& & 1 \arrow[r, dash, dotted] & 0.
\end{tikzcd}$$

The first page of the spectral sequence:

$$\begin{array}{|ccccccc}
[3,2,1,0\,|\,|\,] & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & [3,2,0,-1\,|\,|\,]\to & [3,1,0,-2\,|\,|\,]\longrightarrow & [2,1,0,-3\,|\,|\,] & 0 & 0 & 0 \\
0 & 0 & 0 & [3,0,-1,-2\,|\,|\,]\to & [2,0,-1,-3\,|\,|\,]\to & [1,0,-2,-3\,|\,|\,] & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & [0,-1,-2,-3\,|\,|\,]. \\
\hline
\end{array}$$

Compare this to Figure 5.4. Again, two non-standard operators are missing here, and it might not be obvious to guess which ones. They are: $[3,2,1,0\,|\,|\,] \to [3,0,-1,-2\,|\,|\,]$ and $[2,1,0,-3\,|\,|\,] \to [0,-1,-2,-3\,|\,|\,]$. Namely, the Enright-Shelton equivalence says that this orbit should decompose into two disjoint blocks, each of the shape $\bullet \to \bullet \to \bullet \to \bullet$ (on Figure 4.2), according to the parity. ■

The main technical difference between the two kinds of the singularity is the following: In the first kind, all the non-standard operators to be constructed go across zero columns in the first page of the spectral sequence; in the second kind, the wanted non-standard operators go across columns with non-zero entries.

5.2.5 Examples in rank 5

Instead of writing down again the complete relative BGG resolutions, we just give the degrees of the surviving higher direct images. They are presented on Figure 5.8 and Figure 5.9, from $\Sigma = \{\alpha_1\}$ to $\Sigma = \{\alpha_5\}$, respectively. The non-standard operators to be constructed are presented with the dashed arrows. It is clearly visible that for the first kind, the surviving part of the relative BGG resolution together with the announced non-standard operators, has the same shape as the relative BGG, but of rank lower by 1. For the second kind, every object of the relative BGG survives, and they organize into two parts, each of which has the same shape as before.

5.3 Construction of non-standard operators, first kind

In this section we work with the singularity of the first kind:

$$\Sigma = \{\alpha_k\}, \quad 1 \leq k < n.$$

This means that $\lambda + \rho$ has a repeated coordinate entry on the positions $(k, k+1)$. We work with the minimal such weight, which is given in (5.1). Recall that we start with $\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ on G/R, where

$$\tilde{\lambda} + \rho = [n-k \,|\, n-1, n-2, \ldots, 2, 1], \quad 1 \leq k < n. \tag{5.12}$$

Then, on G/Q we construct the relative BGG resolution, which is parametrized by $W_{\mathfrak{r}}^{\mathfrak{q}}$ given in (5.6). Finally, we calculate the higher direct images of the relative BGG resolution. The result is the following:

Proposition 5.12. *In the singularity of the first kind:*

(a) The objects in the relative BGG resolution that survive a higher direct image are parametrized by the following LS words:

$$\overline{0\, d_1 \ldots d_{k-1}\, 1\, d_{k+1} \ldots d_{n-1}}. \tag{5.13}$$

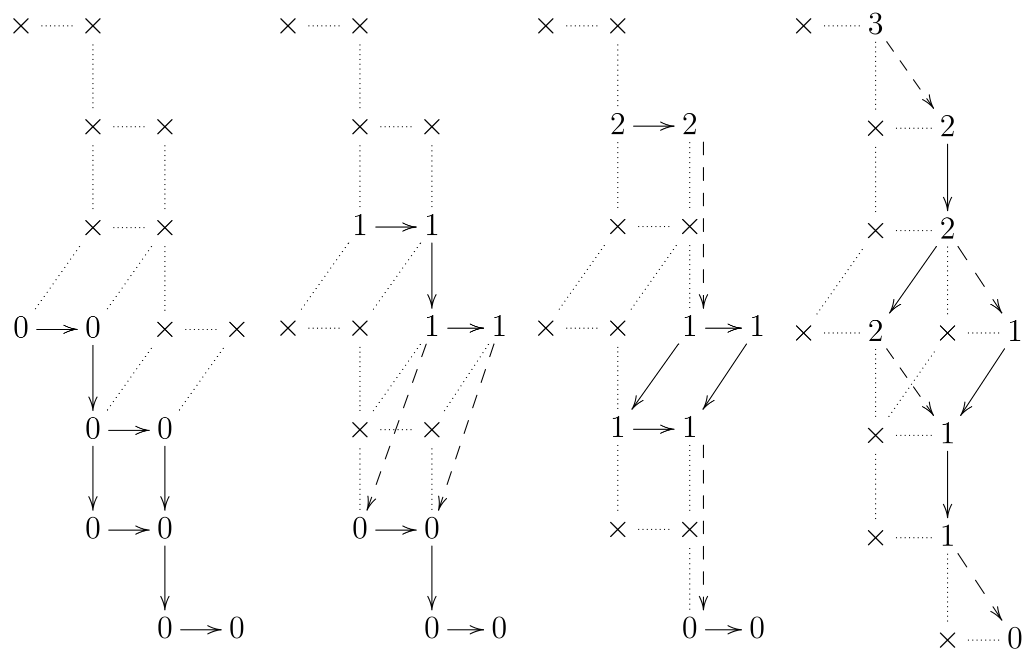


Figure 5.8: Relative and singular BGG complexes in rank 5, first kind

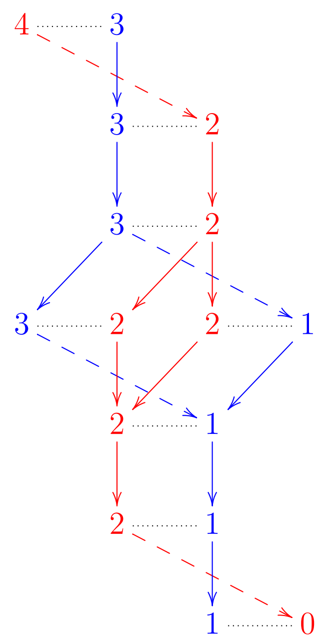


Figure 5.9: Relative and singular BGG complexes in rank 5, second kind

(b) *The degree of the surviving higher direct image of an object in (5.13) is equal to the number of the digits* 0 *among* $\overline{d_1 \dots d_{k-1}}$. *The result of the direct image in this degree corresponds to*

$$\overline{d_1 \dots d_{k-1}\, 01\, d_{k+1} \dots d_{n-1}} \in W^{\mathfrak{p},\Sigma}. \tag{5.14}$$

(c) *The first page of the spectral sequence agrees with the singular orbit, including both the objects and the standard operators.*

