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FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

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DOCTORAL DISSERTATION

Zagreb, 2022.

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Sveučilište u Zagrebu

PRIRODOSLOVNO–MATEMATIČKI FAKULTET

MATEMATIČKI ODSJEK

Marina Dajaković

Elementi teorije obnavljanja za točkovne procese s klasterima

DOKTORSKI RAD

Mentor:

prof.dr.sc. Bojan Basrak

Zagreb, 2022.

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Summary

The main goal of this thesis is to discuss renewal theorems in the framework of the cluster point processes on the real line. More precisely, under appropriate integrability assumptions, we describe the asymptotic distribution of the shifted renewal cluster point process and, as a consequence, we show that its mean measure on a bounded interval is asymptotically proportional to the length of the interval. Also, we prove the equivalent version of the latter result that allows one to determine the asymptotic behavior of the functions of special convolution form. These generalizations of the well-known results from the standard renewal theory are obtained using the methods and tools of modern probability theory, such as point process theory, especially vague convergence, and coupling method.

Keywords: cluster point processes, Blackwell's renewal theorem, the key renewal theorem, the extended renewal theorem

Sažetak

Glavni cilj ove disertacije je diskutirati teoreme obnavljanja u kontekstu točkovnih procesa s klasterima na skupu realnih brojeva. Preciznije, uz određene pretpostavke na prve momente, opisujemo asimptotsku distribuciju pomaknutog točkovnog procesa s klasterima te, kao posljedicu, pokazujemo da je srednja mjera takvog procesa na ograničenom intervalu proporcionalna duljini intervala. Također, dokazujemo ekvivalentnu verziju prethodnog rezultata koja omogućuje određivanje asimptotskog ponašanja konvolucije funkcija specijalnog oblika. Navedene generalizacije dobro poznatih rezultata iz standardne teorije obnavljanja su dobivene korištenjem metoda i alata moderne teorije vjerojatnosti, kao što su teorija točkovnih procesa, posebno *vague* konvergencija, te metoda sparivanja.

Ključne riječi: točkovni procesi s klasterima, Blackwellov teorem obnavljanja, ključni teorem obnavljanja, prošireni teorem obnavljanja

Contents

Chapter 1

Introduction

1.1 Motivation

Renewal theory has a long history and many applications in different areas of mathematics and engineering. In its simplest form, it studies the so-called renewal process, i.e. a process of renewal times T_n, which satisfy $T_n = X_0 + X_1 + \cdots + X_n$ for some sequence of i.i.d. nonnegative random variables $(X_n)_{n\geq 1}$ independent of the initial nonnegative random variable X_0. Blackwell's renewal theorem shows that the expected number of renewals in an interval of a bounded length is asymptotically proportional to that length, provided that the distribution of X_n's is nonarithmetic and has a finite mean. Several related results soon appeared in the literature, for instance concerning the limiting distribution of the overshoot over a given level t and also concerning the asymptotic behaviour of the solutions of the so-called renewal equation. The latter result is known as the key renewal theorem. Although their statements are intuitively simple, initial proofs of various renewal theorems were rather analytical until the arrival of the so-called coupling method to probability theory, as described in the books by Lindvall [Lin02] and Thorisson [Tho00].

First generalizations of the renewal theorems to random walks appeared soon after the original results, see e.g. [Bla53]. Since then, renewal theorems have been extended to a number of other stochastic processes, for instance to Markov renewal processes, see [Ç69], to nonlinear renewal sequences, see [LS77] and [LS79], and recently to iterated random walks, see [IRS22] and [BIMR22].

It appears to be quite natural to discuss renewal theory from the point process viewpoint. In [Kal17], a so-called extended renewal theorem is presented, which describes the asymptotic distribution of renewal times after time t, using the language of point process theory. Furthermore, in the last couple of years Iksanov and coauthors derived renewal theorems for random processes with immigration, see [Iks16b] and [IMM17]. Continuing their research, Marynych, see [Mar15], extended these results by relaxing the independence assumptions.

In some applications, it makes sense to assume that the renewal times are accompanied by a cluster of points. For instance, Mikosch in his non-life insurance mathematics book [Mik09] devoted a whole chapter to cluster point processes. Furthermore, Daley and Vere-Jones in [DVJ03] explain how Poisson cluster processes have been proved useful in a variety of fields when modelling events which cluster either in space or time. For applications in financial mathematics, see e.g. [BMM15], in seismology, see e.g. [CS18], in neurology, see e.g. [WDWL20], or in teletraffic engineering, see e.g. [FGAMS06].

Initial renewal results for some special cluster point processes appeared already in the 1960s. For instance, Lewis in [Lew64] studied the limiting distribution of the overshoot over a given level t, relying on analytical methods and special Poissonian character of a process. However, the corresponding renewal theory remains comparatively undeveloped in the setting of cluster point processes.

1.2 Chapter overview

This thesis is divided in four chapters. In Chapter 2 we introduce the basic theoretical concepts of point processes which will be used throughout the thesis, including stationarity and convergence in distribution. Since the latter depends on the choice of topology on the space of point measures $M_p(\mathbb{S})$, we describe the vague topology on this space. Furthermore, we present some important classes of point processes: Poisson process, the renewal point process and the marked point process, whose study will be continued in subsequent chapters.

Chapter 3 deals with the main results of renewal theory, including the stationary version of a given renewal process, equivalent versions of the renewal theorem, as well as its extended version. The above-mentioned subjects are a basis for a more sophisticated renewal theory for the renewal cluster point processes studied in the final chapter.

Chapter 4 is the central part of the thesis where we define renewal cluster point process on the real line and give, under certain first moment conditions, extension of the classical renewal theorems to this case. We apply this results to some special cluster point processes and conduct a short simulation study. Also, as an auxiliary result, we give an alternative proof of the extended renewal theorem for the marked point processes based on the coupling method.

Chapter 2

Point processes

This chapter is intended to be a brief but comprehensive introduction to the study of point processes on general Polish spaces. We give formal definitions and overview of the results from the existing literature necessary for understanding the material in the central part of this thesis. Furthermore, we introduce the notation which will be used in the subsequent chapters. This part of the thesis is influenced mostly by the books by Kallenberg [Kal17], Resnick [Res87] and Karr [Kar91], as well as the short note by Basrak and Planinić [BP19].

2.1 Boundedness

Let $\mathbb{S}$ be a Polish space, meaning that $\mathbb{S}$ is a topological space whose topology is induced by some metric d on $\mathbb{S}$ and $(\mathbb{S}, d)$ is a complete separable metric space. It is well-known that $\mathbb{R}$ with the usual topology induced by the Euclidean norm is Polish, as well as $\mathbb{N}$ with the discrete topology. Furthermore, the product of countably many Polish spaces is Polish. In particular, $\mathbb{R}^{\mathbb{N}}$ equipped with the product topology of the usual topologies is Polish. The σ-algebra generated by the topology of $\mathbb{S}$ is called the Borel σ-algebra of $\mathbb{S}$ and denoted by $\mathcal{B}(\mathbb{S})$.

Definition 2.1.1. *A (Borel) boundedness $\mathcal{B}_b(\mathbb{S})$ on $\mathbb{S}$ is a family of Borel subsets of $\mathbb{S}$, called bounded sets, satisfying the following conditions:*

(i) every subset of a bounded set is a bounded set,

(ii) a finite union of bounded sets is a bounded set.

For notational convenience, when there is no fear of confusion, we will omit the state space $\mathbb{S}$ and simply write $\mathcal{B}$ instead of $\mathcal{B}(\mathbb{S})$ and $\mathcal{B}_b$ instead of $\mathcal{B}_b(\mathbb{S})$.

A boundedness is said to properly localize $\mathbb{S}$ if there exists a sequence of open bounded sets $(U_n)_{n\geq 1}$ such that for every bounded set B there exists $n \in \mathbb{N}$ such that $B \subseteq U_n$, $\mathbb{S} \subseteq \bigcup_{n\in\mathbb{N}} U_n$ and $\overline{U}_n \subseteq U_{n+1}$, for all $n \in \mathbb{N}$, where $\overline{U}_n$ denotes the closure of a set U_n in $\mathbb{S}$. Sequence $(U_n)_n$ is called a proper localizing sequence.

Theorem 2.1.2. *Boundedness $\mathcal{B}_b$ properly localizes $\mathbb{S}$ if and only if there exists metric on $\mathbb{S}$ which generates the topology of $\mathbb{S}$ and under which the metrically bounded Borel subsets of $\mathbb{S}$ coincide with $\mathcal{B}_b$.*

For the proof of the above theorem we refer to Theorem 2.1 in [BP19]. We now give examples of boundednesses which will be used in the sequel, as well as the one which is important in the theory of regular variation.

Example 2.1.3.

(*i*) Let $\mathbb{S} = \mathbb{R}$ and $\mathcal{B}_b = \{B \in \mathcal{B} : \exists h > 0 \text{ such that } B \subseteq [-h, h]\}$. Observe that such $\mathcal{B}_b$ properly localizes $\mathbb{R}$ since one can take $U_n = \langle -n, n\rangle$, $n \in \mathbb{N}$, as a proper localizing sequence.

(*ii*) Denote by $\mathbb{M}$ a general Polish space. Let $\mathbb{S} = \mathbb{R} \times \mathbb{M}$ and $\mathcal{B}_b = \{B \in \mathcal{B} : \exists h > 0 \text{ such that } B \subseteq [-h, h] \times \mathbb{M}\}$. Observe that such $\mathcal{B}_b$ properly localizes $\mathbb{R} \times \mathbb{M}$ since one can take $U_n = \langle -n, n\rangle \times \mathbb{M}$, $n \in \mathbb{N}$, as a proper localizing sequence.

(*iii*) Let $\mathbb{S} = \mathbb{R}\backslash\{0\}$ and $\mathcal{B}_b = \{B \in \mathcal{B} : \exists \varepsilon > 0 \text{ such that } |x| > \varepsilon, \text{ for all } x \in B\}$. Observe that such $\mathcal{B}_b$ properly localizes $\mathbb{R}\backslash\{0\}$ since one can take $U_n = \langle -\infty, -\frac{1}{n}\rangle \cup \langle \frac{1}{n}, +\infty\rangle$, $n \in \mathbb{N}$, as a proper localizing sequence.

From now on, we will always assume that the space $\mathbb{S}$ is properly localized by a family of bounded sets $\mathcal{B}_b$.

2.2 Vague convergence

A Borel measure m on $\mathbb{S}$ is said to be locally finite if $m(B) < \infty$ for all $B \in \mathcal{B}_b$. The space of all such measures will be denoted by $M(\mathbb{S})$. Let $\mathcal{M}(\mathbb{S})$ denote the smallest σ-algebra on $M(\mathbb{S})$ making the mappings $m \mapsto m(B)$ measurable for all $B \in \mathcal{B}_b$, i.e.

$$\mathcal{M}(\mathbb{S}) = \sigma\{m \mapsto m(B) : B \in \mathcal{B}_b\}\,.$$

Definition 2.2.1. *For measures $m, m_1, m_2, \ldots \in M(\mathbb{S})$, a sequence $(m_n)_{n\geq 1}$ is said to converge vaguely to m, denoted by $m_n \xrightarrow{v} m$, if*

$$\int_{\mathbb{S}} f(x) m_n(\mathrm{d}x) \to \int_{\mathbb{S}} f(x) m(\mathrm{d}x)\,,$$

as $n \to \infty$, for all continuous and bounded real-valued functions f on $\mathbb{S}$ with bounded support.

Let $CB_b(\mathbb{S})$ denote the family of all functions as in Definition 2.2.1 and $CB_b^+(\mathbb{S})$ the subset of all nonnegative functions in $CB_b(\mathbb{S})$. In the sequel, we will occasionally use compact integral notation $m(f) = \int_{\mathbb{S}} f(x)m(\mathrm{d}x)$.

Remark 2.2.2. There are other approaches to vague convergence of locally finite measures on Polish spaces. The above presented concept of vague convergence puts emphasize on the notion of bounded set which is defined without reference to a metric. Just by choosing a different family of bounded sets in $\mathbb{S}$ one changes the space of locally finite measures and consequently the corresponding notion of vague convergence of locally finite measures. On the other hand, concept of vague convergence in [Kal17] is based on metrically bounded Borel sets with respect to a metric generating the topology of $\mathbb{S}$. However, the assumption that the boundedness $\mathcal{B}_b$ properly localizes $\mathbb{S}$ and Theorem 2.1.2 allow one to unite the two concepts. More on this subject, as well as the connection with other frequently used types of vague convergence from the literature can be found in [BP19], see Section 2.

The following result gives some useful characterizations of vague convergence in terms of sets. For the proof see Lemma 4.1 in [Kal17].

Theorem 2.2.3. *(Portmanteau theorem) Let $m, m_1, m_2, \ldots \in M(\mathbb{S})$. The following statements are equivalent:*

(i) $m_n \xrightarrow{v} m$,

(ii) $\lim_{n\to\infty} m_n(B) = m(B)$, *for all* $B \in \mathcal{B}_b$ *such that* $m(\partial B) = 0$,

(iii) $m(B^o) \leq \liminf_{n\to\infty} m_n(B) \leq \limsup_{n\to\infty} m_n(B) \leq m(\overline{B})$, *for all* $B \in \mathcal{B}_b$,

where B^o denotes the interior and ∂B the boundary of B.

On $M(\mathbb{S})$ we introduce the topology giving the notion of vague convergence in the following way. The family of all finite intersections of sets of the form

$$\{m \in M(\mathbb{S}) : a < m(f) < b\}\,,$$

for any $f \in CB_b^+(\mathbb{S})$ and $0 < a < b$ forms a base for a topology on $M(\mathbb{S})$. This topology is equivalent to the topology generated by the mappings $m \mapsto m(f)$, $f \in CB_b^+(\mathbb{S})$, i.e. the smallest topology making these mappings continuous. By Lemma 4.7 in [Kal17], σ-algebra generated by the vaguely open sets in $M(\mathbb{S})$ coincides with the above introduced σ-algebra $\mathcal{M}(\mathbb{S})$.

Recall that for every $x \in \mathbb{S}$, mapping $\delta_x : \mathcal{B} \to \{0, 1\}$ given by

$$\delta_x(B) = \begin{cases} 1\,, & x \in B\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

defines the Dirac measure concentrated at x.

Definition 2.2.4. *A point measure on $\mathbb{S}$ is a locally finite measure given by*

$$m = \sum_{i=1}^{l} \delta_{x_i}\,,$$

for some $0 \le l \le +\infty$ and not necessarily distinct points $x_1, x_2, \ldots, x_l \in \mathbb{S}$.

Note that in case of point measures

$$m(f) = \sum_{i=1}^{l} f(x_i)\,,$$

for all measurable functions $f : \mathbb{S} \to \mathbb{R}$. Furthermore, we say that m is simple if $m(\{x\}) \le 1$, for all $x \in \mathbb{S}$.

The space of all point measures on $\mathbb{S}$ will be denoted by $M_p(\mathbb{S})$. Note that one can equivalently define point measures as integer-valued measures in $M(\mathbb{S})$, see Theorem 2.18 in [Kal21]. Denote by $\mathcal{M}_p(\mathbb{S})$ the induced sub σ-algebra of $\mathcal{M}(\mathbb{S})$ on $M_p(\mathbb{S})$.

The following result gives a characterization of vague convergence of point measures in terms of points, see Proposition 2.8 in [BP19], cf. also Proposition 3.13 in [Res87].

Proposition 2.2.5. *Let $m, m_1, m_2, \ldots \in M_p(\mathbb{S})$. Then $m_n \xrightarrow{v} m$, as $n \to \infty$, if and only if for every $B \in \mathcal{B}_b$ such that $m(\partial B) = 0$ there exist $k, n_0 \in \mathbb{N}$ and a labeling of points of m and m_n, $n \ge n_0$, in B such that*

$$m_n\Big|_B = \sum_{i=1}^{k} \delta_{x_i^{(n)}} \quad \text{and} \quad m\Big|_B = \sum_{i=1}^{k} \delta_{x_i}$$

and

$$x_i^{(n)} \to x_i \text{ in } \mathbb{S}\,, \quad \text{for all} \quad i = 1, 2, \ldots, k\,,$$

as $n \to \infty$, where $m\Big|_B$ denotes the restriction of a measure m to the set B.

We now give a proof of a fundamental topological result, see also Proposition 9.1.IV. (*iii*) in [DVJ08].

Proposition 2.2.6. *The space $M_p(\mathbb{S})$ equipped with the vague topology is Polish.*

Proof. By Theorem 4.2 in [Kal17], see also Section 3 in [BP19], $M(\mathbb{S})$ is Polish. Adapting the proof of Proposition 3.14 in [Res87] on a locally compact state space with countable base to our setting, we show that $M_p(\mathbb{S})$ is vaguely closed in $M(\mathbb{S})$, hence Polish itself.

Let $m_1, m_2, \ldots \in M_p(\mathbb{S})$, $m \in M(\mathbb{S})$ and $m_n \xrightarrow{v} m$. Denote

$$\mathcal{B}_0 = \{B \in \mathcal{B}_b : m(\partial B) = 0\} .$$

Then, by Theorem 2.2.3, $m_n(B) \to m(B)$, for all $B \in \mathcal{B}_0$, as $n \to \infty$. Since $\mathbb{N}$ is closed in $\mathbb{R}_+$, $m(B)$ is a nonnegative integer, for all $B \in \mathcal{B}_0$.

Let $(U_n)_{n\geq 1}$ be a proper localizing sequence and set

$$B_n = \begin{cases} U_n \,, & m(\partial U_n) = 0 \,, \\ U_n^{\varepsilon} \,, & m(\partial U_n) > 0 \,, \end{cases}$$

for all $n \in \mathbb{N}$, where

$$U_n^{\varepsilon} = \{x \in \mathbb{S} : d(x, U_n) < \varepsilon\} ,$$

and d is a metric from Theorem 2.1.2. Note that by choosing the metric d from Theorem 2.1.2, one can find $\varepsilon_0 > 0$ such that $U_n^{\varepsilon_0} \in \mathcal{B}_b$ and $U_n^{\varepsilon_0} \subseteq U_{n+1}$. Observe that for all $0 < \varepsilon \leq \varepsilon_0$, $\partial U_n^{\varepsilon} \subseteq \{x \in \mathbb{S} : d(x, U_n) = \varepsilon\}$ and therefore $\{\partial U_n^{\varepsilon} : 0 < \varepsilon \leq \varepsilon_0\}$ is a disjoint family of sets. Then $U_n^{\varepsilon} \in \mathcal{B}_0$ for all but at most countably many $0 < \varepsilon \leq \varepsilon_0$, otherwise $m(U_n^{\varepsilon_0}) = \infty$. Hence, $(B_n)_{n\geq 1} \subseteq \mathcal{B}_0$ and $\mathbb{S} \subseteq \bigcup_{n\in\mathbb{N}} B_n$.

Fix $n \in \mathbb{N}$ and let

$$\mathcal{B}_n = \{B \in \mathcal{B} : m(B \cap B_n) \text{ is a nonnegative integer}\} .$$

Note that $B \in \mathcal{B}_0$ implies $B \cap B_n \in \mathcal{B}_0$, hence $\mathcal{B}_0 \subseteq \mathcal{B}_n$. One easily checks that $\mathcal{B}_0$ is a $\pi-$system and $\mathcal{B}_n$ is a $\lambda-$system. Hence Dynkin's $\pi - \lambda$ theorem, see e.g. Theorem A.1.4 in [Dur10], yields $\sigma(\mathcal{B}_0) \subseteq \mathcal{B}_n$. Furthermore, by [Kal17] Lemma 1.9 (v) and the discussion before it, we conclude that $\sigma(\mathcal{B}_0) = \mathcal{B}$. Hence, $\mathcal{B} = \mathcal{B}_n$.

It now immediately follows that, for every $B \in \mathcal{B}$,

$$m(B) = \lim_{n\to\infty} m(B \cap B_n) \in \{0, 1, \ldots, +\infty\} ,$$

which completes the proof. $\square$

The next result, which is a simple generalisation of Proposition 3.18 in [Res87], gives a sufficient condition for continuity of functionals on $M_p(\mathbb{S})$.

Lemma 2.2.7. *Let $\mathbb{S}'$ and $\mathbb{S}$ be Polish spaces. Suppose $f : \mathbb{S}' \to \mathbb{S}$ is a continuous function which satisfies*

$$f^{-1}(B) \in \mathcal{B}_b(\mathbb{S}'),\ \textit{for all } B \in \mathcal{B}_b(\mathbb{S})\,, \tag{2.1}$$

then $T_f : M_p(\mathbb{S}') \to M_p(\mathbb{S})$ defined by

$$T_f(m) = m \circ f^{-1}\,,$$

i.e.

$$T_f\left(\sum_{i=1}^{l} \delta_{x_i}\right) = \sum_{i=1}^{l} \delta_{f(x_i)}$$

is continuous.

Proof. Let $m, m_1, m_2, \ldots \in M_p(\mathbb{S}')$ and $m_n \xrightarrow{v} m$. For $g \in CB_b^+(\mathbb{S})$, we obtain

$$\begin{aligned}(T_f(m_n))(g) = m_n \circ f^{-1}(g) &= \int_{\mathbb{S}} g(x) m_n \circ f^{-1}(\mathrm{d}x) \\ &= \int_{\mathbb{S}'} (g \circ f)(x) m_n(\mathrm{d}x) = m_n(g \circ f)\,,\end{aligned}$$

where we have used the change of variable formula in the third equality. Obviously, $g \circ f$ is continuous, bounded and nonnegative. Denote the bounded support of g with $\operatorname{supp} g$. Since $x \in f^{-1}(\operatorname{supp} g)$ if $g(f(x)) \neq 0$ and $f^{-1}(\operatorname{supp} g)$ is bounded by assumption (2.1), we conclude that $g \circ f \in CB_b^+(\mathbb{S}')$. Therefore, $m_n \xrightarrow{v} m$ implies

$$(T_f(m_n))(g) = m_n(g \circ f) \to m(g \circ f) = (T_f(m))(g)\,,$$

as $n \to \infty$, which proves the claim. □

For a continuous function f, requirement (2.1) is automatically satisfied whenever $\mathbb{S}'$ is bounded.

2.3 Fundamentals of point processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random measure ξ on $\mathbb{S}$ is a measurable map from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(M(\mathbb{S}), \mathcal{M}(\mathbb{S}))$. Equivalently, we may define a random measure on $\mathbb{S}$ as a locally finite kernel from Ω to $\mathbb{S}$. Thus,

- for fixed $\omega \in \Omega$, the mapping $B \mapsto \xi(\omega, B)$ is a locally finite measure,
- for fixed $B \in \mathcal{B}$, the mapping $\omega \mapsto \xi(\omega, B)$ is a random variable.

Definition 2.3.1. *A point process ξ on $\mathbb{S}$ is an integer-valued random measure.*

Alternatively, a point process on $\mathbb{S}$ can be defined as a measurable mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(M_p(\mathbb{S}), \mathcal{M}_p(\mathbb{S}))$. Furthermore, by Theorem 2.18 in [Kal21], one can write

$$\xi = \sum_{i=1}^{L} \delta_{X_i}\,, \tag{2.2}$$

for some random elements $X_1, X_2, \ldots \in \mathbb{S}$ and a random variable L with values in $\{0, 1, \ldots, +\infty\}$. We will call $(X_i)_i$ the points of a point process ξ. However, we will use a slightly different notation and terminology on $\mathbb{R}$, since it is a custom to denote the points of a point process on $\mathbb{R}$ with a nondecreasing sequence of arrival times $(T_i)_i$. A point process ξ is said to be simple if

$$\mathbb{P}(\xi(\{x\}) \leq 1, \text{ for all } x \in \mathbb{S}) = 1\,.$$

Definition 2.3.2. *The mean measure of a point process ξ on $\mathbb{S}$ is the measure $\mathbb{E}\xi$ on $\mathbb{S}$ given by*

$$\mathbb{E}\xi(B) = \mathbb{E}[\xi(B)]\,, \quad \textit{for all } B \in \mathcal{B}\,.$$

Note that the mean measure of a point process ξ is not necessarily a locally finite measure, since the expected value of a random variable $\xi(B)$ can be infinite, even for $B \in \mathcal{B}_b$.

Let ξ be a point process on $\mathbb{R}^d$, $d \in \mathbb{N}$. If there exists a nonnegative measurable function $\lambda : \mathbb{R}^d \to \mathbb{R}$ such that

$$\mathbb{E}\xi(B) = \int_B \lambda(x)\mathrm{d}x\,, \quad \text{for all } B \in \mathcal{B}\,,$$

then the function λ is called the intensity function of ξ. If λ is a constant, then it is called the intensity of ξ. Note that in this case mean measure is a multiple of the Lebesgue measure, which will be denoted by *Leb* in the sequel.

Example 2.3.3 (**Poisson process**). Assume that Λ is a given locally finite measure on $\mathbb{S}$. The point process ξ on $\mathbb{S}$ is called a Poisson (point) process or, synonymously, Poisson random measure with mean measure Λ if it satisfies

(i) for all $k \in \mathbb{N}$ and for all $B \in \mathcal{B}_b$

$$\mathbb{P}(\xi(B) = k) = e^{-\Lambda(B)} \frac{\Lambda(B)^k}{k!}\,, \tag{2.3}$$

(ii) for all $k \in \mathbb{N}$ and for all disjoint sets $B_1, \dots, B_k \in \mathcal{B}_b$

$$\xi(B_1), \dots, \xi(B_k)$$

are independent random variables.

Note that the local finiteness of the mean measure Λ ensures that, for any $B \in \mathcal{B}_b$, $\xi(B) < \infty$ a.s., as required in the definition of a point process.

Example 2.3.4 (**Homogeneous Poisson process on** $\mathbb{R}_+$). Poisson process ξ on $\mathbb{R}_+ = [0, \infty\rangle$ with mean measure λLeb, for some $\lambda > 0$, is called the homogeneous Poisson (point) process on $\mathbb{R}_+$ with intensity λ. It can be equivalently defined, see for instance Proposition 4.2.1 in [Res92], as a point process

$$\xi = \sum_{i=1}^{\infty} \delta_{T_i}\,,$$

for $T_i, i \geq 1$ given by $T_i = \sum_{j=1}^{i} E_j$, where $E_j, j \geq 1$ are i.i.d. exponential random variables with rate λ.

Example 2.3.5 (**Renewal point process on** $\mathbb{R}_+$). Given a renewal sequence $(T_i)_{i \geq 0}$, the associated renewal point process ξ is defined by

$$\xi = \sum_{i=0}^{\infty} \delta_{T_i}\,.$$

By Lemma 3.1.5 it follows that $\xi(B) < \infty$ a.s., for all $B \in \mathcal{B}_b$, hence ξ is well-defined.

Definition 2.3.6. *The distribution of a point process ξ on $\mathbb{S}$ defined over $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability measure $\mathbb{P}_\xi$ on $M_p(\mathbb{S})$ given by*

$$\mathbb{P}_\xi(A) = \mathbb{P}(\xi \in A)\,, \quad \textit{for all } A \in \mathcal{M}_p(\mathbb{S})\,.$$

We say that point processes ξ_1 and ξ_2, not necessarily defined over the same probability space, are equal in distribution if $\mathbb{P}_{\xi_1} = \mathbb{P}_{\xi_2}$ and we denote it by $\xi_1 \overset{d}{=} \xi_2$.

Definition 2.3.7. *The Laplace functional of a point process ξ on $\mathbb{S}$ is the mapping $\mathcal{L}_\xi$ defined by*

$$\mathcal{L}_\xi(f) = \mathbb{E}e^{-\xi(f)}\,,$$

for all nonnegative measurable functions $f : \mathbb{S} \to \mathbb{R}$.

Laplace functional $\mathcal{L}_\xi$ uniquely determines the distribution $\mathbb{P}_\xi$ of a point process ξ, see Proposition 3.5 in [Res87]. However, as will be stated below, analogous result is valid even for some smaller families of functions.

Example 2.3.8. The Laplace functional of a Poisson process ξ on $\mathbb{S}$ with mean measure Λ is given by

$$\mathcal{L}_\xi(f) = \exp\left\{ -\int_{\mathbb{S}} (1 - e^{-f(x)})\Lambda(\mathrm{d}x) \right\},$$

for all nonnegative measurable functions $f : \mathbb{S} \to \mathbb{R}$.

Definition 2.3.9. *The avoidance (or void) probability function of a point process ξ on $\mathbb{S}$ is the function v_ξ defined by*

$$v_\xi(B) = \mathbb{P}(\xi(B) = 0), \quad \textit{for all } B \in \mathcal{B}.$$

Example 2.3.10. For a Poisson process ξ on $\mathbb{S}$ with mean measure Λ it follows directly from (2.3) that $v_\xi(B) = e^{-\Lambda}(B)$, for all $B \in \mathcal{B}_b$.

The next theorem gives some useful characterizations of the distribution of a point process. For the proof see Theorem 2.2 and Corollary 2.3 in [Kal17].

Theorem 2.3.11. *For point processes ξ_1 and ξ_2 on $\mathbb{S}$ the following assertions are equivalent:*

(i) $\xi_1 \stackrel{d}{=} \xi_2$,

(ii) $(\xi_1(B_1), \dots, \xi_1(B_k)) \stackrel{d}{=} (\xi_2(B_1), \dots, \xi_2(B_k))$, *for all* $k \in \mathbb{N}$ *and all* $B_1, \dots, B_k \in \mathcal{B}_b$,

(iii) $\xi_1(f) \stackrel{d}{=} \xi_2(f)$, *for all* $f \in CB_b^+(\mathbb{S})$,

(iv) $\mathcal{L}_{\xi_1}(f) = \mathcal{L}_{\xi_2}(f)$, *for all* $f \in CB_b^+(\mathbb{S})$.

For simple point processes ξ_1 and ξ_2 on $\mathbb{S}$, additional equivalent assertion is given by

(v) $v_{\xi_1}(B) = v_{\xi_2}(B)$, *for all* $B \in \mathcal{B}_b$.

We associate with a point process ξ on $\mathbb{S}$ a σ-algebra

$$\mathcal{F}^\xi = \sigma\{\xi(B) : B \in \mathcal{B}\},$$

which represents complete knowledge of ξ.

Definition 2.3.12. *A family $\{\xi_i : i \in I\}$ of point processes on $\mathbb{S}$, defined on the same probability space, for an arbitrary set I, is said to be independent if the family of σ-algebras $\{\mathcal{F}^{\xi_i} : i \in I\}$ is independent.*

2.4 Convergence in distribution

The following definition is a standard definition of convergence in distribution of a sequence of random elements in a Polish space.

Definition 2.4.1. *A sequence $(\xi_n)_{n\geq 1}$ of point processes on $\mathbb{S}$ converges in distribution to a point process ξ on $\mathbb{S}$, denoted by $\xi_n \xrightarrow{d} \xi$, if*

$$\mathbb{E}f(\xi_n) \to \mathbb{E}f(\xi)\,,$$

as $n \to \infty$, for all bounded and continuous real-valued functions f on $M_p(\mathbb{S})$.

Note that,

$$\mathbb{E}f(\xi) = \int_\Omega f(\xi(\omega))\mathbb{P}(\mathrm{d}\omega) = \int_{M_p(\mathbb{S})} f(m)\mathbb{P}_\xi(\mathrm{d}m)\,,$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space for ξ. It is apparent that point processes $\xi, \xi_1, \xi_2, \dots$ in Definition 2.4.1 do not have to be defined over the same probability space.

Since continuity is defined with respect to the topology and a point process on $\mathbb{S}$ can be regarded as a random element with values in $M_p(\mathbb{S})$, convergence in distribution of point processes on $\mathbb{S}$ therefore depends on the choice of topology on $M_p(\mathbb{S})$. We will consider convergence in distribution in $M_p(\mathbb{S})$ with respect to the vague topology. The following result gives some basic characterizations of this convergence, for the proof see Theorems 4.11 and 4.15 in [Kal17].