Proof. An element of the form $w = \overline{0\, d_1 \dots d_k \dots d_{n-1}} \in W^{\mathfrak{q}}_{\mathfrak{r}}$ will make the weight (5.12) $\mathfrak{p}$-regular if and only if it makes the coordinate entry $n-k$ (after the bar) negative. This will happen if and only if $d_k = 1$, so (a) follows. The number of the transpositions of adjacent coordinates needed to make $w(\tilde{\lambda}+\rho)$ $\mathfrak{p}$-dominant is equal to the number of the coordinates in $w(\tilde{\lambda}+\rho)$ greater than $n-k$ (those can be only on the positions 2 to k), and this is the same as the number of the digits 0 among $\overline{d_1 \dots d_{k-1}}$ (the digit 1 makes the coordinate entry negative). It is easy to see that the result agrees with (5.14) applied to $\tilde{\lambda}+\rho$, hence (b) follows. The part (c) follows from comparing (5.14) to Proposition 5.2. □

From Proposition 5.12 (a) follows that we have a bijection from the surviving part of the relative BGG resolution (and the singular orbit), to the regular Hasse diagram of the rank $n-2$, given by:

$$\overline{0\, d_1 \dots d_{k-1}\, 1\, d_{k+1} \dots d_{n-1}} \mapsto \overline{d_1 \dots d_{k-1} d_{k+1} \dots d_{n-1}}. \tag{5.15}$$

However, this is not a directed-graph isomorphism. We need to "add" more arrows to the left-hand side, to make this map a directed-graph isomorphism (to satisfy the Enright-Shelton equivalence). Those arrows are the missing non-standard operators, constructed in the following theorem.

Theorem 5.13. *There are non-standard invariant differential operators*

$$\mathcal{O}_{\mathfrak{p}}(\nu) \to \mathcal{O}_{\mathfrak{p}}(\nu'') \tag{5.16}$$

for all the pairs ν, ν'' *in the singular orbit given by*

$$\nu = \overline{d_1 \dots d_{k-2}\, 0\mathbf{011}\, d_{k+3} \dots d_n} \cdot \lambda, \quad \nu'' = \overline{d_1 \dots d_{k-2}\, 1\mathbf{010}\, d_{k+3} \dots d_n} \cdot \lambda. \tag{5.17}$$

For $k=1$ *this condition is empty, and for* $k = n-1$ *we ignore the last coordinate (that is,* $\nu = \overline{d_1 \dots d_{n-3}\, 0\mathbf{01}} \cdot \lambda$ *and* $\nu'' = \overline{d_1 \dots d_{n-3}\, 1\mathbf{01}} \cdot \lambda$*).*

If λ *is minimal as in (5.1), the operator (5.16) is of the order* 2.

Proof. Take X' to be an open ball inside the big affine cell in G/P, and consider the Penrose transform over the restricted double fibration (5.5). In the relative BGG resolution, we find and fix the following sequence:

$$\begin{array}{c} \mu = \overline{0\,d_1 \dots d_{k-2}\,0\mathbf{1}1\,d_{k+2} \dots d_{n-1}} \\ \big\downarrow \\ \mu' = \overline{0\,d_1 \dots d_{k-2}\,1\mathbf{0}1\,d_{k+2} \dots d_{n-1}} \\ \big\downarrow \\ \mu'' = \overline{0\,d_1 \dots d_{k-2}\,1\mathbf{1}0\,d_{k+2} \dots d_{n-1}}. \end{array}$$

Denote $q = 1+$the number of the digits 0 in $\overline{d_1 \dots d_{k-2}}$. Consider the (part of the) Čech bi-complex that calculates the higher direct images, described in Figure 5.10. Here the

$$\begin{tikzcd}
\check{C}^{q+1}_{\mu} \arrow[r] & \check{C}^{q+1}_{\mu'} \arrow[r] & \check{C}^{q+1}_{\mu''} \\
\check{\mathbf{C}}^{\mathbf{q}}_{\mu} \arrow[r] \arrow[u] & \check{C}^{q}_{\mu'} \arrow[r] \arrow[u] & \check{C}^{q}_{\mu''} \arrow[u] \\
\check{C}^{q-1}_{\mu} \arrow[r] \arrow[u] & \check{C}^{q-1}_{\mu'} \arrow[r] \arrow[u] & \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''} \arrow[u] \\
\check{C}^{q-2}_{\mu} \arrow[r] \arrow[u] & \check{C}^{q-2}_{\mu'} \arrow[r] \arrow[u] & \check{C}^{q-2}_{\mu''} \arrow[u] \\
\mathcal{O}_{\mathfrak{q}}(\mu) \arrow[r] \arrow[u, dotted, no head] & \mathcal{O}_{\mathfrak{q}}(\mu') \arrow[r] \arrow[u, dotted, no head] & \mathcal{O}_{\mathfrak{q}}(\mu'') \arrow[u, dotted, no head]
\end{tikzcd}$$

Figure 5.10: Čech bi-complex (1)

horizontal morphisms, denoted by d_h, are induced from the differentials of the relative BGG resolution. The vertical morphisms, denoted d_v, are the usual differentials in the Čech resolution. We have $d_v^2 = 0$, $d_h^2 = 0$, and for each square in the diagram, the horizontal and the vertical morphisms anticommute: $d_h d_v = -d_v d_h$. By the definition, the vertical cohomologies are equal to the higher direct images of the corresponding homogeneous sheaf. By Proposition 5.12 (b),

$$H^q(\check{C}^{\bullet}_{\mu}, d_v) = \tau_*^q \mathcal{O}_{\mathfrak{q}}(\mu) = \mathcal{O}_{\mathfrak{p}}(\nu) \quad \text{and} \quad H^{q-1}(\check{C}^{\bullet}_{\mu''}, d_v) = \tau_*^{q-1} \mathcal{O}_{\mathfrak{q}}(\mu'') = \mathcal{O}_{\mathfrak{p}}(\nu''). \quad (5.18)$$

Those cochain spaces are denoted with the bold font. All other vertical cohomologies are trivial, including the complete middle column.