Theorem 2.4.2. *Let $\xi, \xi_1, \xi_2, \dots$ be point processes on $\mathbb{S}$. The following statements are equivalent:*

(i) $\xi_n \xrightarrow{d} \xi\,,$

(ii) $(\xi_n(B_1), \dots, \xi_n(B_k)) \xrightarrow{d} (\xi(B_1), \dots, \xi(B_k))$, *for all $k \in \mathbb{N}$ and all $B_1, \dots, B_k \in \mathcal{B}_b$ such that $\xi(\partial B_i) = 0$ a.s., for each $i = 1, \dots, k$,*

(iii) $\xi_n(f) \xrightarrow{d} \xi(f)$, *for all $f \in CB_b^+(\mathbb{S})$,*

(iv) $\mathcal{L}_{\xi_n}(f) \to \mathcal{L}_\xi(f)$, *for all $f \in CB_b^+(\mathbb{S})$,*

as $n \to \infty$. If ξ is a simple point process on $\mathbb{S}$, additional equivalent assertion is given by *(v)* $v_{\xi_n}(B) \to v_\xi(B)$ *, for all* $B \in \mathcal{B}_b$ *such that* $\xi(\partial B) = 0$ *a.s.* *and*

$$\limsup_{n\to\infty} \mathbb{P}(\xi_n(B) > 1) \le \mathbb{P}(\xi(B) > 1)\,, \text{ for all } B \in \mathcal{B}_b \text{ such that } \xi(\partial B) = 0 \text{ a.s.}$$

In the following example, as a consequence of Theorem 2.4.2, we obtain a necessary condition for convergence in distribution of point processes on $\mathbb{R}_+$.

Example 2.4.3. Let $\xi = \sum_{i=1}^\infty \delta_{T_i}$ with $0 \le T_1 \le T_2 \le \ldots$ and $\xi_n = \sum_{i=1}^\infty \delta_{T_{ni}}$ with $0 \le T_{n1} \le T_{n2} \le \ldots$, $n \ge 1$, be point processes on $\mathbb{R}_+$. Then $\xi_n \xrightarrow{d} \xi$ implies

$$(T_{n1}, \ldots, T_{nk}) \xrightarrow{d} (T_1, \ldots, T_k)\,,$$

for all $k \in \mathbb{N}$, as $n \to \infty$. Indeed, let $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$. Then, by Theorem 2.4.2 $(i) \Rightarrow (ii)$, for all $k \in \mathbb{N}$ and all $0 \le x_1 \le \ldots \le x_k$ such that $\xi(\{0, x_i\}) = 0$ a.s., for $i = 1, \ldots, k$

$$(\xi_n\langle 0, x_1], \ldots, \xi_n\langle 0, x_k]) \xrightarrow{d} (\xi\langle 0, x_1], \ldots, \xi\langle 0, x_k])\,,$$

thus

$$\mathbb{P}(\xi_n\langle 0, x_1] = 1, \ldots, \xi_n\langle 0, x_k] = k) \to \mathbb{P}(\xi\langle 0, x_1] = 1, \ldots, \xi\langle 0, x_k] = k)\,,$$

and

$$\mathbb{P}(T_{n1} \le x_1, \ldots, T_{nk} \le x_k) \to \mathbb{P}(T_1 \le x_1, \ldots, T_k \le x_k)\,,$$

as $n \to \infty$, which proves the statement.

In the above example and in the sequel, we use abbreviation $\xi\langle 0, x]$ for $\xi(\langle 0, x])$ to avoid abundance of parentheses. Analogous abbreviations, for all $a, b \in \mathbb{R} \cup \{\pm\infty\}$ such that $a \le b$ and all types of intervals, will be used in the sequel without any further notice.

2.5 Stationarity

In this part we consider point processes on finite-dimensional Euclidean spaces. For any constant $t \in \mathbb{R}^d$, $d \in \mathbb{N}$, let θ_t denote the shift operator defined by

$$\theta_t s = s - t\,,$$

for all $s \in \mathbb{R}^d$. For an arbitrary m in $M_p(\mathbb{R}^d)$, the shifted measure $\theta_t m$ is defined by

$$\theta_t m(B) = m(B + t)\,,$$

for all $B \in \mathcal{B}$, where $B + t = \{b + t : b \in B\}$. Equivalently, shifted measure $\theta_t m$ can be defined in terms of integrals by $\theta_t m(f) = m(f \circ \theta_t)$, for all nonnegative measurable functions f on $\mathbb{R}^d$.

Definition 2.5.1. *A point process ξ on $\mathbb{R}^d$ is stationary if*

$$\theta_t \xi \stackrel{d}{=} \xi \,,$$

for all $t \in \mathbb{R}^d$.

Note that $\theta_t \xi$ is indeed a point process, as a composition of two measurable mappings and since $B + t \in \mathcal{B}_b$, for all $B \in \mathcal{B}_b$ and $t \in \mathbb{R}$. Proof that, for all $t \in \mathbb{R}^d$, $\theta_t : M_p(\mathbb{R}^d) \to M_p(\mathbb{R}^d)$ is measurable can be found, for instance, in [DVJ08] (see Lemma 12.1.I).

To be consistent with the terminology introduced in Section 2.3, we will denote $\lambda := \mathbb{E}\xi[0,1]^d$ and call it the intensity of ξ. Stationarity of ξ implies that its mean measure is shift invariant, that is $\mathbb{E}\xi(B+t) = \mathbb{E}\xi(B)$, for all $B \in \mathcal{B}$ and all $t \in \mathbb{R}^d$. If additionally ξ has a finite intensity λ, then it follows e.g. from Theorem 5.8 (*ii*) in [Sch17] that the mean measure of a stationary point process ξ on $\mathbb{R}^d$ is a multiple of a Lebesgue measure, i.e.

$$\mathbb{E}\xi(B) = \lambda Leb(B) \qquad \text{for all } B \in \mathcal{B} \,.$$

The converse is generally not true. However, by Proposition 8.3 in [LP18], a Poisson process on $\mathbb{R}^d$ with finite intensity λ is stationary if and only if its mean measure equals λLeb.

By the so-called $0 - \infty$ law (see Lemma 5.2 in [Kal17]), any stationary point process $\xi \neq 0$ on $\mathbb{R}^d$ has infinitely many points. Additionally, if $d = 1$ and point process is simple, points of ξ may be enumerated as

$$\ldots < T_{-2} < T_{-1} < 0 \leq T_0 < T_1 < T_2 < \ldots \,.$$

Remark 2.5.2. Note that it is a standard in the point process literature to enumerate the first point on $\mathbb{R}_+$ with T_1. However, that differs from the usual enumeration of points in the renewal theory, where the first point on $\mathbb{R}_+$ is denoted as T_0. Careful reader may have noticed that we have already used both enumerations, see e.g. Examples 2.3.4 and 2.3.5. Since our main interest are point processes with certain renewal structure, we embrace the latter enumeration.

The above described concept of stationarity will sometimes be referred to as time stationarity to avoid confusion with point stationarity. The latter one can be found in the literature also by the name cycle stationarity.

Definition 2.5.3. *A simple point process $\tilde{\xi} = \sum_{i\in\mathbb{Z}} \delta_{\tilde{T}_i}$ on $\mathbb{R}$ with $\ldots < \tilde{T}_{-1} < 0 \le \tilde{T}_0 < \tilde{T}_1 < \ldots$ is point stationary if*

$$\theta_{\widetilde{T}_k} \tilde{\xi} \stackrel{d}{=} \tilde{\xi}\,,$$

for all $k \in \mathbb{Z}$.

It is obvious that a simple point stationary point process $\tilde{\xi}$ has a point at 0 a.s. Furthermore, note that random variables $\widetilde{T}_k - \widetilde{T}_{k-1}$, for all k, form a stationary sequence. In fact, it is easy to show that a simple point process with these properties is necessarily point stationary.

A simple point process on $\mathbb{R}$ can be stationary with respect to time shifts, which yields the concept of time stationarity, or with respect to point shifts, which yields the concept of point stationary. These two forms of stationarity are mutually exclusive.

Example 2.5.4. Let $E_i, i \in \mathbb{Z}$ be i.i.d. exponential random variables with rate $\lambda > 0$.

(i) Time stationary Poisson process on $\mathbb{R}$ with intensity λ is given by

$$\xi = \sum_{i\in\mathbb{Z}} \delta_{T_i}\,,$$

where $T_0 = E_0$, $T_i = \sum_{j=0}^{i} E_j$ and $T_{-i} = -\sum_{j=-i}^{-1} E_j$, $i \ge 1$.

(ii) Point stationary Poisson process on $\mathbb{R}$ with intensity λ is given by

$$\tilde{\xi} = \sum_{i\in\mathbb{Z}} \delta_{\tilde{T}_i}\,,$$

where $\tilde{T}_0 = 0$, $\tilde{T}_i = \sum_{j=1}^{i} E_j$ and $\tilde{T}_{-i} = -\sum_{j=-i+1}^{0} E_j$, $i \ge 1$.

2.6 Marked point processes

To each point of a point process may be attached an attribute. In many examples, this attribute is a way of adding some additional data or information about the point. For instance, to each point of the point process of claim arrival times we may attach the claim size.

Definition 2.6.1. *Let $\zeta = \sum_{i=1}^{L} \delta_{X_i}$ be a point process on $\mathbb{S}$ and let $(W_i)_i$ be a random sequence with values in a Polish space $\mathbb{M}$. A point process*

$$\xi = \sum_{i=1}^{L} \delta_{(X_i, W_i)}$$

on $\mathbb{S} \times \mathbb{M}$ is called a marked point process on $\mathbb{S}$ with marks in $\mathbb{M}$.

We will refer to the point process ζ as the ground process, to the random element W_i in $\mathbb{M}$ as the mark associated to X_i and to the space $\mathbb{M}$ as the mark space.

We will be mostly interested in the marked point processes with ground process on $\mathbb{R}$. In Example 2.1.3 (ii), we have seen that the boundedness $\mathcal{B}_b = \{B \in \mathcal{B} : \exists h > 0 \text{ such that } B \subseteq [-h, h] \times \mathbb{M}\}$ properly localizes $\mathbb{R} \times \mathbb{M}$. Hence, if ξ takes values in $M_p(\mathbb{R} \times \mathbb{M})$ then $\xi(A \times \mathbb{M}) < \infty$ a.s., for an arbitrary $A \in \mathcal{B}_b(\mathbb{R})$.

Typical examples of mark spaces are $\mathbb{M} = \mathbb{N}$ or $\mathbb{M} = \mathbb{R}^d$. However, marks can be very general random elements. By Proposition 2.2.6, $M_p(\mathbb{S})$ is a Polish space, hence marks can even be point processes.

Example 2.6.2 (**Independently marked point process**). Let $\zeta = \sum_{i=1}^{L} \delta_{X_i}$ be a point process on $\mathbb{S}$ and let $(W_i)_i$ be a sequence of i.i.d. random variables independent of ζ with values in a Polish space $\mathbb{M}$. Then the marked point process $\xi = \sum_{i=1}^{L} \delta_{(X_i, W_i)}$ is called an independently marked point process on $\mathbb{S}$ with marks in $\mathbb{M}$.

Both concepts of stationarity extend immediately to marked point processes with ground process on $\mathbb{R}$. A marked point process $\xi = \sum_{i \in \mathbb{Z}} \delta_{(T_i, W_i)}$ on $\mathbb{R}$ with marks in $\mathbb{M}$ is said to be (time) stationary if

$$\theta_t \xi = \sum_{i \in \mathbb{Z}} \delta_{(T_i - t, W_i)} \overset{d}{=} \xi \,,$$

for all $t \in \mathbb{R}$. Furthermore, we say that a marked point process ξ on $\mathbb{R}$ with marks in $\mathbb{M}$ is simple if $\mathbb{P}(\xi(\{t\} \times \mathbb{M}) \leq 1\,,\ \text{for all } t \in \mathbb{R}) = 1$. A simple marked point process $\tilde{\xi} = \sum_{i \in \mathbb{Z}} \delta_{(\widetilde{T}_i, \widetilde{W}_i)}$ on $\mathbb{R}$ with marks in $\mathbb{M}$ is said to be point stationary if

$$\theta_{\widetilde{T}_k} \tilde{\xi} = \sum_{i \in \mathbb{Z}} \delta_{(\widetilde{T}_i - \widetilde{T}_k, \widetilde{W}_i)} \overset{d}{=} \tilde{\xi} \,,$$

for all $k \in \mathbb{Z}$.

Starting from a simple marked point stationary process $\tilde{\xi}$ on $\mathbb{R}$ with marks in $\mathbb{M}$ it is relatively easy to see that

$$\mathbb{E} f(\xi) = \frac{1}{\mu} \mathbb{E} \int_0^{\widetilde{T}_1} f(\theta_t \tilde{\xi}) \mathrm{d}t \,, \tag{2.4}$$

where $\mu = \mathbb{E}\widetilde{T}_1 \in \langle 0, \infty \rangle$ and f is an arbitrary nonnegative measurable function on $M_p(\mathbb{R} \times \mathbb{M})$, defines a distribution of a stationary marked point process ξ on $\mathbb{R}$ with marks in $\mathbb{M}$. Indeed, for any $n \in \mathbb{N}$ and an arbitrary nonnegative bounded measurable function f on $M_p(\mathbb{R} \times \mathbb{M})$,

$$n\mathbb{E}f(\xi) = \frac{1}{\mu}\mathbb{E}n \int_0^{\widetilde{T}_1} f(\theta_t\tilde{\xi})\mathrm{d}t = \frac{1}{\mu}\mathbb{E}\int_0^{\widetilde{T}_n} f(\theta_t\tilde{\xi})\mathrm{d}t\,,$$

where we have used the assumption of point stationarity. Also, for any $s \in \mathbb{R}$,

$$n\mathbb{E}f(\theta_s\xi) = \frac{1}{\mu}\mathbb{E}\int_0^{\widetilde{T}_n} f(\theta_{t+s}\tilde{\xi})\mathrm{d}t = \frac{1}{\mu}\mathbb{E}\int_s^{\widetilde{T}_n+s} f(\theta_t\tilde{\xi})\mathrm{d}t\,,$$

by the change of variable formula. Therefore,

$$\begin{aligned}
n\left|\mathbb{E}f(\theta_s\xi) - \mathbb{E}f(\xi)\right| &= \frac{1}{\mu}\left(\left|\mathbb{E}\int_{\widetilde{T}_n}^{\widetilde{T}_n+s} f(\theta_t\tilde{\xi})\mathrm{d}t - \mathbb{E}\int_0^s f(\theta_t\tilde{\xi})\mathrm{d}t\right|\right)\\
&\leq \frac{1}{\mu}\left(\left|\mathbb{E}\int_{\widetilde{T}_n}^{\widetilde{T}_n+s} f(\theta_t\tilde{\xi})\mathrm{d}t\right| + \left|\mathbb{E}\int_0^s f(\theta_t\tilde{\xi})\mathrm{d}t\right|\right)\\
&\leq \frac{2}{\mu}\|f\||s| < \infty\,,
\end{aligned}$$

where $\|f\| = \sup_{m\in M_p(\mathbb{R}\times\mathbb{M})} f(m)$. As $n \to \infty$, we get $\mathbb{E}f(\theta_s\xi) = \mathbb{E}f(\xi)$, for all $s \in \mathbb{R}$ and all bounded measurable $f \geq 0$. Let now f denote an arbitrary, possibly unbounded, nonnegative measurable function. Define

$$f^n(m) = \min\left(f(m), n\right)\,,$$

for all $n \in \mathbb{N}$ and for all $m \in M_p(\mathbb{R} \times \mathbb{M})$. Observe that $\|f^n\| \leq n < +\infty$, for all $n \in \mathbb{N}$ and $(f^n)_{n\in\mathbb{N}}$ is a nondecreasing sequence such that $\lim_{n\to\infty} f^n(m) = f(m)$, for all $m \in M_p(\mathbb{R} \times \mathbb{M})$. By the monotone convergence theorem,

$$\mathbb{E}f(\theta_s\xi) = \lim_{n\to\infty} \mathbb{E}f^n(\theta_s\xi) = \lim_{n\to\infty} \mathbb{E}f^n(\xi) = \mathbb{E}f(\xi)\,,$$

for all $s \in \mathbb{R}$. Hence, ξ is a stationary marked point process on $\mathbb{R}$ with marks in $\mathbb{M}$.

Remark 2.6.3. In the Palm theory for stationary point processes, equality (2.4) is known under the name of inversion formula. The above procedure is an adaptation of arguments from [Kal17], see Theorem 5.4, to our setting.

Chapter 3

Standard renewal theory

Renewal theory has its origins in the study of some particular probability problems connected with the failure and replacement of industrial components, such as light bulbs. Over the years, it was noticed that essentially the same problems arise also in a wide range of other applied probability areas. On the other hand, mathematical results of renewal theory are of great importance on their own and constitute a fundamental part of the theory of probability.

Due to its great importance and long history, there is an abundance of literature concerning renewal theory. Classical references used in this chapter include [Cox62] and [Fel71], more modern ones, suitable even for the beginners in the field, are [Res92] and [Iks16a], while a yet unpublished manuscript [Als13] takes the most general approach of all of the above.

3.1 Renewal processes

Definition 3.1.1. *Renewal process is a random sequence $(T_i)_{i\geq 0}$ defined by*

$$T_i = X_0 + X_1 + \cdots + X_i\,,$$

for all $i \geq 0$, where $(X_i)_{i\geq 1}$ is a sequence of nonnegative i.i.d. random variables with a positive (possibly infinite) mean μ, independent of a nonnegative random variable X_0.

Renewal process $(T_i)_{i\geq 0}$ is called pure or zero-delayed if $X_0 = 0$ and delayed otherwise. Random variable T_i is usually thought of as a time of occurrence of some phenomenon and is called the i-th renewal time, while the common distribution function of $(X_i)_{i\geq 1}$ is called the interarrival distribution.

Example 3.1.2. Suppose we are given an infinite supply of light bulbs of the same type produced at the same factory. It is only natural to assume that their lifetimes $(X_i)_{i\geq 1}$ are independent and identically distributed. Suppose that light bulbs are used one at a time until they fail, when they are instantaneously replaced. If the initial light bulb is fresh then $X_0 = 0$, otherwise $X_0 > 0$ represents the time until the first replacement. Failure times $T_i = X_0 + X_1 + X_2 + \dots X_i$, for $i \geq 0$, constitute a renewal process.

Alsmeyer in [Als13] listed questions which are naturally related to this example. These turn out to be the most important questions of renewal theory.

- Is the number of renewals up to time t finite almost surely, for all $t \geq 0$?
- What is the long run average number of renewals per unit of time?
- What can be said about the asymptotic behaviour of the expected number of renewals in a fixed length interval?

All of the above will be answered in this chapter.

As seen in Example 2.3.5, it is quite natural to embed the renewal process in the point process framework. Recall, given the renewal process $(T_i)_{i\geq 0}$, the associated point process $\xi = \sum_{i=0}^{\infty} \delta_{T_i}$ is called the renewal point process.

Definition 3.1.3. *Stochastic process*

$$\nu(t) = \sum_{i=0}^{\infty} \mathbb{1}_{\{T_i \leq t\}}\,, \quad t \geq 0\,,$$

is called the renewal counting process.

For each $t \geq 0$, the random variable $\nu(t)$ counts the number of events of the renewal point process ξ in the closed interval $[0, t]$. Note that

$$\nu(t) = \inf\{i \geq 0 : T_i > t\}\,,$$

hence $\nu(t)$ can be interpreted as the level t first-passage time. Furthermore, observe that for all $i \geq 0$, $\{\nu(t) \leq i\} = \{\nu(t) > i\}^c = \{T_i \leq t\}^c$, thus $\nu(t)$ is a stopping time with respect to the natural filtration $(\mathcal{F}_k)_{k\in\mathbb{N}_0}$ of the renewal process $(T_i)_{i\geq 0}$, defined by

$$\mathcal{F}_k = \sigma\{T_0, T_1, \dots, T_k\}\,, \quad k \in \mathbb{N}_0\,.$$

Definition 3.1.4. *Function $U : [0, \infty\rangle \to [0, \infty]$ defined by*

$$U(t) = \mathbb{E}\nu(t)$$

is called the renewal function.

Hereafter, let G denote the distribution of X_0 and F the common distribution function of $X_i, i \geq 1$. It will be convenient to express the renewal function in terms of the distribution functions of X_i, $i \geq 0$. Observe that T_i, for $i \geq 1$, is the sum of independent random variables. Hence, for $T_0 = 0$,

$$U(t) = \mathbb{E}\left[\sum_{i=0}^{\infty} \mathbb{1}_{\{T_i \leq t\}}\right] = \sum_{i=0}^{\infty} \mathbb{P}(T_i \leq t) = \sum_{i=0}^{\infty} F^{i*}(t) \tag{3.1}$$

and, for $T_0 > 0$,

$$U(t) = \mathbb{E}\left[\sum_{i=0}^{\infty} \mathbb{1}_{\{T_i \leq t\}}\right] = \sum_{i=0}^{\infty} \mathbb{P}(T_i \leq t) = \sum_{i=0}^{\infty} G * F^{i*}(t) = G * \sum_{i=0}^{\infty} F^{i*}(t)\,, \tag{3.2}$$

where F^{i*} is the i-fold convolution of F with itself, defined recursively by

$$F^{0*}(t) = \mathbb{1}_{[0,\infty\rangle}(t)\,,$$
$$F^{i*}(t) = F^{(i-1)*} * F(t) = \int_{[0,t]} F^{(i-1)*}(t-s)\mathrm{d}F(s)\,, \quad i \geq 1\,.$$

Lemma 3.1.5. *The renewal function U is finite, for all $t \geq 0$.*

For the proof of Lemma 3.1.5 in the zero-delayed case see e.g. Theorem 3.3.1 in [Res92]. Statement in the delayed case then follows by relation (3.2).

It is now immediate that $\nu(t) < \infty$ almost surely for all $t \geq 0$. Furthermore, Lemma 3.1.5 and the dominated convergence theorem yield the right-continuity of the renewal function U and since it is obviously nondecreasing, we can associate with it an unique measure μ_U on $[0, \infty\rangle$ such that $\mu_U[0, t] = U(t)$, for all $t \geq 0$. However, it is common in the literature to use the same notation for the renewal function and this measure, called the renewal measure, since the context prevents the confusion. Hence, in the sequel we will denote both with U.

The following theorem is the simplest renewal theorem. For the proof see e.g. Theorem 3.3.3 in [Res92].

Theorem 3.1.6 (Elementary renewal theorem)**.** *The renewal function U satisfies*

$$\lim_{t\to\infty} \frac{U(t)}{t} = \frac{1}{\mu}\,.$$

Remark 3.1.7. In case $\mu = \infty$, we henceforth interpret the ratio $\frac{1}{\infty}$ as 0.

3.2 Stationary renewal processes

Intuitively, for a renewal process to be stationary we require that, if for any $t > 0$ we shift the origin to t, distributions of the renewal times are left unchanged.

Definition 3.2.1. *The delayed renewal process is called stationary if*

$$(T_{\nu(t)+k} - t)_{k\geq 0} \stackrel{d}{=} (T_k)_{k\geq 0}\,, \tag{3.3}$$

for all $t \geq 0$*.*

By differencing successive terms on both sides of (3.3), we get

$$T_{\nu(t)} - t, X_{\nu(t)+1}, X_{\nu(t)+2}, \ldots \stackrel{d}{=} X_0, X_1, X_2, \ldots\,.$$

Breaking things down according to the value of $\nu(t)$, it is easy to see that $T_{\nu(t)} - t$ is independent of $(X_{\nu(t)+k})_{k\geq 1}$ and that the latter sequence has the same distribution as $(X_k)_{k\geq 1}$. Hence, by a suitable choice of the delay distribution, a renewal process can be made stationary. The following theorem is proved in e.g. [Bré20], see Theorem 4.1.20.

Theorem 3.2.2. *For a delayed renewal process to be stationary it is necessary and sufficient that*

$$\mu < \infty \quad \text{ and } \quad \mathbb{P}(X_0 \leq x) = F_0(x)\,,$$

where F_0 *is the integrated tail distribution given by*

$$F_0(x) = \frac{1}{\mu}\int_0^x (1 - F(y))\mathrm{d}y\,, \quad x \geq 0\,. \tag{3.4}$$

Example 3.2.3. Let X_0 and $(X_i)_{i\geq 1}$ have exponential distribution with strictly positive and finite parameter $0 < \lambda < \infty$. It is easy to show that

$$F_0(x) = 1 - e^{-\lambda x}\,, \quad x \geq 0\,.$$

Hence, by the previous theorem, $(X_i)_{i\geq 0}$ is a stationary renewal process. Obviously, the associated renewal point process is the homogeneous Poisson process on $\mathbb{R}_+$ with intensity λ, which was introduced in Example 2.3.4. Moreover, in the notation of this chapter, we know that $U(t) = \frac{t}{\mu}$, for all $t \geq 0$.

The last statement in the above example is valid for general stationary renewal processes. In fact, even more can be said.

Theorem 3.2.4. *The delayed renewal process is stationary if and only if*

$$U(t) = \frac{t}{\mu},$$

for all $t \geq 0$.

For the proof of necessity see e.g. [Iks16a], page 18, and for sufficiency see e.g. [Dur10], page 179.

3.3 Blackwell's renewal theorem

The next definition gives an important classification of distribution functions in the context of renewal theory.

Definition 3.3.1. *Let F be a distribution function of an arbitrary nonnegative random variable X. If there is no $d > 0$ such that $\sum_{k=0}^{\infty} \mathbb{P}(X = kd) = 1$, then F is said to be nonarithmetic. In the contrary case, F is said to be arithmetic.*

If the common distribution function F of the interarrival times X_i, $i \geq 1$ is nonarithmetic, we say that the renewal process in nonarithmetic. As observed in the literature (see e.g. [Kal17], page 489), results in the arithmetic case are similar and more elementary then the ones in the nonarithmetic case, though technically cumbersome and therefore skipped in the sequel. Interested reader can find the proofs of the most important results from the standard renewal theory in the arithmetic case in, for instance, Section 3.8. in [Res92] or Sections 3.1 and 4.1 in [Iks16a].

Blackwell's renewal theorem is one of the most fundamental results in the probability theory. It is named after the mathematician who first proved it in the nonarithmetic case, see [Bla48]. Since the 1948th, several proofs have been obtained, however neither of those is really elementary, even though the statement of the theorem is intuitively simple. Initial proofs were rather analytical, relaying on, for instance, Fourier analysis (in [FO61]), Choquet-Deny theorem (in [Fel71], pages 364-366) or harmonic analysis (in [McD75]). Finally, usage of the coupling method resulted in the probabilistic proofs of the theorem, the first one given in [Lin77] and the self-contained one in [Tho87]. Longer historical account and more references important in the development of the modern day proof of Blackwell's renewal theorem can be found in [Als13].

Theorem 3.3.2 (Blackwell's renewal theorem)**.** *Suppose that U is the renewal measure of a nonarithmetic renewal process, then*

$$\lim_{t\to\infty} U\langle t, t+x] = \frac{x}{\mu}\,,$$

for all $x > 0$.

Note that Blackwell's renewal theorem is stronger then the elementary renewal theorem. An alternate proof of the elementary renewal theorem which uses Theorem 3.3.2 can be found e.g. in Section 3.7 in [Res92].

3.4 Key renewal theorem

It is a simple observation that

$$\lim_{t\to\infty}\int_{[0,t]} \mathbb{1}_{[0,x\rangle}(t-y)\mathrm{d}U(y) = \lim_{t\to\infty} U\langle t-x, t] = \frac{x}{\mu} = \frac{1}{\mu}\int_0^\infty \mathbb{1}_{[0,x\rangle}(y)\mathrm{d}y\,, \tag{3.5}$$

for all $x > 0$, where we have used Blackwell's renewal theorem in the second equality. Hence, it is natural to ask to which class of functions $g : \mathbb{R} \to \mathbb{R}$ it is possible to generalize (3.5) in the sense that

$$\lim_{t\to\infty}\int_{[0,\infty\rangle} g(t-y)\mathrm{d}U(y) = \frac{1}{\mu}\int_{-\infty}^{\infty} g(y)\mathrm{d}y\,.$$

First, we recall the notion of Riemann integrability. A function $g : \mathbb{R} \to \mathbb{R}_+$ is said to be Riemann integrable on the closed interval $[a, b]$, for $-\infty < a < b < \infty$, if

$$\lim_{h\to 0+}(\overline{\sigma}_R(h) - \underline{\sigma}_R(h)) = 0\,,$$

where

$$\overline{\sigma}_R(h) = h\sum_{k:a+h\le kh\le b}\ \sup_{(k-1)h\le y<kh} g(y) \quad \text{and} \quad \underline{\sigma}_R(h) = h\sum_{k:a+h\le kh\le b}\ \inf_{(k-1)h\le y<kh} g(y)\,.$$

Riemann integral is then defined by

$$\int_a^b g(y)\mathrm{d}y = \lim_{h\to 0+}\overline{\sigma}_R(h)\,.$$

Furthermore, $g : \mathbb{R} \to \mathbb{R}_+$ is said to be improperly Riemann integrable on $\mathbb{R}$ if it is Riemann integrable on closed intervals $[a, c]$, $[c, b]$ and the limit $\lim_{a\to-\infty}\int_a^c g(y)\mathrm{d}y + \lim_{b\to\infty}\int_c^b g(y)\mathrm{d}y$ exists, for every $c \in \langle a, b\rangle$. In this case

$$\int_{-\infty}^{\infty} g(y)\mathrm{d}y = \lim_{a\to-\infty}\int_a^c g(y)\mathrm{d}y + \lim_{b\to\infty}\int_c^b g(y)\mathrm{d}y\,,$$

for an arbitrary $c \in \langle a, b\rangle$. Essentially, improper Riemann integral is defined as a limit of Riemann integrals on closed intervals which are made bigger and bigger. These definitions are easily extended to functions $g : \mathbb{R} \to \mathbb{R}$ since every real-valued function can be expressed as a difference of two nonnegative functions, namely $g = g^+ - g^-$, where $g^+(x) = \max(g(x), 0)$ and $g^-(x) = \max(-g(x), 0)$.

As we will see shortly, definition of Riemann integrability makes too many functions integrable. Hence, it is preferable to naively extend the original definition to infinite intervals. This notion of integrability, know as direct Riemann integrability, was first introduced and thus named in [Fel71].

Definition 3.4.1. *A function $g : \mathbb{R} \to \mathbb{R}_+$ is called directly Riemann integrable on $\mathbb{R}$ if*

$$\overline{\sigma}(h) < \infty, \ \textit{for all } h > 0 \tag{3.6}$$

and

$$\lim_{h \to 0+} (\overline{\sigma}(h) - \underline{\sigma}(h)) = 0 \tag{3.7}$$

where

$$\overline{\sigma}(h) = h \sum_{k \in \mathbb{Z}} \sup_{(k-1)h \le y < kh} g(y) \quad \textit{and} \quad \underline{\sigma}(h) = h \sum_{k \in \mathbb{Z}} \inf_{(k-1)h \le y < kh} g(y)\,.$$

Analogous definitions of direct Riemann integrability on $\mathbb{R}_- = \langle -\infty, 0]$ and $\mathbb{R}_+$ are obtained by changing the bounds of summation in the definition of $\overline{\sigma}$ and $\underline{\sigma}$. A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be directly Riemann integrable on $\mathbb{R}_+$ if (3.6) and (3.7) are valid, where

$$\overline{\sigma}(h) = h \sum_{k=1}^{\infty} \sup_{(k-1)h \le y < kh} g(y) \quad \text{and} \quad \underline{\sigma}(h) = h \sum_{k=1}^{\infty} \inf_{(k-1)h \le y < kh} g(y)\,. \tag{3.8}$$

Similarly, a function $g : \mathbb{R}_- \to \mathbb{R}_+$ is said to be directly Riemann integrable on $\mathbb{R}_-$ if (3.6) and (3.7) are valid, where

$$\overline{\sigma}(h) = h \sum_{k=-\infty}^{0} \sup_{(k-1)h \le y < kh} g(y) \quad \text{and} \quad \underline{\sigma}(h) = h \sum_{k=-\infty}^{0} \inf_{(k-1)h \le y < kh} g(y)\,.$$

A function $g : \mathbb{R} \to \mathbb{R}$ is said to be directly Riemann integrable on a (semi-)infinite interval, if so are the nonnegative functions g^+ and g^-.