We will define the operator (5.16) on the representatives of the cohomology classes in $H^q(\check{C}^\bullet_\mu, d_v)$. Take a representative $x \in \check{\mathbf{C}}^{\mathbf{q}}_\mu$ of such class. This means that x is a cocycle: $d_v(x) = 0$. But $d_v d_h(x) = -d_h d_v(x) = 0$ implies that $d_h(x) \in \check{C}^q_{\mu'}$ is also a cocycle. Since also $H^q(\check{C}^\bullet_{\mu'}, d_v) = \tau^q_* \mathcal{O}_{\mathfrak{q}}(\mu') = 0$, it follows that $d_h(x) \in \operatorname{Im} d_v$. More precisely, there is $y \in \check{C}^{q-1}_{\mu'}$ such that $d_v(y) = d_h(x)$. Then, $d_h(y) \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''}$, in the correct cochain space. Diagrammatically,

$$\begin{tikzcd}
0 & 0 & \\
x \in \check{\mathbf{C}}^{\mathbf{q}}_\mu \arrow[u, mapsto] \arrow[r, mapsto] \arrow[drr, dashed, mapsto] & d_h(x) \in \check{C}^q_{\mu'} \arrow[u, mapsto] \arrow[r, mapsto] & 0 \\
& y \in \check{C}^{q-1}_{\mu'} \arrow[u, mapsto] \arrow[r, mapsto] & d_h(y) \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''}. \arrow[u, mapsto]
\end{tikzcd} \tag{5.19}$$

Let us check that $d_h(y)$ is a cocycle: $d_v d_h(y) = -d_h d_v(y) = -d_h^2(x) = 0$. Next, we check that we have a well defined map on the cohomology classes

$$[x] \mapsto [d_h(y)],$$

which does not depend on the choices of x and y. Take another cocycle x' in the same cohomology class $[x]$, and find y' so that $d_v(y') = d_h(x')$. We are required to show that $[d_h(y)] = [d_h(y')]$, that is $d_h(y) - d_h(y') \in \operatorname{Im} d_v$. We assumed that $x - x' = d_v(t)$ for some $t \in \check{C}^{q-1}_\mu$. Observe that

$$d_v(y - y' + d_h(t)) = d_h(x - x') - d_h d_v(t) = 0, \tag{5.20}$$

so we conclude that $y - y' + d_h(t) = d_v(t')$ for some $t' \in \check{C}^{q-2}_{\mu'}$. Finally,

$$d_v(-d_h(t')) = d_h(d_v(t')) = d_h(y) - d_h(y') + d_h^2(t) = d_h(y) - d_h(y') \in \operatorname{Im} d_v.$$

By (5.18), we have a well defined map $D\colon \mathcal{O}_{\mathfrak{p}}(\nu) \to \mathcal{O}_{\mathfrak{p}}(\nu'')$ given by $D([x]) = [d_h(y)]$, which is by the construction local, and G-invariant. By Remark 1.21, it is a differential operator. Its order is given by the difference of the generalized conformal weights, which is easily seen to be 2 in the minimal case. □

Definition 5.14. *In the singularity of the first kind, the singular orbit with all the non-standard operators constructed in Theorem 5.13 included in it, is called the* ***singular BGG complex*** *of the infinitesimal character $\lambda + \rho$.*

Theorem 5.15. *For the singularity of the first kind:*

(a) The singular BGG complexes of the rank n are directed-graph isomorphic to the regular one of the rank $n - 2$.

(b) *Every square in the singular BGG complex anticommutes.*

(c) *If we add up all objects of the singular BGG complex of the same degree*[1]*, we get a cochain complex.*

Proof. (a) This is easy to check: now (5.15) is a directed-graph isomorphism.

(b) The standard operators anticommute, since this was already true in the relative BGG resolution. There are no squares with all the operators non-standard. Therefore, we only check the combination of a non-standard and a standard one. This is the typical situation in the relative BGG that induces such square:

$$
\begin{tikzcd}
\overline{0\dots0\mathbf{11}\dots01\dots} \arrow[r] \arrow[d] & \overline{0\dots1\mathbf{01}\dots01\dots} \arrow[r] \arrow[d] & \overline{0\dots1\mathbf{10}\dots01\dots} \arrow[d] \\
\overline{0\dots0\mathbf{11}\dots10\dots} \arrow[r] & \overline{0\dots1\mathbf{01}\dots10\dots} \arrow[r] & \overline{0\dots1\mathbf{10}\dots10\dots}.
\end{tikzcd}
$$

(Other possible situations start with $\overline{0\dots01\dots0\mathbf{11}\dots}$, or $\overline{0\dots0\mathbf{11}\dots0}$).

Denote by μ, μ', μ'' the objects in the first row, and by θ, θ', θ'' the objects in the second row, in any of these situations, and consider the part of the Čech bi-complex above it, with the same notation as in the proof of Theorem 5.13. Denote by d all the standard operators $\mu \to \theta$, $\mu' \to \theta'$ and $\mu'' \to \theta''$ in the relative BGG. These are just horizontal differentials, but they go in a different direction then those we denoted by d_h in the definition of D. The maps d anticommute with both d_h and d_v. Let us draw a diagram relating the elements from the definition of D:

$$
\begin{tikzcd}
 & d(x) \in \check{\mathbf{C}}^{\mathbf{q}}_{\theta} \arrow[r, mapsto] & d_h(d(x)) & \\
x \in \check{\mathbf{C}}^{\mathbf{q}}_{\mu} \arrow[ur, mapsto] \arrow[r, mapsto] & d_h(x) & y' \arrow[u, mapsto] \arrow[r, mapsto] & d_h(y') \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\theta''} \\
 & y \arrow[u, mapsto] \arrow[r, mapsto] & d_h(y) \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''}. \arrow[ur, mapsto, dashed, "-d"'] &
\end{tikzcd}
\tag{5.21}
$$

This is the part of the Čech bi-complex that is mapped to a square in the singular orbit with two parallel standard, and two parallel non-standard operators. We need to show that $[d(d_h(y))] = -[d_h(y')]$.

First note that for y' we can take $d(y)$ without changing the cohomology class

[1]The degree in the singular BGG complex is defined in the obvious way: the length of a minimal path from the trivial object. Alternatively, use Proposition 5.12 and the directed-graph isomorphism (5.15).

$D([d(x)])$. Indeed,

$$d_v(d(y)) = -d(d_v(y)) = -d(d_h(x)) = d_h(d(x)).$$

Therefore, $d_h(y') = d_h(d(y)) = -d(d_h(y))$. This proves (b).

(c) This follows from (b), since $(\sum d_i)^2 = \sum_{i \neq j} d_i d_j = \sum_{i<j}(d_i d_j + d_j d_i) = 0$.

$\square$

Remark 5.16 This proof of 5.15 (b) is a special case of One-step push forward lemma from [PS16]. $\blacksquare$

5.4 Construction of non-standard operators, second kind

In this section we work with the singularity of the second kind:

$$\Sigma = \{\alpha_n\}.$$

This means that $\lambda + \rho$ has 0 on the last coordinate. We work with the minimal such weight, which is $\lambda + \rho = [n-1, n-2, \ldots, 1, 0]$. Recall that we start with $\mathcal{O}_{\mathfrak{r}}(\tilde{\lambda})$ on G/R, where

$$\tilde{\lambda} + \rho = [0 \,|\, n-1, n-2, \ldots, 2, 1]. \tag{5.22}$$

Then, on G/Q we construct the relative BGG resolution, which is parametrized by $W_{\mathfrak{r}}^{\mathfrak{q}}$ given in (5.6). Finally, we calculate the higher direct images of the resolution. The result is the following:

Proposition 5.17. *In the singularity of the second kind:*

(a) All the objects in the relative BGG resolution survive a higher direct image.