Example 3.4.2. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$g(y) = \begin{cases} h_k \frac{y-(c_k-2^{-1}l_k)}{2^{-1}l_k}\,, & y \in [c_k - 2^{-1}l_k, c_k\rangle\,, \\ h_k \frac{(c_k+2^{-1}l_k)-y}{2^{-1}l_k}\,, & y \in [c_k, c_k + 2^{-1}l_k\rangle\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{3.9}$$

where $(\langle c_k - 2^{-1}l_k, c_k + 2^{-1}l_k\rangle)_{k\in\mathbb{N}}$ is a disjoint sequence of intervals, $(c_k)_{k\in\mathbb{N}}$ is an increasing sequence such that $\inf_{k\in\mathbb{N}}(c_{k+1} - c_k) > 0$, $(h_k)_{k\in\mathbb{N}}$ is a nondecreasing sequence such that $\sum_{k=1}^{\infty} h_k = \infty$ and $\sum_{k=1}^{\infty} l_k h_k < \infty$. One can think of l_k as the basis of the k-th

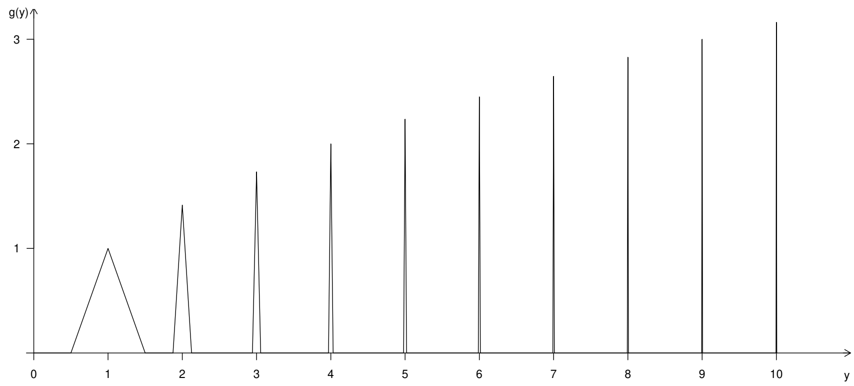

Figure 3.1: Graph of g on interval $[0, 10.5\rangle$, for $c_k = k$, $l_k = k^{-2}$ and $h_k = k^{\frac{1}{2}}$.

non-overlapping isosceles triangle centered at c_k with height h_k, for all $k \in \mathbb{N}$ (see Figure 3.1). Then,

$$\int_0^{\infty} g(y)\mathrm{d}y = 2^{-1}\sum_{k=1}^{\infty} h_k l_k < \infty\,,$$

and g is Riemann integrable on $\mathbb{R}_+$. However, g is not directly Riemann integrable on $\mathbb{R}_+$, since

$$\overline{\sigma}(h) = h \sum_{k=1}^{\infty} \sup_{(k-1)h\le y<kh} g(y) \geq h \sum_{k=1}^{\infty} h_k = \infty\,,$$

for $h = \inf_{k\in\mathbb{N}}(c_{k+1} - c_k)$.

We state here some necessary and sufficient conditions for direct Riemann integrability which will be needed in the sequel. More criteria can be found in the literature, see for instance Section 3.10 in [Res92], Section 4.1 in [Als13], Section 4.2 in [Iks16a] or Section 4.2 in [Bré20].

Lemma 3.4.3. *Suppose that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is directly Riemann integrable on $\mathbb{R}_+$, then g is Riemann integrable on $\mathbb{R}_+$ and*

$$\lim_{h \to 0+} \overline{\sigma}(h) = \int_0^\infty g(y) \mathrm{d}y \,.$$

For the proof of the above lemma see e.g. Remark 3.10.2 in [Res92]. Note that Example 3.4.2 and Lemma 3.4.3 imply that the class of directly Riemann integrable functions on $\mathbb{R}_+$ is a proper subset of the class of Riemann integrable functions on $\mathbb{R}_+$.

Lemma 3.4.4. *(i) Suppose that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing and Riemann integrable on $\mathbb{R}_+$, then g is directly Riemann integrable on $\mathbb{R}_+$.*

(ii) Suppose that $g : \mathbb{R}_- \to \mathbb{R}_+$ is nondecreasing and Riemann integrable on $\mathbb{R}_-$, then g is directly Riemann integrable on $\mathbb{R}_-$.

Proof. (i) See e.g. Remark 3.10.3 in [Res92].

(ii) Define $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(y) = g(-y)$ and apply (i) to f.

□

The following lemma can be found, for instance, in [Res92] (see Remark 3.10.5).

Lemma 3.4.5. *Suppose that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a Riemann integrable function on $\mathbb{R}_+$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a directly Riemann integrable function on $\mathbb{R}_+$ such that $g(y) \leq h(y)$, for all $y \geq 0$. Then g is directly Riemann integrable on $\mathbb{R}_+$.*

Most of the literature considers only directly Riemann integrable functions on $\mathbb{R}_+$, hence we first state the key renewal theorem in its most familiar form. Proof can be found e.g. in [Res92], see Theorem 3.10.1 (ii). Note that Resnick gives the proof only in the zero-delayed case, however the proof in the delayed case is essentially the same.

Theorem 3.4.6 (Key renewal theorem)**.** *Suppose that U is the renewal measure of a nonarithmetic renewal process and that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a directly Riemann integrable function on $\mathbb{R}_+$, then*

$$\lim_{t \to \infty} \int_{[0,t]} g(t-y) \mathrm{d}U(y) = \frac{1}{\mu} \int_0^\infty g(y) \mathrm{d}y \,.$$

The first proof of the key renewal theorem was given by W. L. Smith for nonincreasing bounded Lebesgue integrable functions on $\mathbb{R}_+$, see [Smi54]. In the same article, Smith also gave a version for functions of bounded variation vanishing outside an arbitrary finite interval, not necessarily confined to a positive half line.

Analogous versions of the key renewal theorem, involving directly Riemann integrable functions on $\mathbb{R}_-$ and $\mathbb{R}$, are obtained by simple and obvious modifications in the proof.

Theorem 3.4.7 (Key renewal theorem on a negative half line/real line)**.**

(i) Suppose that U is the renewal measure of a nonarithmetic renewal process and that $g : \mathbb{R}_- \to \mathbb{R}_+$ is a directly Riemann integrable function on $\mathbb{R}_-$, then

$$\lim_{t\to\infty} \int_{[t,\infty\rangle} g(t-y)\mathrm{d}U(y) = \frac{1}{\mu}\int_{-\infty}^{0} g(y)\mathrm{d}y\,.$$

(ii) Suppose that U is the renewal measure of a nonarithmetic renewal process and that $g : \mathbb{R} \to \mathbb{R}_+$ is a directly Riemann integrable function on $\mathbb{R}$, then

$$\lim_{t\to\infty} \int_{[0,\infty\rangle} g(t-y)\mathrm{d}U(y) = \frac{1}{\mu}\int_{-\infty}^{\infty} g(y)\mathrm{d}y\,.$$

Remark 3.4.8. Blackwell's renewal theorem and the key renewal theorem are equivalent. From the proof of Theorem 3.10.1 $(i) \Rightarrow (ii)$ in [Res92], i.e. reference for the proof of Theorem 3.4.6, it follows that Blackwell's renewal theorem implies the key renewal theorem. The converse implication follows by choosing $g(y) = \mathbb{1}_{[0,x\rangle}(y)$ or $g(y) = \mathbb{1}_{[x,0\rangle}(y)$, depending on the domain of g.

The next example shows that the key renewal theorem can fail if the function g oscillates too much in the neighbourhood of infinity.

Example 3.4.9. Let F concentrate on $\{\alpha, 1-\alpha\}$, for some irrational $\alpha \in \langle 0, 1\rangle$. Then the renewal function U of the pure renewal process is piecewise constant with jumps at the points of the form $k_1\alpha + k_2(1-\alpha)$, for $k_1, k_2 \in \mathbb{N}_0$. Define function $g : \mathbb{R}_+ \to \mathbb{R}_+$ by (3.9) with $c_k = k_1\alpha + k_2(1-\alpha)$, for $k_1, k_2 \in \mathbb{N}_0$ such that $k_1 + k_2 = k$, and $h_k = 1$, for all $k \in \mathbb{N}$. Note that we have already shown in Example 3.4.2 that this function is Riemann integrable, but it is not directly Riemann integrable on $\mathbb{R}_+$. If the key renewal theorem was true, then

$$\lim_{t\to\infty} \int_{[0,t]} g(t-y)\mathrm{d}U(y) = \frac{1}{\mu}\int_{0}^{\infty} g(y)\mathrm{d}y$$

would be finite. However, for an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}\int_{[0,n]} g(n-y)\mathrm{d}U(y) &= \sum_{\substack{k_1,k_2\in\mathbb{N}_0\\ k_1\alpha+k_2(1-\alpha)\le n}} g(n-(k_1\alpha+k_2(1-\alpha)))U(k_1\alpha+k_2(1-\alpha))\\ &= \sum_{\substack{k_1,k_2\in\mathbb{N}_0\\ k_1\alpha+k_2(1-\alpha)< n}} \sum_{i=0}^{\infty}\mathbb{P}(T_i = k_1\alpha+k_2(1-\alpha))\\ &> U(n-1)\to\infty\,,\end{aligned}$$

as $n\to\infty$. The last equality follows since $g(0)=0$ and $g(n-(k_1\alpha+k_2(1-\alpha))) = g((n-k_1)\alpha+(n-k_2)(1-\alpha)) = 1$, for $k_1\alpha+k_2(1-\alpha)<n$.

3.5 Extended renewal theorem

The following renewal theorem describes the asymptotic behavior of the renewal point process. Unlike the other renewal theorems, this one is not as common in the literature. For the proof see Theorem 12.7 in [Kal17], case $m<\infty$, cf. also Lemma 5.1 in [IMM17].

Theorem 3.5.1 (Extended renewal theorem)**.** *Suppose that ξ' is a point process associated to the renewal process $(T_i')_{i\ge0}$ with an arbitrary initial distribution and nonarithmetic interarrival distribution F with finite mean. Furthermore, suppose that ξ is a point process associated to the stationary renewal process $(T_i)_{i\ge0}$ with the initial distribution given by (3.4) and the same interarrival distribution F. Then*

$$\theta_t\xi' \xrightarrow{d} \xi\,,$$

as $t\to\infty$, with respect to the vague topology on $M_p(\mathbb{R}_+)$.

As noted in [Kal17], in the case of a finite interarrival mean it is possible to deduce Blackwell's renewal theorem from the extended renewal theorem. Indeed, by Lemma 12.9 in [Kal17], $\xi'\langle t,t+x],t\ge0$ are uniformly integrable random variables for a fixed $x>0$, hence Theorem 3.3.2 follows by Lemma 5.11 in [Kal21].

3.6 Renewal equation

Starting from relation (3.1), we get the following well-known equation for the renewal function U of the zero-delayed renewal process

$$\begin{aligned} U(t) &= \sum_{i=0}^{\infty} F^{i*}(t) = F^{0*}(t) + \sum_{i=1}^{\infty} F^{i*}(t) \\ &= 1 + F * \sum_{i=1}^{\infty} F^{(i-1)*}(t) = 1 + F * U(t)\,, \quad t \geq 0\,. \end{aligned} \tag{3.10}$$

Similar equation for the renewal function in the delayed case is obtained from relations (3.2) and (3.10)

$$\begin{aligned} U(t) &= G * \sum_{i=0}^{\infty} F^{i*}(t) = G * \left(1 + F * \sum_{i=0}^{\infty} F^{i*}\right)(t) \\ &= G(t) + F * \left(G * \sum_{i=0}^{\infty} F^{i*}\right)(t) = G(t) + F * U(t)\,, \quad t \geq 0\,, \end{aligned} \tag{3.11}$$

where we have used the algebraic properties of convolution in the third equality. We have just showed that the renewal function of an arbitrary renewal process, both pure and delayed, satisfies the renewal equation.

Definition 3.6.1. *The convolution equation*

$$f = g + f * F\,,$$

that is

$$f(t) = g(t) + \int_{[0,t]} f(t-y)\mathrm{d}F(y)\,, \quad t \geq 0\,, \tag{3.12}$$

where $f : \mathbb{R}_+ \to \mathbb{R}$ is an unknown function, $g : \mathbb{R}_+ \to \mathbb{R}$ is a known measurable function and F is a known nondecreasing right-continuous function on $\mathbb{R}_+$ such that $\lim_{x\to\infty} F(x) < \infty$ *and $F(0) = 0$, is called the renewal equation.*

Note that in case $\lim_{x\to\infty} F(x) = 1$, F is a distribution function of a nonnegative random variable. Denote $c = \lim_{x\to\infty} F(x)$. Regarding the value of c, a renewal equation is called defective if $c < 1$, proper if $c = 1$ and excessive if $c > 1$.

We now give two more examples of the renewal equation.

Example 3.6.2 (**Forward and backward recurrence times**). Let $(T_i)_{i\geq 0}$ be a renewal process. Define

$$R_F(t) = T_{\nu(t)} - t\,, \qquad \text{for all } t \geq 0\,,$$

and

$$R_B(t) = t - T_{\nu(t)-1}\,, \qquad \text{for all } t \geq T_0\,.$$

The forward recurrence time or overshoot $R_F(t)$ is the time until the next renewal, while the backward recurrence time or undershoot $R_B(t)$ is the time since the last renewal. It can be shown that, in the zero-delayed case, for a fixed $x > 0$

$$\mathbb{P}\left(R_F(t) > x\right) = 1 - F(t+x) + \int_{[0,t]} \mathbb{P}\left(R_F(t-y) > x\right) \mathrm{d}F(y) \tag{3.13}$$

and

$$\mathbb{P}\left(R_B(t) \leq x\right) = (1 - F(t))\mathbb{1}_{[0,x]}(t) + \int_{[0,t]} \mathbb{P}\left(R_B(t-y) \leq x\right) \mathrm{d}F(y)\,, \tag{3.14}$$

where F is the interarrival distribution. The proof is based on the renewal argument, meaning that one conditions on the value in T_1 and then observes that the process starts from scratch at T_1. Indeed,

$$\mathbb{P}(R_F(t) > x) = \mathbb{P}(R_F(t) > x, T_1 > t) + \mathbb{P}(R_F(t) > x, T_1 \leq t)\,,$$

where

$$\mathbb{P}(R_F(t) > x, T_1 > t) = \mathbb{P}(T_1 - t > x, T_1 > t) = \mathbb{P}(T_1 > t + x) = 1 - F(t+x)$$

and

$$\begin{aligned}
\mathbb{P}(R_F(t) > x, T_1 \leq t) &= \mathbb{P}(T_{\nu(t)} - t > x, \nu(t) \geq 2) \\
&= \sum_{i=2}^{\infty} \mathbb{P}(T_i - t > x, T_{i-1} \leq t < T_i) \\
&= \sum_{i=2}^{\infty} \int_{[0,t]} \mathbb{P}(y + \sum_{j=2}^{i} X_i - t > x, y + \sum_{j=2}^{i-1} X_i \leq t < y + \sum_{j=2}^{i} X_i)\mathrm{d}F(y) \\
&= \sum_{i=2}^{\infty} \int_{[0,t]} \mathbb{P}(T_{i-1} - (t-y) > x, T_{i-2} \leq t - y < T_{i-1})\mathrm{d}F(y) \\
&= \sum_{i=2}^{\infty} \int_{[0,t]} \mathbb{P}(T_{\nu(t-y)} - (t-y) > x, \nu(t-y) = i-1)\mathrm{d}F(y) \\
&= \int_{[0,t]} \mathbb{P}(R_F(t-y) > x, \nu(t-y) \geq 1)\mathrm{d}F(y) \\
&= \int_{[0,t]} \mathbb{P}\left(R_F(t-y) > x\right) \mathrm{d}F(y)\,.
\end{aligned}$$

This shows relation (3.13). One similarly obtains (3.14), for the details see e.g. Example 3.5.3 in [Res92].

Actually, many renewal quantities may be described as solutions to some renewal equation, hence it is only natural to ask about existence and uniqueness of a solution. Recall that a function $g : \mathbb{R}_+ \to \mathbb{R}$ is called locally bounded if g is bounded on finite intervals. The following theorem is proved in e.g. [Res92], see Theorem 3.5.1.

Theorem 3.6.3. *Suppose $g : \mathbb{R}_+ \to \mathbb{R}$ is a locally bounded function, then a unique locally bounded solution of the renewal equation (3.12) is given by*

$$U * g(t) = \int_{[0,t]} g(t-y)\mathrm{d}U(y)\,.$$

Note that, in case of a directly Riemann integrable function g, the key renewal theorem describes the asymptotic behavior of the unique solution of the renewal equation.

Chapter 4

Renewal cluster point processes

This chapter is the main part of the thesis in which results from the standard renewal theory are extended to renewal cluster point processes. Following [Mik09], see Chapter 11, we give a simple and intuitive interpretation of points of a renewal cluster point process in terms of insurance mathematics. Assume that at every point of a renewal point process a cluster of activities start. We interpret these points as the arrival times of claims and think of them as the cluster centers. The i-th claim triggers a random stream of payments from the insurer to the insured and the times at which the payments are executed are the members of the i-th cluster. The superposition of the cluster members constitutes the renewal cluster point process.

4.1 Marked renewal point processes

In this section we first construct a stationary marked renewal point process and then give a generalization of the so-called extended renewal theorem (Theorem 3.5.1) to marked renewal point processes, which will be an initial step in the further generalization to the renewal cluster point processes.

Assume that $(\widetilde{W}_i)_{i\geq 0}$ is an i.i.d. $\mathbb{M}$-valued sequence, where $\mathbb{M}$ is a general Polish space. Furthermore, assume that for some measurable function $\varphi : \mathbb{M} \to \mathbb{R}_+$, nonarithmetic random variables

$$\widetilde{X}_i = \varphi(\widetilde{W}_i)\,, \quad i \geq 1\,,$$

are given and satisfy $\mu = \mathbb{E}\widetilde{X}_1 \in \langle 0, \infty \rangle$. Define

$$\widetilde{T}_0 = 0 \quad \text{and} \quad \widetilde{T}_i = \sum_{j=1}^{i} \widetilde{X}_j\,, \quad i \geq 1\,.$$

We now form a two-sided renewal process with arrival times $\cdots < \widetilde{T}_{-2} < \widetilde{T}_{-1} < \widetilde{T}_0 < \widetilde{T}_1 < \widetilde{T}_2 \cdots$, where $\widetilde{T}_0 = 0$, by extending the pure renewal process $(\widetilde{T}_i)_{i \geq 0}$ on $\mathbb{R}_+$ in the obvious way. Precisely, let $\widetilde{W}_{-i}$ be an independent copy of $\widetilde{W}_i$, for all $i \geq 1$, and define

$$\widetilde{T}_{-i} = -\sum_{j=-i+1}^{0} \widetilde{X}_j\,, \quad i \geq 1\,,$$

where $\widetilde{X}_{-i} = \varphi(\widetilde{W}_{-i})$. Finally, set

$$\tilde{\eta} = \sum_{i \in \mathbb{Z}} \delta_{(\widetilde{T}_i, \widetilde{W}_i)}$$

and note that such process is clearly point stationary by construction. It might be even considered as a simplest class of point stationary processes on $\mathbb{R} \times \mathbb{M}$.

Remark 4.1.1. Alternatively, we may consider i.i.d. pairs $(\widetilde{W}_i, \widetilde{X}_i) \in \mathbb{M} \times \mathbb{R}_+$, as in [Mar15]. However, the two formulations are equivalent, since $\mathbb{M} \times \mathbb{R}_+$ is also Polish and we may use the projection on the second coordinate instead of φ. In particular, this implies that one can assume that φ above is continuous with no loss of generality in the sequel.

We start the construction of the stationary version of $\tilde{\eta}$ by picking the renewal interval in which origin will land. However, one must be careful and observe that origin is more likely to fall into a long renewal interval than into a short one. This phenomenon is known as size-biasing (see e.g. [AGK19]). Therefore, we define an additional random element W^* in $\mathbb{M}$ independent of the sequence $(\widetilde{W}_i)_{i \neq 0}$ with the distribution $\widetilde{W}_1$ biased by $\widetilde{X}_1$, or more precisely

$$\mathbb{E}f(W^*) = \frac{1}{\mu}\mathbb{E}\left[\widetilde{X}_1 f(\widetilde{W}_1)\right]\,,$$

for all bounded measurable real-valued functions f. Denote $X^* = \varphi(W^*)$. Next, we randomly pick the relative position of the origin in the landing interval of the length X^*. To make this precise, let U denote an uniform random variable on $[0, 1]$ independent of $W^*, (\widetilde{W}_i)_{i \neq 0}$ and set

$$T_0 = UX^* \quad \text{and} \quad T_{-1} = -(1-U)X^*\,. \tag{4.1}$$

Now we define recursively

$$T_i = T_0 + \sum_{j=1}^{i} \widetilde{X}_j\,, \quad T_{-(i+1)} = T_{-1} - \sum_{j=-i}^{-1} \widetilde{X}_j\,, \qquad i \geq 1 \tag{4.2}$$

and set

$$W_0 = W^*\,, \quad W_i = \widetilde{W}_i\,, \quad \text{for all } i \neq 0\,. \tag{4.3}$$

Then

$$\eta = \sum_{i\in\mathbb{Z}} \delta_{(T_i,W_i)} \tag{4.4}$$

is a stationary marked renewal point process. Indeed, one can check that an alternative representation of the distribution of η is given by (2.4), i.e.

$$\mathbb{E}f(\eta) = \frac{1}{\mu}\mathbb{E}\int_0^{\widetilde{T}_1} f\left(\sum_i \delta_{(\widetilde{T}_i - t, \widetilde{W}_i)}\right)\mathrm{d}t\,,$$

for an arbitrary nonnegative measurable function on $M_p(\mathbb{R}\times\mathbb{M})$. Then, stationarity of η follows as observed in Section 2.6.

Let point process $\hat{\eta} = \sum_{i\in\mathbb{Z}} \delta_{(\hat{T}_i,\hat{W}_i)}$ with $\ldots \hat{T}_{-1} < 0 \leq \hat{T}_0 < \hat{T}_1 < \ldots$ satisfy (2.4) and denote $\hat{X}_0 = \hat{T}_0$, $\hat{X}_{-1} = -\hat{T}_{-1}$, $\hat{X}_i = \hat{T}_i - \hat{T}_{i-1}$ and $\hat{X}_{-(i+1)} = -(\hat{T}_{-(i+1)} - \hat{T}_{-i})$, for $i \geq 1$. Let $x_{-i}, \ldots, x_i \in \mathbb{R}_+$ be arbitrary nonnegative numbers and $A_{-i}, \ldots, A_i$ arbitrary sets in $\mathcal{B}(\mathbb{M})$, $i \geq 0$. Observe that

$$\begin{aligned}
&\mathbb{P}(\hat{X}_{-i} \leq x_{-i}, \hat{W}_{-i} \in A_{-i}, \ldots, \hat{X}_i \leq x_i, \hat{W}_i \in A_i) \\
&= \frac{1}{\mu}\mathbb{E}\int_0^{\widetilde{T}_1} \mathbb{1}_{\{\widetilde{X}_{-i+2} \leq x_{-i}\}} \mathbb{1}_{\{\widetilde{W}_{-i+1} \in A_{-i}\}} \cdots \mathbb{1}_{\{\widetilde{X}_0 \leq x_{-2}\}} \mathbb{1}_{\{\widetilde{W}_{-1} \in A_{-2}\}} \mathbb{1}_{\{t \leq x_{-1}\}} \mathbb{1}_{\{\widetilde{W}_0 \in A_{-1}\}} \\
&\qquad \mathbb{1}_{\{\widetilde{T}_1 - t \leq x_0\}} \mathbb{1}_{\{\widetilde{W}_1 \in A_0\}} \mathbb{1}_{\{\widetilde{X}_2 \leq x_1\}} \mathbb{1}_{\{\widetilde{W}_2 \in A_1\}} \cdots \mathbb{1}_{\{\widetilde{X}_{i+1} \leq x_i\}} \mathbb{1}_{\{\widetilde{W}_{i+1} \in A_i\}} \mathrm{d}t \\
&= \frac{1}{\mu}\mathbb{E}\Big[\mathbb{1}_{\{\widetilde{X}_{-i+2} \leq x_{-i}\}} \mathbb{1}_{\{\widetilde{W}_{-i+1} \in A_{-i}\}} \cdots \mathbb{1}_{\{\widetilde{X}_0 \leq x_{-2}\}} \mathbb{1}_{\{\widetilde{W}_{-1} \in A_{-2}\}} \mathbb{1}_{\{\widetilde{W}_0 \in A_{-1}\}} \mathbb{1}_{\{\widetilde{W}_1 \in A_0\}} \\
&\qquad \mathbb{1}_{\{\widetilde{X}_2 \leq x_1\}} \mathbb{1}_{\{\widetilde{W}_2 \in A_1\}} \cdots \mathbb{1}_{\{\widetilde{X}_{i+1} \leq x_i\}} \mathbb{1}_{\{\widetilde{W}_{i+1} \in A_i\}} \int_0^{\widetilde{X}_1} \mathbb{1}_{\{t \leq x_{-1}, \widetilde{X}_1 - t \leq x_0\}} \mathrm{d}t\Big] \\
&= \frac{1}{\mu}\mathbb{P}(\widetilde{X}_{-i+2} \leq x_{-i})\mathbb{P}(\widetilde{X}_{-i+1} \leq x_{-i-1}, \widetilde{W}_{-i+1} \in A_{-i}) \cdots \mathbb{P}(\widetilde{X}_0 \leq x_{-2}, \widetilde{W}_0 \in A_{-1}) \\
&\qquad \mathbb{P}(\widetilde{X}_2 \leq x_1, \widetilde{W}_2 \in A_1) \cdots \mathbb{P}(\widetilde{X}_{i+1} \leq x_i, \widetilde{W}_{i+1} \in A_i)\mathbb{E}\Big[\mathbb{1}_{\{\widetilde{W}_1 \in A_0\}} \int_{(\widetilde{X}_1 - x_0)_+}^{\widetilde{X}_1 \wedge x_{-1}} \mathrm{d}t\Big]
\end{aligned}$$

where we have used (2.4) in the first equality and the independence of $(\widetilde{X}_i, \widetilde{W}_i)$, for all i, in the last equality. As usual in the literature, symbol $\wedge$ denotes minimum, while $\vee$

denotes maximum. Furthermore,

$$\begin{aligned}
\frac{1}{\mu}\mathbb{E}\Big[\mathbb{1}_{\{\widetilde{W}_1\in A_0\}}\int_{(\widetilde{X}_1-x_0)_+}^{\widetilde{X}_1\wedge x_{-1}}\mathrm{d}t\Big] &= \frac{1}{\mu}\mathbb{E}\Big[\mathbb{1}_{\{\widetilde{W}_1\in A_0\}}\widetilde{X}_1\int_{\left(1-\frac{x_0}{\widetilde{X}_1}\right)_+}^{1\wedge\frac{x_{-1}}{\widetilde{X}_1}}\mathrm{d}u\Big]\\
&= \frac{1}{\mu}\mathbb{E}\Big[\mathbb{1}_{\{\widetilde{W}_1\in A_0\}}\widetilde{X}_1\int_0^1 \mathbb{1}_{\{\widetilde{X}_1\le\frac{x_0}{1-u}\}}\mathbb{1}_{\{\widetilde{X}_1\le\frac{x_{-1}}{u}\}}\mathrm{d}u\Big]\\
&= \int_0^1\frac{1}{\mu}\mathbb{E}\Big[\widetilde{X}_1\mathbb{1}_{\{\widetilde{W}_1\in A_0\}}\mathbb{1}_{\{\widetilde{X}_1\le\frac{x_0}{1-u}\}}\mathbb{1}_{\{\widetilde{X}_1\le\frac{x_{-1}}{u}\}}\Big]\mathrm{d}u\\
&= \int_0^1\mathbb{P}\Big(X^*\le\frac{x_0}{1-u}, X^*\le\frac{x_{-1}}{u}, W^*\in A_0\Big)\mathrm{d}u\\
&= \mathbb{P}((1-U)X^*\le x_0, UX^*\le x_{-1}, W^*\in A_0)\\
&= \mathbb{P}(T_0\le x_0, -T_{-1}\le x_{-1}, W_0\in A_0)\,,
\end{aligned}$$

where the last equality follows from (4.1) and (4.3). Since $(\widetilde{X}_i,\widetilde{W}_i)_i$ is an i.i.d. sequence and moreover $(\widetilde{X}_i,\widetilde{W}_i)_{i\neq 0}$ is independent of T_{-1},T_0 and W_0, we have shown that

$$\begin{aligned}
&\mathbb{P}(\hat{X}_{-i}\le x_{-i},\hat{W}_{-i}\in A_{-i},\dots,\hat{X}_i\le x_i,\hat{W}_i\in A_i)\\
&\quad=\mathbb{P}(\widetilde{X}_{-i+1}\le x_{-i},\widetilde{W}_{-i}\in A_{-i},\dots,\widetilde{X}_{-1}\le x_{-2},\widetilde{W}_{-2}\in A_{-2},-T_{-1}\le x_{-1},\widetilde{W}_{-1}\in A_{-1},\\
&\qquad\quad T_0\le x_0,W_0\in A_0,\widetilde{X}_1\le x_1,\widetilde{W}_1\in A_1,\dots,\widetilde{X}_i\le x_i,\widetilde{W}_i\in A_i)\,,
\end{aligned}$$

which implies that

$$(\hat{T}_{-i},\hat{W}_{-i},\dots,\hat{T}_i,\hat{W}_i)\stackrel{d}{=}(T_{-i},W_{-i},\dots,T_i,W_i)\,,$$

for all $i\ge 0$. Now, as a consequence of Theorem 2.3.11 $(ii)\Rightarrow(i)$, we conclude that

$$\hat{\eta}\stackrel{d}{=}\eta\,.$$

We now turn to the announced generalization of the extended renewal theorem to marked renewal point processes. Consider a stationary marked renewal point process

$$\eta=\sum_{i\in\mathbb{Z}}\delta_{(T_i,W_i)}$$

introduced in (4.4) above. Recall that $(W_i)_{i\neq 0}$ is an i.i.d. $\mathbb{M}$-valued sequence and that for some measurable $\varphi:\mathbb{M}\to\mathbb{R}_+$, $X_i=T_i-T_{i-1}$, $i\neq 0$ satisfy $X_i=\varphi(W_i)$, $i\in\mathbb{Z}$.

Furthermore, let (X_0',W_0') be a random element in $\mathbb{R}_+\times\mathbb{M}$ which is independent of an i.i.d. $\mathbb{M}$-valued sequence $(W_i')_{i\ge 1}$ with the same distribution as $\widetilde{W}_1$. Denote $X_i'=\varphi(W_i')$, for all $i\ge 1$ and $T_i'=\sum_{j=0}^i X_j'$, for all $i\ge 0$. Consider now marked renewal point process

$$\eta'=\sum_{i=0}^{\infty}\delta_{(T_i',W_i')}\,.$$

We sometimes identify η' with the sequence $(T_i', W_i')_{i\geq 0}$.

In the context of Theorem 3.5.1, the following lemma is not overly surprising. Actually, we learnt that it was already proved analytically by Marynych for a very complicated but special space of marks (see Lemma 3.1 in [Mar15]). However, we give a completely different probabilistic proof based on the coupling method.

Proposition 4.1.2 (Extended renewal theorem for marked renewal point processes)**.** *Under the assumptions above, it holds that*

$$\theta_t \eta' \xrightarrow{d} \eta\,,$$

as $t \to \infty$ with respect to the vague topology on $M_p(\mathbb{R} \times \mathbb{M})$.

To prove Proposition 4.1.2, we need two auxiliary results. Before we state the first one, recall that a discrete random variable ϑ is said to be Rademacher if

$$\mathbb{P}(\vartheta = 1) = \mathbb{P}(\vartheta = -1) = 1/2$$

and a sequence of independent Rademacher variables is called a Rademacher sequence, see e.g. Chapter 4 in [LT91].

Lemma 4.1.3. *Suppose that β is a random element in some Polish space $\mathbb{M}$ independent of a Rademacher sequence $(\vartheta_k)_{k\in\mathbb{N}}$ and let $\mathcal{F}_n = \sigma\{\beta, \vartheta_1, \dots, \vartheta_n\}$. Assume that τ is $\{\mathcal{F}_n\}$–stopping time, then the sequence*

$$(\beta, \vartheta_1, \dots, \vartheta_\tau, -\vartheta_{\tau+1}, -\vartheta_{\tau+2}, \dots)$$

has the same distribution as $(\beta, \vartheta_1, \vartheta_2, \vartheta_3, \dots)$.