(b) The degree of the surviving higher direct image of an object parametrized by the LS word $w = \overline{0\, d_1 \ldots d_{n-1}}$ is equal to the number of the digits 0 among $\overline{d_1 \ldots d_{n-1}}$. The result of the direct image in this degree corresponds to

$$\overline{d_1 \ldots d_{n-1}\, 0} \in W^{\mathfrak{p},\Sigma}. \tag{5.23}$$

(c) The first page of the spectral sequence agrees with the singular orbit, including both the objects and the standard operators.

Proof. Since no permutation and sign changes can make two coordinates of $\tilde{\lambda}+\rho$ equal, all elements in the $W_{\mathfrak{r}}^{\mathfrak{q}}$-orbit of $\tilde{\lambda}+\rho$ are $\mathfrak{p}$-regular, therefore we have (a). The number of the transpositions of adjacent coordinates needed to put 0 to its correct position in $w(\tilde{\lambda}+\rho)$ is of course equal to the number of the positive coordinates, a this is the same as the number of the digits 0 in w not counting the first one. The result agrees with (5.23) applied to $\lambda+\rho$ by direct checking, so we have (b). The part (c) follows directly from (b) and Proposition 5.2. ☐

According to Definition 4.6, we split the singular orbit into the even and the odd part, parametrized respectively by the following subsets of the singular Hasse diagram:

$$W_{\epsilon}^{\mathfrak{p},\Sigma} = \left\{\overline{d_1 \dots d_{n-1}\,0} \text{ with the number of digits 1 of parity } \epsilon\right\}, \quad \epsilon \in \{\text{even}, \text{odd}\}.$$

Recall again that the number of the digits 1 in $w \in W^{\mathfrak{p},\Sigma}$ is equal to the number of the negative coordinates in $w\lambda$. Both $W_{\epsilon}^{\mathfrak{p},\Sigma}$ are in a bijection with the regular Hasse diagram of rank $n-2$, given by the same formula:

$$\overline{d_1 \dots d_{n-2} d_{n-1}\,0} \mapsto \overline{d_1 \dots d_{n-2}}. \tag{5.24}$$

Again, the idea is to add enough arrows on the left-hand side to make (5.24) a directed-graph isomorphism. By inspection, the missing arrows should occur in these situations: $\overline{\dots 000} \to \overline{\dots 110}$. Before we proceed to construct them, we need a crucial fact about the singular orbit of the second kind (already stated in 1.43). This is the only thing we formally use from Enright-Shelton's theory:

Lemma 5.18 (Enright-Shelton)**.** *There are no non-trivial morphisms between subquotients of objects from the blocks with different parities.*

Proof. See [ES87, p. 63]. ☐

Theorem 5.19. *There are non-standard invariant differential operators*

$$\mathcal{O}_{\mathfrak{p}}(\nu) \to \mathcal{O}_{\mathfrak{p}}(\nu''') \tag{5.25}$$

for all the pairs ν, ν''' in the singular orbit given by

$$\nu = \overline{d_1 \dots d_{n-3}\,00\mathbf{0}} \cdot \lambda, \quad \nu''' = \overline{d_1 \dots d_{n-3}\,11\mathbf{0}} \cdot \lambda. \tag{5.26}$$

If λ is minimal as in (5.2), the operator (5.25) is of the order 3.

Proof. Take X' to be an open ball inside the big affine cell in G/P, and consider the Penrose transform over the restricted double fibration (5.5). In the relative BGG resolution, we find and fix the following sequence:

$$\overline{0\,d_1\ldots d_{n-3}\,00}\to\overline{0\,d_1\ldots d_{n-3}\,01}\to\overline{0\,d_1\ldots d_{n-3}\,10}\to\overline{0\,d_1\ldots d_{n-3}\,11}.$$

Denote the weights corresponding the the above sequence μ,μ',μ'' and μ''', respectively. Denote $q=2+$the number of the digits 0 in $\overline{d_1\ldots d_{n-3}}$. Denote also $\nu'=\overline{d_1\ldots d_{n-3}\,010}$ and $\nu''=\overline{d_1\ldots d_{n-3}\,100}$ in $W^{\mathfrak{p},\Sigma}$. Consider the (part of the) Čech bi-complex that calculates the higher direct images, described in Figure 5.11. Again, the horizontal

$$\begin{tikzcd}
\check{C}^{q+1}_{\mu} \arrow[r] & \check{C}^{q+1}_{\mu'} \arrow[r] & \check{C}^{q+1}_{\mu''} \arrow[r] & \check{C}^{q+1}_{\mu'''} \\
\check{\mathbf{C}}^{\mathbf{q}}_{\mu} \arrow[r] \arrow[u] & \check{C}^{q}_{\mu'} \arrow[r] \arrow[u] & \check{C}^{q}_{\mu''} \arrow[r] \arrow[u] & \check{C}^{q}_{\mu'''} \arrow[u] \\
\check{C}^{q-1}_{\mu} \arrow[r] \arrow[u] & \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu'} \arrow[r] \arrow[u] & \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''} \arrow[r] \arrow[u] & \check{C}^{q-1}_{\mu'''} \arrow[u] \\
\check{C}^{q-2}_{\mu} \arrow[r] \arrow[u] & \check{C}^{q-2}_{\mu'} \arrow[r] \arrow[u] & \check{C}^{q-2}_{\mu''} \arrow[r] \arrow[u] & \check{\mathbf{C}}^{\mathbf{q-2}}_{\mu'''} \arrow[u] \\
\mathcal{O}_{\mathfrak{q}}(\mu) \arrow[r] \arrow[u, dotted, no head] & \mathcal{O}_{\mathfrak{q}}(\mu') \arrow[r] \arrow[u, dotted, no head] & \mathcal{O}_{\mathfrak{q}}(\mu'') \arrow[r] \arrow[u, dotted, no head] & \mathcal{O}_{\mathfrak{q}}(\mu''') \arrow[u, dotted, no head]
\end{tikzcd}$$