Proof. Select arbitrary $i_1, i_2, \dots, i_n \in \{-1, 1\}$ and observe that both $A_n = \{\tau > n-1\}$ and $A_n^c = \{\tau < n\}$ are in $\mathcal{F}_{n-1}$ and therefore independent of ϑ_n. Hence for any measurable $B \subseteq \mathbb{M}$

$$\begin{aligned}
&\mathbb{P}\left(\vartheta_j(-1)^{\mathbb{1}_{\{\tau<j\}}} = i_j,\ j = 1, 2, \dots, n, \beta \in B\right)\\
&\quad = \mathbb{P}\left(\vartheta_n = i_n\,,\, \vartheta_j(-1)^{\mathbb{1}_{\{\tau<j\}}} = i_j,\ j = 1, 2, \dots, n-1, \beta \in B, A_n\right)\\
&\quad + \mathbb{P}\left(\vartheta_n = -i_n\,,\, \vartheta_j(-1)^{\mathbb{1}_{\{\tau<j\}}} = i_j,\ j = 1, 2, \dots, n-1, \beta \in B, A_n^c\right)\\
&\quad = \mathbb{P}\left(\vartheta_n = i_n\right) \mathbb{P}\left(\vartheta_j(-1)^{\mathbb{1}_{\{\tau<j\}}} = i_j,\ j = 1, 2, \dots, n-1, \beta \in B, A_n\right)\\
&\quad + \mathbb{P}\left(\vartheta_n = -i_n\right) \mathbb{P}\left(\vartheta_j(-1)^{\mathbb{1}_{\{\tau<j\}}} = i_j,\ j = 1, 2, \dots, n-1, \beta \in B, A_n^c\right)\\
&\quad = \mathbb{P}\left(\vartheta_n = i_n\right) \mathbb{P}\left(\vartheta_j(-1)^{\mathbb{1}_{\{\tau<j\}}} = i_j,\ j = 1, 2, \dots, n-1, \beta \in B\right)\,.
\end{aligned}$$

Repeating the argument for $j = 1, 2, \ldots, n-1$, yields the claim. $\square$

Note that, as kindly suggested by prof. Alexander V. Marynych, the above result is valid for any sequence of i.i.d. random variables with a symmetric distribution.

Lemma 4.1.4. *Let $\mathbb{M}$ be an arbitrary Polish space. Assume that for an arbitrary non-negative continuous real-valued function f on $\mathbb{M}$ and a sequence $(t_0^{(n)}, (w_i^{(n)})_{i\geq 0})_{n\geq 1}$ in $\mathbb{R}_+ \times \mathbb{M}^{\mathbb{N}_0}$ the following conditions hold:*

(i) $t_0^{(n)} \to t_0$ *in* $\mathbb{R}_+$ *and* $w_i^{(n)} \to w_i$ *in* $\mathbb{M}\,,$ *for all* $i \geq 0\,,$ *as* $n \to \infty\,,$

(ii) $\sum_{j=1}^{i} f(w_j^{(n)}) \to \infty$ *and* $\sum_{j=1}^{i} f(w_j) \to \infty\,,$ *for all* $n \in \mathbb{N}\,,$ *as* $i \to \infty\,.$

Then

$$\sum_{i=0}^{\infty} \delta_{(t_0^{(n)}+\sum_{j=1}^{i} f(w_j^{(n)}), w_i^{(n)})} \xrightarrow{v} \sum_{i=0}^{\infty} \delta_{(t_0+\sum_{j=1}^{i} f(w_j), w_i)}\,,$$

as $n \to \infty$.

Proof. Denote

$$m_n = \sum_{i=0}^{\infty} \delta_{(t_0^{(n)}+\sum_{j=1}^{i} f(w_j^{(n)}), w_i^{(n)})}\,, \text{ for all } n \in \mathbb{N} \quad \text{and} \quad m = \sum_{i=0}^{\infty} \delta_{(t_0+\sum_{j=1}^{i} f(w_j), w_i)}\,.$$

From assumption (ii), it follows that integer-valued measures m_n and m are locally finite on $\mathbb{R}_+ \times \mathbb{M}$. Furthermore, we conclude from (i) and (ii) that for an arbitrary $h > 0$ such that $m(\partial([0,h] \times \mathbb{M})) = 0$, there exist $k, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$m_n\Big|_{[0,h]\times\mathbb{M}} = \sum_{i=0}^{k} \delta_{(t_0^{(n)}+\sum_{j=1}^{i} f(w_j^{(n)}), w_i^{(n)})} \quad \text{and} \quad m\Big|_{[0,h]\times\mathbb{M}} = \sum_{i=0}^{k} \delta_{(t_0+\sum_{j=1}^{i} f(w_j), w_i)}\,.$$

Moreover, by (i)

$$t_0^{(n)} + \sum_{j=1}^{i} f(w_j^{(n)}) \to t_0 + \sum_{j=1}^{i} f(w_j) \text{ in } \mathbb{R}_+ \quad \text{and} \quad w_i^{(n)} \to w_i \text{ in } \mathbb{M}\,,$$

for all $i = 0, 1, \ldots, k$, since f is continuous. Thus, by Proposition 2.2.5

$$m_n \xrightarrow{v} m\,,$$

as $n \to \infty$. $\square$

We are now in position to prove Proposition 4.1.2.

Proof of Proposition 4.1.2. Like several other proofs of different types of renewal theorem, we base our argument on the coupling method. In the initial steps, it follows a coupling of the type used by Thorisson [Tho00] in sections 2.7 and 2.8 and Kallenberg [Kal17] in the proof of Theorem 12.7.

Consider an i.i.d. sequence $(\widetilde{W}_i, \widetilde{X}_i)_{i\geq 1}$ with the same distribution as $(W_i, X_i)_{i\geq 1}$ and an independent Rademacher sequence $(\vartheta_i)_{i\geq 1}$.

Step 1. We first construct copies of η and η' on the same probability space. Assume that (W_0, X_0) and (W_0', X_0') have the desired distribution, but are independent of the sequences $(\widetilde{W}_i, \widetilde{X}_i)_{i\geq 1}$ and $(\vartheta_i)_{i\geq 1}$. Set $K_0 = K_0'' = 0$ and then for $i \geq 1$

$$K_i = \inf\{k > K_{i-1} \,:\, \vartheta_k = 1\} \quad \text{and} \quad K_i'' = \inf\{k > K_{i-1}'' \,:\, \vartheta_k = -1\}\,.$$

Alternatively, one can view K_i and K_i'' as the indices of the i-th 1 and -1, respectively, in the sequence $(\vartheta_i)_{i\geq 1}$. Let $T_0 = X_0$, $T_0'' = X_0'$ and $W_0'' = W_0'$. Then, for $i \geq 1$ let

$$T_i = T_0 + \sum_{j=1}^{K_i} \widetilde{X}_j \mathbb{1}_{\{\vartheta_j = 1\}} = T_0 + \sum_{j=1}^{i} \widetilde{X}_{K_j} =: T_{i-1} + Y_i\,, \quad \text{and} \quad W_i = \widetilde{W}_{K_i}\,, \tag{4.5}$$

$$T_i'' = T_0'' + \sum_{j=1}^{K_i''} \widetilde{X}_j \mathbb{1}_{\{\vartheta_j = -1\}} = T_0'' + \sum_{j=1}^{i} \widetilde{X}_{K_j''}\,, \quad \text{and} \quad W_i'' = \widetilde{W}_{K_i''}\,. \tag{4.6}$$

Note that $Y_i = \widetilde{X}_{K_i}$ in (4.5) are i.i.d., for all $i \geq 1$. As explained above, one can extend the sequence $(T_i, W_i)_{i\geq 0}$ to a sequence indexed over $\mathbb{Z}$ so that the point process $\eta = \sum_{i\in\mathbb{Z}} \delta_{(T_i, W_i)}$ becomes stationary. It is immediate then that η and

$$\eta'' = \sum_{i\geq 0} \delta_{(T_i'', W_i'')}$$

have the same distribution as the η and η' in the statement of the theorem.

Step 2. Set

$$L_i = |\{k \leq i : \vartheta_k = 1\}| \quad \text{and} \quad L_i' = i - L_i\,, \quad i \geq 0\,,$$

where $|\cdot|$ denotes cardinality of a set. By (4.5) and (4.6), the difference between the arrival times T_{L_i} and $T''_{L_i'}$ equals

$$V_i = T_0 - T_0'' + \sum_{j=1}^{i} \vartheta_j \widetilde{X}_j\,, \quad i \geq 0\,.$$

In particular $(V_i)_{i\geq 0}$ is a random walk with nonarithmetic and symmetric steps. By the Chung-Fuchs theorem, see Theorem 4 in [CF51], $(V_i)_{i\geq 0}$ is recurrent and for all $x \in \mathbb{R}$ and

$\varepsilon > 0$

$$\mathbb{P}(|V_i - x| < \varepsilon \text{ i.o.}) = 1\,.$$

Therefore, for any $\varepsilon > 0$, with probability 1

$$\tau = \inf\{i : V_i \in [0, \varepsilon\rangle\} < \infty\,.$$

Step 3. Let now

$$\begin{aligned}
&T'_i = T''_i \quad \text{and} \quad W'_i = W''_i\,, && \text{if } i \le L'_\tau \\
&T'_i = T'_{i-1} + Y_{k+L_\tau} = T'_{i-1} + \widetilde{X}_{K_{k+L_\tau}} \quad \text{and} \quad W'_i = W_{k+L_\tau}\,, && \text{if } i = L'_\tau + k\,, k > 0\,.
\end{aligned}$$

Then in particular, for any $k \ge 0$

$$T_{k+L_\tau} - T'_{k+L'_\tau} \in [0, \varepsilon\rangle \quad \text{and} \quad W_{k+L_\tau} = W'_{k+L'_\tau}\,.$$

That is, after coming ε–close at times L_τ and L'_τ, the marked point processes η and

$$\eta' = \sum_{i \ge 0} \delta_{(T'_i, W'_i)}$$

stay ε–close in time with exactly the same marks indefinitely. Denote now by $(\widetilde{\vartheta}_i)_{i\ge1}$ Rademacher sequence

$$\vartheta_1, \vartheta_2, \vartheta_3, \ldots, \vartheta_\tau, -\vartheta_{\tau+1}, -\vartheta_{\tau+2}, -\vartheta_{\tau+3}, \ldots$$

Applying Lemma 4.1.3 to the sequence $\beta = (\widetilde{W}_i, \widetilde{X}_i)_{i\ge1}$, the sequence $(\vartheta_i)_{i\ge1}$ and stopping time τ, we see that $(\beta, \widetilde{\vartheta}_1, \widetilde{\vartheta}_2, \ldots)$ has the same distribution as $(\beta, \vartheta_1, \vartheta_2, \ldots)$. Let $K'_0 = 0$ and $K'_i = \inf\{k > K'_{i-1} : \widetilde{\vartheta}_k = -1\}$, $i \ge 1$. Since

$$T'_i = T'_0 + \sum_{j=1}^{K'_i} \widetilde{X}_j \mathbb{1}_{\{\widetilde{\vartheta}_j = -1\}} \quad \text{and} \quad W'_i = \widetilde{W}_{K'_i}\,,$$

a quick comparison with (4.6) shows that η' has the same distribution as in the statement of the theorem.

Step 4. Observe that $V_\varepsilon := T_{L_\tau} \vee T''_{L'_\tau} < \infty$ a.s. Denote for $t > 0$

$$\sigma(t) = \inf\{k \ge 0 : T_k > t\} \quad \text{and} \quad \sigma'(t) = \inf\{k \ge 0 : T'_k > t\}\,.$$

For arbitrary $u \ge 0$ and a measurable set $A_0 \subseteq \mathbb{M}$, we have

$$\begin{aligned}
&\mathbb{P}\left(T_{\sigma(t)} - t > u + \varepsilon,\, W_{\sigma(t)} \in A_0\right) - \mathbb{P}(V_\varepsilon > t) \\
&\quad \le \ \mathbb{P}\left(T'_{\sigma'(t)} - t > u,\, W'_{\sigma'(t)} \in A_0\right) \\
&\quad \le \mathbb{P}\left(T_{\sigma(t)} - t \in [0, \varepsilon\rangle\right) + \mathbb{P}\left(T_{\sigma(t)} - t > u,\, W_{\sigma(t)} \in A_0\right) + \mathbb{P}(V_\varepsilon > t)\,.
\end{aligned} \tag{4.7}$$

Clearly, $\mathbb{P}(V_\varepsilon > t) \to 0$ as $t \to \infty$. Moreover, by the stationarity of the process η

$$\mathbb{P}\left(T_{\sigma(t)} - t > u,\, W_{\sigma(t)} \in A_0\right) = \mathbb{P}\left(T_0 > u,\, W_0 \in A_0\right) .$$

Observe further that the function $u \mapsto \mathbb{P}\left(T_0 \le u,\, W_0 \in A_0\right)$ is continuous, for any choice of A_0. Therefore, letting first $t \to \infty$ and then $\varepsilon \to 0$ on the left and right hand side of (4.7), yields

$$\left(T'_{\sigma'(t)} - t, W'_{\sigma'(t)}\right) \xrightarrow{d} (T_0, W_0) \,, \tag{4.8}$$

as $t \to \infty$. On the other hand, observe that

$$\left(W'_{\sigma'(t)+i}\right)_{i\ge1} \stackrel{d}{=} (W_i)_{i\ge1} \tag{4.9}$$

with the left hand side independent of $\left(T'_{\sigma'(t)} - t, W'_{\sigma'(t)}\right)$ and the right hand side independent of (T_0, W_0).

Step 5. Because $\mathbb{R} \times \mathbb{M}$ is Polish space, by the Skorokhod's representation theorem (see e.g. [Kal21], Theorem 5.31), there exists a probability space where one can find random elements $(\hat{T}'_{\sigma'(t)} - t, \hat{W}_{\sigma'(t)}, (\hat{W}_{\sigma'(t)+i})_{i\ge1})$, $t > 0$, and $(\hat{T}_0, \hat{W}_0, (\hat{W}_i)_{i\ge1})$ which have the same joint distribution as the random elements in (4.8) and (4.9), but satisfy

$$\left(\hat{T}'_{\sigma'(t)} - t, (\hat{W}_{\sigma'(t)+i})_{i\ge0}\right) \xrightarrow{a.s.} \left(\hat{T}_0, (\hat{W}_i)_{i\ge0}\right) ,$$

as $t \to \infty$, in the product topology on $\mathbb{R} \times \mathbb{M}^{\mathbb{N}_0}$. Moreover, by the strong law of large numbers (see e.g. Theorem 2.4.1 in [Dur10])

$$\sum_{j=1}^{i} \varphi(\hat{W}_{\sigma'(t)+j}) = \sum_{j=1}^{i} \hat{X}_{\sigma'(t)+j} \quad \text{and} \quad \sum_{j=1}^{i} \varphi(\hat{W}_j) = \sum_{j=1}^{i} \hat{X}_j$$

tend to ∞ almost surely, as $i \to \infty$. Observe that by Lemma 4.1.4

$$\begin{aligned} \theta_t \eta' &\stackrel{d}{=} \sum_{i=0}^{\infty} \delta_{\left(\hat{T}'_{\sigma'(t)} - t + \sum_{j=1}^{i} \varphi\left(\hat{W}'_{\sigma'(t)+j}\right), \hat{W}'_{\sigma'(t)+i}\right)} \\ &\xrightarrow{a.s.} \sum_{i=0}^{\infty} \delta_{\left(\hat{T}_0 + \sum_{j=1}^{i} \varphi(\hat{W}_j), \hat{W}_i\right)} \stackrel{d}{=} \eta\Big|_{[0,\infty\rangle \times \mathbb{M}} , \end{aligned}$$

as $t \to \infty$. This now yields the statement of the theorem on the state space $[0, \infty\rangle \times \mathbb{M}$.

Step 6. To extend convergence in $M_p([0, \infty\rangle \times \mathbb{M})$ to the convergence in distribution with respect to the vague topology on $M_p(\mathbb{R} \times \mathbb{M})$, consider an arbitrary continuous bounded function $f : \mathbb{R} \times \mathbb{M} \to \mathbb{R}_+$ with a support on a set of the form $[-h, h] \times \mathbb{M}$, for some $h > 0$.