Figure 5.11: Čech bi-complex (2)

differentials will be denoted by d_h, and the vertical ones by d_v. By Proposition 5.17 (b), we have the following:

$$\begin{aligned}
H^q(\check{C}^{\bullet}_{\mu},d_v)&=\tau^q_*\mathcal{O}_{\mathfrak{q}}(\mu)=\mathcal{O}_{\mathfrak{p}}(\nu),\\
H^{q-1}(\check{C}^{\bullet}_{\mu'},d_v)&=\tau^{q-1}_*\mathcal{O}_{\mathfrak{q}}(\mu')=\mathcal{O}_{\mathfrak{p}}(\nu'),\\
H^{q-1}(\check{C}^{\bullet}_{\mu''},d_v)&=\tau^{q-1}_*\mathcal{O}_{\mathfrak{q}}(\mu'')=\mathcal{O}_{\mathfrak{p}}(\nu''),\\
H^{q-2}(\check{C}^{\bullet}_{\mu'''},d_v)&=\tau^{q-2}_*\mathcal{O}_{\mathfrak{q}}(\mu''')=\mathcal{O}_{\mathfrak{p}}(\nu'''),
\end{aligned}$$

and all other vertical cohomologies are trivial. The cochain spaces with non-trivial vertical cohomology are denoted with the bold font. Note that we also have a standard operator $\mathcal{O}_{\mathfrak{q}}(\mu')\to\mathcal{O}_{\mathfrak{q}}(\mu'')$ that survives on the $(q-1)$-th cohomology,

$$\begin{gathered}
d\colon H^{q-1}(\check{C}^{\bullet}_{\mu'},d_v)\to H^{q-1}(\check{C}^{\bullet}_{\mu''},d_v),\\
d([y])=[d_h(y)].
\end{gathered}\tag{5.27}$$

We will define the operator (5.25) on the representatives of the cohomology classes in $H^q(\check{C}^{\bullet}_{\mu},d_v)$. Take a representative $x\in\check{\mathbf{C}}^{\mathbf{q}}_{\mu}$ of a class. Since x is a cocycle, $d_h(x)\in\check{C}^q_{\mu'}$

is also a cocycle. Since also $H^q(\check{C}^\bullet_{\mu'}, d_v) = 0$, it follows that there is $y \in \check{C}^{q-1}_{\mu'}$ such that $d_v(y) = d_h(x)$. Then, $d_h(y) \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''}$ is also a cocycle. See again (5.19). But since $H^{q-1}(\check{C}^\bullet_{\mu''}, d_v) \neq 0$, we cannot immediately conclude that $d_h(y) \in \operatorname{Im} d_v$ and proceed in the same way. However, the map

$$[x] \mapsto [d_h(y)] + \operatorname{Im} d, \tag{5.28}$$

$$H^q(\check{C}^\bullet_{\mu}, d_v) \to H^{q-1}(\check{C}^\bullet_{\mu''}, d_v)\Big/\operatorname{Im} d$$

is well-defined. To see this, take $x' = x + d_v(t)$ and choose y' so that $d_h(x') = d_v(y')$. The line (5.20) shows that $[y - y' + d_h(t)] \in H^{q-1}(\check{C}^\bullet_{\mu'}, d_v)$. Moreover, observe that we have $d([y - y' + d_h(t)]) = [d_h(y)] - [d_h(y')] \in \operatorname{Im} d$, which proves our claim.

Obviously, ν and ν'' are of different parity, so Lemma 5.18 implies that the map (5.28) is trivial. Unwinding, this means that we can find a cocycle $y'' \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu'}$ so that $d_h(y) - d_h(y'') \in \operatorname{Im} d_v$. Consequently, we can replace y by $y - y''$ and continue our diagram chase downwards, since now:

$$\begin{aligned} &d_v(y - y'') = d_h(x), \\ &d_h(y - y'') = d_v(z), \text{ for some } z \in \check{C}^{q-2}_{\mu''}. \end{aligned}$$

It is easy to see now that $d_h(z) \in \check{\mathbf{C}}^{\mathbf{q-2}}_{\mu'''}$ is a cocycle. Diagrammatically,

$$\begin{tikzcd}
0 & & & \\
x \in \check{\mathbf{C}}^{\mathbf{q}}_{\mu} \arrow[u, mapsto] \arrow[r, mapsto] & d_h(x) & & \\
& y - y'' \arrow[u, mapsto] \arrow[r, mapsto] & d_h(y - y'') \in \operatorname{Im} d_v & 0 \\
& & z \arrow[u, mapsto] \arrow[r, mapsto] & d_h(z) \in \check{\mathbf{C}}^{\mathbf{q-2}}_{\mu'''} \arrow[u, mapsto]
\end{tikzcd}$$

We would like to define $D([x]) = [d_h(z)]$. It still needs to be checked that D does not depend on the various choices we made. For this, we introduce another auxiliary map

$$\tilde{d}\colon \operatorname{Ker} d \to H^{q-2}(\check{C}^\bullet_{\mu'''}, d_v), \tag{5.29}$$

defined as follows. For $[y] \in \operatorname{Ker} d$, we can choose $z \in \check{C}^{q-2}_{\mu''}$ such that $d_v(z) = d_h(y)$. We put $\tilde{d}([y]) := [d_h(z)]$. It is not hard to check that $\tilde{d}$ is well defined (in the same way as for D in the proof of Theorem 5.13). Since ν' and ν''' are of different parity, Lemma 5.18 implies that the map $\tilde{d}$ is trivial.

Let us return to the proof that D is well-defined. Suppose we have two representatives $x, x' \in \check{\mathbf{C}}^{\mathbf{q}}_{\mu}$ of the same cohomology class. Then $x - x' = d_v(t)$ for some $t \in \check{C}^{q-1}_{\mu}$. Consider different choices for defining D of this class:

$$\begin{tikzcd}
x \arrow[r, mapsto] & d_h(x) & & & x' \arrow[r, mapsto] & d_h(x') & & \\
& y \arrow[u, mapsto] \arrow[r, mapsto] & d_h(y) \in \operatorname{Im} d_v & & & y' \arrow[u, mapsto] \arrow[r, mapsto] & d_h(y') \in \operatorname{Im} d_v & \\
& & z \arrow[u, mapsto] \arrow[r, mapsto] & d_h(z), & & & z' \arrow[u, mapsto] \arrow[r, mapsto] & d_h(z').
\end{tikzcd}$$

We need to show that $d_h(z)$ and $d_h(z')$ define the same cohomology class. The line (5.20) again shows that $[y - y' + d_h(t)] \in H^{q-1}(\check{C}^{\bullet}_{\mu'}, d_v)$, and moreover,

$$d([y - y' + d_h(t)]) = [d_h(y) - d_h(y')] = [d_v(z - z')] = 0 \in H^{q-1}(\check{C}^{\bullet}_{\mu''}, d_v).$$

So, $[y - y' + d_h(t)] \in \operatorname{Ker} d$, and therefore

$$0 = \tilde{d}([y - y' + d_h(t)]) = [d_h(z - z')] = [d_h(z)] - [d_h(z')] \;\Rightarrow\; [d_h(z)] = [d_h(z')].$$

We have a well-defined map $D\colon \mathcal{O}_{\mathfrak{p}}(\nu) \to \mathcal{O}_{\mathfrak{p}}(\nu''')$, which is by construction local, and G-invariant. By Remark 1.21, it is a differential operator. Its order is given by the difference of the generalized conformal weights, which is easily seen to be 3 in the minimal case. $\square$

Definition 5.20. *In the singularity of the second kind, the even (resp. the odd) part of the singular orbit, with all the non-standard operators constructed in Theorem 5.19 included in it, is called the* ***even*** *(resp. the* ***odd****)* ***singular BGG complex*** *of the infinitesimal character $\lambda + \rho$.*

The next results are analogous to those in the first kind.

Theorem 5.21. *For the singularity of the second kind:*

(a) The singular orbit of rank n consists of two singular BGG complexes, each of which is directed-graph isomorphic to the regular one of rank $n - 2$.

(b) Every square in the singular BGG complex anticommutes.

(c) If we add up all objects of the singular BGG complex of the same degree, we get a cochain complex.

Proof. Since (5.24) are directed-graph isomorphisms, (a) is trivial, and from (b) we automatically have (c). To prove (b), situations to consider in the relative BGG resolution are the following:

$$\begin{tikzcd}
\overline{0\ldots01\ldots00} \arrow[r] \arrow[d] & \overline{0\ldots01\ldots01} \arrow[r] \arrow[d] & \overline{0\ldots01\ldots10} \arrow[r] \arrow[d] & \overline{0\ldots01\ldots11} \arrow[d] \\
\overline{0\ldots10\ldots00} \arrow[r] & \overline{0\ldots10\ldots01} \arrow[r] & \overline{0\ldots10\ldots10} \arrow[r] & \overline{0\ldots10\ldots11}.
\end{tikzcd}$$

Denote by μ, μ', μ'', μ''' the objects in the first row, and by θ, θ', θ'', θ''' the objects in the second row, and consider the part of the Čech bi-complex above it, with the same notation as in the proof of Theorem 5.19. Denote by d all the standard operators $\mu \to \theta$, and the primed versions, and recall that d anticommutes with both d_h and d_v. Let us draw a diagram relating the elements from the definition of D:

$$\begin{tikzcd}
 & d(x) \in \check{\mathbf{C}}^{\mathbf{q}}_{\theta} \arrow[r, mapsto] & d_h(d(x)) & & \\
x \in \check{\mathbf{C}}^{\mathbf{q}}_{\mu} \arrow[ur, mapsto] \arrow[r, mapsto] & d_h(x) & y' \arrow[u, mapsto] \arrow[r, mapsto] & d_h(y') \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\theta''} & \\
 & y \arrow[u, mapsto] \arrow[r, mapsto] & d_h(y) \in \check{\mathbf{C}}^{\mathbf{q-1}}_{\mu''} & z' \arrow[u, mapsto] \arrow[r, mapsto] & d_h(z') \in \check{\mathbf{C}}^{\mathbf{q-2}}_{\theta'''} \\
 & & z \arrow[u, mapsto] \arrow[r, mapsto] & d_h(z) \in \check{\mathbf{C}}^{\mathbf{q-2}}_{\mu'''}. \arrow[ur, mapsto, dashed, "-d"'] &
\end{tikzcd} \tag{5.30}$$

We need to show that $[d(d_h(z))] = -[d_h(z')]$. This can be done by the same method as in the proof of Theorem 5.15 (b). There, we saw that we can take $y' = d(y)$, and then $d_h(y') = -d(d_h(y))$. But then we see that $d_v(d(z)) = -d(d_v(z)) = -d(d_h(y)) = d_h(y')$, so $z' = d(z)$ is a good candidate. Finally, $d_h(z') = d_h(d(z)) = -d(d_h(z))$, which proves the anticommuting statement. □

5.5 Exactness of the singular BGG complex

One of the classical techniques for calculating sheaf cohomologies is to find a "good" open cover, and calculate the Čech cohomology over this cover. A connection to the sheaf cohomology is provided by the following:

Theorem 5.22 (Leray)**.** *Let T be a topological space, $\mathcal{U} = \{U_\alpha \colon \alpha \in I\}$ its open cover, and $\mathcal{F}$ a sheaf on T. If for any finite subfamily $\{U_{i_0}, \ldots, U_{i_t}\} \subseteq \mathcal{U}$ we have*

$$H^k(U_{i_0} \cap \ldots \cap U_{i_t}, \mathcal{F}) = 0, \quad \textit{for all } k > 0, \tag{5.31}$$

then $H^k(T,\mathcal{F}) \cong \check{H}^k(\mathcal{U},\mathcal{F})$ *for all* $k \geq 0$.

Proof. See [GR12, B.3] or [Tay02, 7.8]. □

Corollary 5.23. *In addition to (5.31), if the cardinality of* $\mathcal{U}$ *is a finite number* n*, then*

$$H^k(T,\mathcal{F}) = 0, \quad \text{for all } k \geq n.$$

Proof. This follows from Leray's theorem and the definition of the Čech cohomology as the cohomology of the complex of the alternating Čech cochains:

$$\check{C}^k(\mathcal{U},\mathcal{F}) = \prod_{i_0<\ldots<i_k} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_k}).$$

This is obviously trivial for $k \geq n$, and so is the cohomology. □

The following vanishing result is crucial for the final part of the Penrose transform we are performing, and so for the proof of the exactness of the singular BGG complexes.

Lemma 5.24. *Let* Z *be the twistor space of the restricted double fibration (5.4). For any coherent sheaf* $\mathcal{F}$ *on* Z *we have:*

$$H^k(Z,\mathcal{F}) = 0, \quad \text{for all } k \geq n. \tag{5.32}$$

Proof. From Proposition 5.3 we see that

$$Z = \left\{ \begin{pmatrix} 1 \\ y_2 \\ \vdots \\ y_n \\ z_1 \\ \vdots \\ z_n \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} y_1 \\ 1 \\ \vdots \\ y_n \\ z_1 \\ \vdots \\ z_n \end{pmatrix} \right\} \cup \ldots \cup \left\{ \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ 1 \\ z_1 \\ \vdots \\ z_n \end{pmatrix} \right\},$$

a union of n open subsets in Z, each of which is isomorphic to $\mathbb{C}^{2n-1}$. So these are affine subsets, and therefore Stein. Any intersection of Stein subsets is again Stein. Cartan's theorem B (Theorem 2.3) gives us the condition (5.31), and the required vanishing now follows from Corollary 5.23. □

The main result is the following theorem:

Theorem 5.25. *In the singularity of either the first or the second kind, each singular BGG complex is exact (in the positive degrees) over the big affine cell* X*.*

Proof. We use the spectral sequence (2.8):

$$E_1^{pq} = \Gamma(X, \tau_*^q \Delta_\eta^p(\lambda)) \implies H^{p+q}(Z, \mathcal{O}_{\mathfrak{r}}(\lambda)).$$

This spectral sequence has on the abutment E_∞ the sections over X of the cohomologies of our singular BGG complex. This follows from the construction: non-standard operators were defined exactly as the induced differentials in the hypercohomology spectral sequence, so they appear on the last page of the spectral sequence before it stabilizes. Moreover, by Cartan's theorem B, the functor $\Gamma(X, -)$ is exact, so it commutes with taking the cohomologies of a cochain complex of sheaves.

Observing Proposition 5.12 and Proposition 5.17, we see that the first non-trivial elements on the first page of the spectral sequence (with the smallest $p+q$) are the following. In the first kind:

$$E_1^{pq} = \begin{array}{|cccc} 0 & \dots & 0 & \overline{0\dots010\dots0}, \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ \hline \end{array}$$

and in the second kind:

$$E_1^{pq} = \begin{array}{|ccc} \overline{0\dots00} & & \\ 0 & \overline{0\dots01} & \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0. \\ \hline \end{array}$$

Using the formula (4.5), it is easy to see that each of those objects has

$$p + q = n - 1.$$

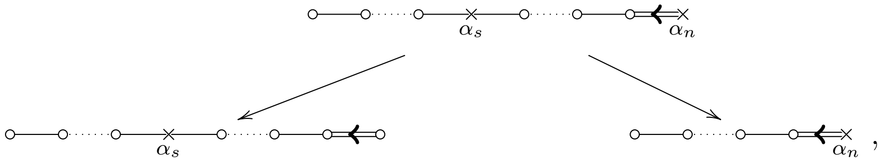
Note that there can be no other objects on this skew-diagonal, because the differentials are of the bi-degree $(r, -r+1)$. The conclusion is that $H^i(Z, \mathcal{O}_{\mathfrak{r}}(\tilde{\lambda}))$ measure the non-exactness of the singular BGG complex, up to the shift in the degree by $n-1$. Because of Lemma 5.24, the singular BGG complex is exact from the degree $n - (n-1) = 1$ above. $\square$

5.5.1 Conjectures

The following result would imply the local exactness of the singular BGG complex, that is, the exactness of the complex in the category of sheaves.

Conjecture 5.26 The conclusion of Lemma 5.24 is true for Z' in (5.5), where X' is a suitably chosen, but arbitrarily small open subset. ■

If λ is of a higher singularity, say $|\Sigma| = s > 1$, the reasonable thing to try would be the Penrose transform over:



(5.33)

and start with the following $\tilde{\lambda} + \rho$:

$$[\text{one of each pair of the repeated coordinates in } \lambda + \rho \,|\, \text{the remaing regular part}].$$

If the long root is singular, one would put also 0 left of the bar. In this setting, the appropriate vanishing result (the analogue of Lemma 5.24) has not yet been achieved. We state it as a conjecture.

Conjecture 5.27 Let $X \subseteq \mathrm{iGr}(n, 2n)$ be the big affine cell, or a ball, or a polydisc inside it, and let Z be the corresponding twistor space via the double fibration (5.33). Then, for any coherent sheaf $\mathcal{F}$ on Z we have

$$H^k(Z, \mathcal{F}) = 0, \quad \text{for all } k > s(n-s) - \frac{s(s-1)}{2}.$$

■

5.6 Example of a non-standard operator

We will study the singularity of the second kind in the rank 3, $\lambda + \rho = [2, 1, 0]$, in more detail. See Example 5.7. There are two singular BGG complexes in this infinitesimal character:

$$[2, 0, 1\,|\,] \xrightarrow{d_1} [1, 0, -2\,|\,] \to 0,$$
$$[2, 1, 0\,|\,] \xrightarrow{d_3} [0, -1, -2\,|\,] \to 0.$$

We are interested in identifying the operators d_1 and d_3 in these complexes. By Theorem 5.25, both of them are surjective over the big affine cell. In particular, they are non-zero. Consider first the standard one, d_1. Let us subtract ρ to make the notation more consistent, and also present the weights in the Dynkin diagram notation. We have:

$$
\begin{tikzcd}
\mathcal{O}_{\mathfrak{p}}([-1,-2,-2\,|\,]) \arrow[r, "d_1"] \arrow[d, equal] & \mathcal{O}_{\mathfrak{p}}([-2,-2,-3\,|\,]) \arrow[d, equal] \\
\mathcal{O}_{\mathfrak{p}}\big(\overset{1}{\circ}\!-\!\!-\!\overset{0}{\circ}\!\Leftarrow\!\overset{-2}{\times}\big) & \mathcal{O}_{\mathfrak{p}}\big(\overset{0}{\circ}\!-\!\!-\!\overset{1}{\circ}\!\Leftarrow\!\overset{-3}{\times}\big).
\end{tikzcd}
$$

By considering the generalized conformal weights (see Proposition 1.46), we see that the order of d_1 is 1. We switch to the generalized Verma modules. The differential operator d_1 corresponds to a homomorphism:

$$
f_1 \;:\; M_{\mathfrak{p}}([-2,-2,-3\,|\,]) \longrightarrow M_{\mathfrak{p}}([-1,-2,-2\,|\,]),
$$

The generalized Verma module $M_{\mathfrak{p}}([-1,-2,-2\,|\,])$ is induced from the standard three-dimensional representation F for $\mathfrak{l}_{\mathrm{ss}} = \mathfrak{sl}(3,\mathbb{C})$, on which the diagonal matrices h_1, h_2 and h_3 from $\mathfrak{l} = \mathfrak{gl}(3,\mathbb{C})$ act as the scalars -1, -2 and -2 respectively. Similarly, $M_{\mathfrak{p}}([-2,-2,-3\,|\,])$ is induced from $\bigwedge^2 F$, on which h_1, h_2, h_3 act as the scalars -2, -2 and -3, respectively. See Remark 3.6. Using the symbol principle, we can find the maximal vector that determines the homomorphism f_1:

$$
Y_{c_{13}} \otimes e_1 + Y_{c_{23}} \otimes e_2 + 2Y_{b_3} \otimes e_3 \in M_{\mathfrak{p}}([-1,-2,-2\,|\,]),
$$

where $Y_\gamma \in \mathfrak{u}^-$ is the standard negative root vector for a root γ, and $\{e_1, e_2, e_3\}$ is the standard basis for F.

The operator d_3 is non-standard, and it is easily seen to be of the order 3. It appears on the third page of the spectral sequence (5.11):

$$
E_3^{pq} = \begin{tikzcd}
{[2,1,0\,|\,]} \arrow[ddrrr, bend left=10, "d_3"] & 0 & 0 & 0 \\
0 & \operatorname{Ker} d_1 & 0 & 0 \\
0 & 0 & 0 & {[0,-1,-2\,|\,].}
\end{tikzcd}
$$

After subtracting ρ, we have:

$$
\begin{tikzcd}
\mathcal{O}_{\mathfrak{p}}([-1,-1,-1\,|\,]) \arrow[r, "d_3"] \arrow[d, equal] & \mathcal{O}_{\mathfrak{p}}([-3,-3,-3\,|\,]) \arrow[d, equal] \\
\mathcal{O}_{\mathfrak{p}}\big(\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ}\!\Leftarrow\!\overset{-1}{\times}\big) & \mathcal{O}_{\mathfrak{p}}\big(\overset{0}{\circ}\!-\!\!-\!\overset{0}{\circ}\!\Leftarrow\!\overset{-3}{\times}\big).
\end{tikzcd}
$$

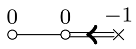
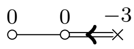
Note that these are actually line bundles. The operator d_3 corresponds to a homomorphism

$$f_3 \; : \; M_{\mathfrak{p}}([-3,-3,-3\,|\,]) \longrightarrow M_{\mathfrak{p}}([-1,-1,-1\,|\,]),$$

where both generalized Verma modules are induced from a one dimensional module $\mathbb{C}$ trivial for $\mathfrak{sl}(3,\mathbb{C})$. The diagonal elements h_1, h_2 and h_3 all act as the scalar -3 in the domain, and as -1 in the codomain. We can find the maximal vector defining the homomorphism f_3:

$$\left(4Y_{b_1}Y_{b_2}Y_{b_3} - Y_{b_1}Y_{c_{23}}^2 - Y_{b_2}Y_{c_{13}}^2 - Y_{b_3}Y_{c_{12}}^2 + Y_{c_{12}}Y_{c_{13}}Y_{c_{23}}\right) \otimes 1 \in M_{\mathfrak{p}}([-1,-1,-1\,|\,]).$$

We can use the duality again, to find the formula for d_3 in the local coordinates on the big affine cell given by $\mathfrak{u}^-$:

$$4\partial_{b_1}\partial_{b_2}\partial_{b_3} - \partial_{b_1}\partial_{c_{23}}^2 - \partial_{b_2}\partial_{c_{13}}^2 - \partial_{b_3}\partial_{c_{12}}^2 + \partial_{c_{12}}\partial_{c_{13}}\partial_{c_{23}} : \mathcal{O}_{\mathfrak{p}}($$  $$) \to \mathcal{O}_{\mathfrak{p}}($$  $$).$$

Conclusion

This dissertation provides the construction of the singular BGG complexes over the Hermitian symmetric space of the type (C_n, A_{n-1}), in a semi-regular infinitesimal character, and proves the exactness of the constructed complexes over the big affine cell. Some work is still left to do in this type:

- Completely avoid the Enright-Shelton's theory in the singularity of the second kind (Lemma 5.18).
- Allow more than one singular root (Conjecture 5.27).
- Get the local exactness of the singular BGG complex (Conjecture 5.26).
- Try higher-graded parabolic subalgebras. Reasonable results are expected for maximal parabolic subalgebras.

For the other (Hermitian) types, nothing is yet known about the singular BGG complexes in (D_n, A_{n-1}), nor the exceptional (E_6, D_5) and (E_7, E_6) types.

Also, an interesting task would be to see how the BGG resolutions look like for disconnected real Lie groups.

In the long term, we would like to extend the Penrose transform to D-modules, and study how it fits into the theory of the Beilinson-Bernstein localization. There, the appropriate generalization of the BGG complex would be the Cousin complex, [Mil].

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List of Figures

Biography

Rafael Mrđen was born on December 31, 1987, in Zadar. He finished the primary school in Biograd na Moru, and a scientific-mathematical gymnasium in Zadar. In 2006 he enrolled in the Undergraduate University Programme *Mathematics*, and in 2009 in the Graduate University Programme *Theoretical Mathematics*, at the Department of Mathematics, Faculty of Science, University of Zagreb. He graduated in July 2011, and earned his Master of Science in Mathematics degree. The subject of his Master's thesis was *Sheaf Cohomology*, supervised by prof.dr.sc. Pavle Pandžić and doc.dr.sc. Zoran Škoda. As a graduate student, he published:

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Rafael Mrđen enrolled in the Croatian Doctoral Program in *Mathematics* at the Department of Mathematics, Faculty of Science, University of Zagreb, in October 2011. There he passed all his exams with the highest grades, two of which were qualifying (*Algebra* and *Geometry and Topology*), and four advanced courses (*Cohomological Induction*, *Spaces with Structure Sheaf*, *Vogan's Theory of K-Types*, and *Representations of Real Reductive Groups*). He gave numerous talks at the *Representation Theory Seminar*.

After working as a teaching assistant in the winter semester of 2011/2012, at the Faculty of Science and the Faculty of Electrical Engineering and Computing, University of Zagreb, he has been working as a teaching assistant at the Department of Mathematics, Faculty of Civil Engineering, University of Zagreb, since January 2012.

Rafael Mrđen participated in 12 international conferences/workshops, and 3 scientific visits. He is a coworker on the project funded by Croatian National Foundation, no. 4176: *Dirac Operators and Representation Theory*, lead by prof.dr.sc. Pavle Pandžić. He is also a member of the *Center of Excellence for the Theory of Quantum and Complex Systems and Lie Algebra Representations (QuantiXLie)*, lead by prof.dr.sc. Hrvoje Buljan and prof.dr.sc. Pavle Pandžić.

Životopis

Rafael Mrđen rođen je 31. prosinca 1987. godine u Zadru. Osnovnu školu završio je u Biogradu na Moru, a gimnaziju, prirodoslovno-matematički smjer, u Zadru. Godine 2006. upisao je preddiplomski sveučilišni studij *Matematika*, a 2009. diplomski sveučilišni studij *Teorijska matematika*, na Matematičkom odsjeku Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu. Diplomirao je u srpnju 2011. godine i stekao titulu Magistar matematike. Tema njegovog diplomskog rada je bila *Kohomologija snopova*, pod vodstvom prof.dr.sc. Pavla Pandžića i doc.dr.sc. Zorana Škode. Kao diplomski student, objavio je stručni članak:

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Nakon rada kao naslovni asistent u zimskom semestru akademske godine 2011/2012. na Prirodoslovno-matematičkom fakultetu i na Fakultetu elektrotehnike i računarstva Sveučilišta u Zagrebu, od siječnja 2012. godine radi kao asistent na Zavodu za matematiku Građevinskog fakulteta Sveučilišta u Zagrebu.

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Izjava o izvornosti rada

Ja, Rafael Mrđen, student Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu, s prebivalištem na adresi [redacted], Zagreb, ovim putem izjavljujem pod materijalnom i kaznenom odgovornošću da je moj doktorski rad pod naslovom

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(Singularni BGG kompleksi za simplektički slučaj)

isključivo moje autorsko djelo, koje je u potpunosti samostalno napisano, uz naznaku izvora drugih autora i dokumenata korištenih u radu.

U Zagrebu, 12. listopada 2017.

potpis