Introduce $f^h(u,w) = f(u-h,w)$ which is clearly continuous bounded and supported on $[0,2h] \times \mathbb{M}$. Now for any fixed $h>0$ as $t \to \infty$

$$\begin{aligned}
\mathbb{E}\exp\left(-\theta_t\eta'(f)\right) &= \mathbb{E}\exp\left(-\sum_{i\geq -\sigma'(t)} f\left(T'_{\sigma'(t)+i}-t, W'_{\sigma'(t)+i}\right)\right) \\
&= \mathbb{E}\exp\left(-\sum_{i\geq -\sigma'(t)} f^h\left(T'_{\sigma'(t)+i}-(t-h), W'_{\sigma'(t)+i}\right)\right) \\
&= \mathbb{E}\exp\left(-\sum_{i\geq 0} f^h\left(T'_{\sigma'(t-h)+i}-(t-h), W'_{\sigma'(t-h)+i}\right)\right) \\
&= \mathbb{E}\exp\left(-\theta_{t-h}\eta'(f^h)\right) \longrightarrow \mathbb{E}\exp\left(-\eta(f^h)\right) \\
&= \mathbb{E}\exp\left(-\sum_{i\geq 0} f^h\left(T_i, W_i\right)\right) = \mathbb{E}\exp\left(-\sum_{i\in\mathbb{Z}} f^h\left(T_i, W_i\right)\right) \\
&= \mathbb{E}\exp\left(-\sum_{i\in\mathbb{Z}} f\left(T_i-h, W_i\right)\right) = \mathbb{E}\exp\left(-\eta(f)\right) ,
\end{aligned}$$

where the last equality follows by the stationarity of the process η. Since f was arbitrary, by Theorem 2.4.2, this concludes the proof. □

4.2 Fundamentals of renewal cluster point processes

We now give a detailed description of the central object of interest to which we extend the renewal theorems from the previous chapter. Let $(X'_i)_{i\geq 1}$ be a sequence of nonnegative random variables with a nonarithmetic distribution and a finite strictly positive mean μ and let $(\xi'_i)_{i\geq 1}$ be a sequence of point processes on $\mathbb{R}$ such that the pairs $W'_i = (\xi'_i, X'_i)$ are independent and identically distributed. Note that ξ'_1 and X'_1 are allowed to be dependent. Assume further that a random pair $W'_0 = (\xi'_0, X'_0)$ also takes values in $M_p(\mathbb{R}) \times \mathbb{R}_+$ and is independent of the i.i.d. sequence $(W'_i)_{i\geq 1}$. Let $(T'_i)_{i\geq 0}$ be a renewal sequence with increments X'_i, i.e.

$$T'_i = \sum_{j=0}^{i} X'_j\,, \quad i \geq 0\,.$$

The associated renewal cluster point process is given by

$$\xi'(B) = \sum_{i\geq 0} \xi'_i(B - T'_i)\,, \quad B \in \mathcal{B}(\mathbb{R})\,, \tag{4.10}$$

assuming that almost surely $\xi'(B) < \infty$, for all $B \in \mathcal{B}_b(\mathbb{R})$. To be in line with the terminology of [BBK20], we call $\zeta' = \sum_{i\geq 0}\delta_{T'_i}$ the parent process and ξ'_i, $i\geq 0$, the descendant process.

Remark 4.2.1. We would like to mention that the definitions of a cluster process differ in the literature. In [BBK20], parent process ζ' is an arbitrary point process on $\mathbb{R}$ and descendant processes are said to be conditionally independent point processes given ζ' with distribution $p(T'_i, \cdot)$ for a probability kernel $p : \mathbb{R} \to M_p(\mathbb{R})$. Even more general cluster process is defined in [Kal17] on page 245, where descendent processes are not even assumed to be point processes but only random measures. It is interesting that these processes can not be considered as generalizations of a renewal cluster point process introduced above. Indeed, assume that a parent process ζ' is a Poisson process on $\mathbb{R}_+$ and let

(*i*) $\xi'_i = \delta_0 + \delta_1$,

(*ii*) $\xi'_i = \delta_0 + \delta_{\mathbb{1}_{\{X'_i \leq 1\}} + 2\mathbb{1}_{\{X'_i > 1\}}}$,

(*iii*) $\xi'_i = \delta_0 + \delta_{\sum_{n=0}^{\infty} \mathbb{1}_{\{2n \leq T'_i < 2n+1\}} + 2\sum_{n=1}^{\infty} \mathbb{1}_{\{2n-1 \leq T'_i < 2n\}}}$,

for all $i \geq 0$. Hence, we are studying a process that consist of the points of the parent process followed by one more point. All three definitions cover (*i*). However, note that ξ'_i defined in (*ii*) essentially depends on the difference of T'_i and T'_{i-1}. On the other hand, ξ'_i in (*iii*) depends on the location of the corresponding point of the parent process and not only of the last interarrival time, hence it does not fit the definition of a renewal cluster point process.

By relation (2.2), each ξ'_i has a representation

$$\xi'_i = \sum_{j=1}^{L'_i} \delta_{T'_{ij}} ,$$

where L'_i is a random variable with values in $\{0, 1, \dots, +\infty\}$ and $T'_{i,1}, \dots, T'_{i,L'_i}$ is a sequence of random variables with values in $\mathbb{R}$. We will assume henceforth that L'_i is almost surely finite, for all i. One can identify

$$W'_i = (\xi'_i, X'_i) \quad \text{with} \quad (L'_i, (T'_{ij})_j, X'_i) , \tag{4.11}$$

for all $i \geq 0$. Thus, we study a renewal cluster point process constructed in such a way that around each point T'_i of the parent process one observes L'_i other points translated by random times $(T'_{ij})_j$. More precisely,

$$\xi'(B) = \sum_{i\geq 0}\sum_{j=1}^{L'_i} \delta_{T'_{ij}}(B - T'_i) = \sum_{i\geq 0}\sum_{j=1}^{L'_i} \delta_{T'_i+T'_{ij}}(B)\,,$$

for every $B \in \mathcal{B}(\mathbb{R})$. Although the points of the parent process ζ' are often unobserved, it is possible to include them in the cluster point process, which will be done occasionally in the sequel. We simply set $T'_{i0} = 0$, for every i and write

$$\xi'_i = \sum_{j=0}^{L'_i} \delta_{T'_{ij}} \quad \text{and} \quad \xi' = \sum_{i\geq 0}\sum_{j=0}^{L'_i} \delta_{T'_i+T'_{ij}}\,.$$

In this case, the $i-$th cluster size is $L'_i + 1$. Proofs in this chapter are carried out without counting the points of the parent process, however they are essentially the same in both cases.

By (4.11), we may regard W'_i as a random element in a Polish space $\mathbb{M} = \mathbb{N}_0 \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}_+$, for all $i \geq 0$. Hence, $\sum_{i\geq 0} \delta_{(T'_i,W'_i)}$ is a marked renewal point process on $\mathbb{R}_+$ with marks in $\mathbb{M}$ and we can, as in the previous section, construct its stationary version $\sum_{i\in\mathbb{Z}} \delta_{(T_i,W_i)}$, where $W_i = (L_i,(T_{ij})_j, X_i)$, for all $i \in \mathbb{Z}$. It now follows that the process

$$\xi = \sum_{i\in\mathbb{Z}}\sum_{j=1}^{L_i} \delta_{T_i+T_{ij}}\,,$$

provided that it is finite a.s. on bounded sets, can be referred to as a stationary renewal cluster point process.

Proposition 4.2.2. *Mean measure of a stationary renewal point process ξ equals*

$$\mathbb{E}\xi(B) = \frac{1}{\mu}\mathbb{E}L_1 Leb(B)\,,$$

for all $B \in \mathcal{B}(\mathbb{R})$.

Proof. First, we introduce the notation. Let $B + x = \{b + x : b \in B\}$, for all $B \in \mathcal{B}_b(\mathbb{R})$ and $x \in \mathbb{R}$.

$$\begin{aligned}
\mathbb{E}\xi(B) &= \mathbb{E}\left[\sum_{i\in\mathbb{Z}} \xi_i(B - T_i)\right] = \mathbb{E}\left[\sum_{i\in\mathbb{Z}} f_B(T_i,\xi_i)\right] = \frac{1}{\mu}\int_{\mathbb{R}} \mathbb{E}[f_B(y,\xi'_1)]\mathrm{d}y\\
&= \frac{1}{\mu}\mathbb{E}\left[\int_{\mathbb{R}} \xi'_1(B-y)\mathrm{d}y\right] = \frac{1}{\mu}\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{s\in B-y\}}\mathrm{d}\xi'_1(s)\mathrm{d}y\right]\\
&= \frac{1}{\mu}\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{y\in B-s\}}\mathrm{d}y\mathrm{d}\xi'_1(s)\right] = \frac{1}{\mu}\mathbb{E}\left[\int_{\mathbb{R}} Leb(B)\mathrm{d}\xi'_1(s)\right]\\
&= \frac{1}{\mu}\mathbb{E}\xi'_1(\mathbb{R})Leb(B) = \frac{1}{\mu}\mathbb{E}L'_1 Leb(B)\,,
\end{aligned}$$

where the third equality follows by the Campbell-Little-Mecke formula (see e.g. relation (1.2.19) in [BB03]) applied to a stationary marked point process $\sum_{i\in\mathbb{Z}} \delta_{(T_i,\xi_i)}$ for a non-negative measurable function $f_B : \mathbb{R} \times M_p(\mathbb{R}) \to \mathbb{R}$ given by $f_B(y,m) = m(B-y)$. Note that $\mathbb{E}\left[\sum_{i\in\mathbb{Z}} \delta_{T_i}[0,1]\right] = \frac{1}{\mu}$, see Theorem 3.2.4. By construction, $X_1 \overset{d}{=} X_1'$ and $L_1 \overset{d}{=} L_1'$, which completes the proof. □

Remark 4.2.3. It follows immediately from the previous proposition that

$$\mathbb{E}L_1 < \infty$$

implies $\mathbb{E}\xi(B) < \infty$, hence $\mathbb{P}(\xi(B) < \infty) = 1$, for all $B \in \mathcal{B}_b(\mathbb{R})$. Thus, finite mean cluster size is a simple sufficient condition for the existence of a stationary renewal cluster point process. However, as stated in [DVJ03] on page 177, this condition is in general not necessary. For instance, assume that at every point of a two-sided Poisson parent process L_i points is observed (i.e. all the points of the descendant process are located at distance 0 from the T_i), where $\mathbb{P}(L_i = n) = \frac{1}{n(n+1)}$, for all $n \in \mathbb{N}$. Obviously, such a process is finite a.s. on bounded sets, but $\mathbb{E}L_i = \infty$, for all i.

4.3 Extended renewal theorem

It is useful in the sequel to introduce the following notation

$$R_i' = \sup_{j\le L_i'} |T_{ij}'|\,,$$

for all $i \geq 0$. Observe that $R_i', i \geq 1$ are i.i.d. nonnegative random variables. Recall that $X_+ := X \vee 0$ is a standard abbreviation for a positive part and $X_- := (-X) \wedge 0$ for a negative part of a general random variable X, which will be used henceforth.

Theorem 4.3.1 (Extended renewal theorem for cluster point processes)**.** *Assume that ξ' is the renewal cluster point process introduced in Section 4.2 and ξ is its stationary version. If $\mathbb{E}R_1'$ is finite, then*

$$\theta_t\xi' \xrightarrow{d} \xi\,,$$

i.e.

$$\sum_{i\ge0}\sum_{j=1}^{L_i'} \delta_{T_i'-t+T_{ij}'} \xrightarrow{d} \sum_{i\in\mathbb{Z}}\sum_{j=1}^{L_i} \delta_{T_i+T_{ij}}\,,$$

as $t \to \infty$ with respect to the vague topology on $M_p(\mathbb{R})$.

Proof. Step 1. Note that the mark space $\mathbb{M} = \mathbb{N}_0 \times \mathbb{R}^{\mathbb{N}} \times [0, \infty\rangle$ of the corresponding marked renewal point processes is again Polish, hence Proposition 4.1.2 can be applied, yielding

$$\sum_{i \geq 0} \delta_{(T'_i - t, W'_i)} \xrightarrow{d} \sum_{i \in \mathbb{Z}} \delta_{(T_i, W_i)} ,$$

as $t \to \infty$ in $M_p(\mathbb{R} \times \mathbb{M})$. By the definition of convergence in distribution,

$$\sum_{i \geq 0} \delta_{(T'_i - t, L'_i, (T'_{ij})_j)} \xrightarrow{d} \sum_{i \in \mathbb{Z}} \delta_{(T_i, L_i, (T_{ij})_j)} , \tag{4.12}$$

as $t \to \infty$ in $M_p(\mathbb{R} \times \mathbb{M}^*)$, where $\mathbb{M}^* = \mathbb{N}_0 \times \mathbb{R}^{\mathbb{N}}$.

Step 2. Fix $K > 0$ and consider the mapping $T_K : M_p(\mathbb{R} \times \mathbb{M}^*) \to M_p(\mathbb{R} \times \mathbb{R})$ given by

$$T_K \left(\sum_i \delta_{(x_i, y_i, (z_{ij})_j)} \right) = \sum_i \sum_{j=1}^{y_i} \delta_{(x_i, z_{ij})} \mathbb{1}_{\{|x_i| \leq K\}} .$$

It is well defined, i.e. $\forall m \in M_p(\mathbb{R} \times \mathbb{M}^*)$, $T_K(m)$ is a locally finite point measure on $\mathbb{R} \times \mathbb{R}$. Denote

$$N_K = \{ m \in M_p(\mathbb{R} \times \mathbb{M}^*) : m(\{\pm K\} \times \mathbb{M}^*) = 0 \} .$$

We will show that T_K is continuous on N_K. Let $m_1, m_2, \ldots \in M_p(\mathbb{R} \times \mathbb{M}^*)$, $m \in N_K$ and $m_n \xrightarrow{v} m$. Then, by Proposition 2.2.5, there exist integers n_0, P and a labelling of the points of m and m_n, $n \geq n_0$ in $[-K, K] \times \mathbb{M}^*$ such that

$$m_n\big|_{[-K,K] \times \mathbb{M}^*} = \sum_{i=1}^{P} \delta_{(x_i^{(n)}, y_i^{(n)}, (z_{ij}^{(n)})_j)} ,$$

$$m\big|_{[-K,K] \times \mathbb{M}^*} = \sum_{i=1}^{P} \delta_{(x_i, y_i, (z_{ij})_j)}$$

and for all $i = 1, 2, \ldots, P$

$$x_i^{(n)} \to x_i , \quad y_i^{(n)} \to y_i , \quad (z_{ij}^{(n)})_j \to (z_{ij})_j ,$$

as $n \to \infty$. Observe that

$$T_K(m_n) = \sum_{i=1}^{P} \sum_{j=1}^{y_i^{(n)}} \delta_{(x_i^{(n)}, z_{ij}^{(n)})} , \qquad \text{for all } n \geq n_0 ,$$

and

$$T_K(m) = \sum_{i=1}^{P} \sum_{j=1}^{y_i} \delta_{(x_i, z_{ij})} ,$$

have only finitely many terms, hence it holds that $T_K(m_n) \to T_K(m)$, as $n \to \infty$. Finally, an application of the continuous mapping theorem (see e.g. [Bil68], Corollary 1, p. 31) to (4.12) yields

$$T_K\Big(\sum_{i\geq 0}\delta_{(T_i'-t,L_i',(T_{ij}')_j)}\Big) \xrightarrow{d} T_K\Big(\sum_{i\in\mathbb{Z}}\delta_{(T_i,L_i,(T_{ij})_j)}\Big)\,,$$

thus

$$\sum_{i\geq 0}\sum_{j=1}^{L_i'}\delta_{(T_i'-t,T_{ij}')}\Big|_{[-K,K]\times\mathbb{R}} \xrightarrow{d} \sum_{i\in\mathbb{Z}}\sum_{j=1}^{L_i}\delta_{(T_i,T_{ij})}\Big|_{[-K,K]\times\mathbb{R}}\,, \tag{4.13}$$

as $t \to \infty$ in $M_p(\mathbb{R}\times\mathbb{R})$. Indeed, from Proposition 8.2 in [LP18], it follows that

$$\mathbb{P}\left(\sum_{i\in\mathbb{Z}}\delta_{(T_i,L_i,(T_{ij})_j)} \in N_K\right) = \mathbb{P}\left(\sum_{i\in\mathbb{Z}}\delta_{T_i}(\{\pm K\}) = 0\right) = 1\,.$$

Step 3. Since addition, as a function from $[-K,K]\times\mathbb{R}$ to $\mathbb{R}$, trivially satisfies condition (2.1), we may apply the continuous mapping theorem to (4.13). Then,

$$\sum_{i\geq 0}\sum_{j=1}^{L_i'}\mathbb{1}_{\{|T_i'-t|\leq K\}}\delta_{T_i'-t+T_{ij}'} \xrightarrow{d} \sum_{i\in\mathbb{Z}}\sum_{j=1}^{L_i}\mathbb{1}_{\{|T_i|\leq K\}}\delta_{T_i+T_{ij}}\,,$$

as $t \to \infty$ in $M_p(\mathbb{R})$.

Taking the point process on the right hand side of the above relation and letting $K \to \infty$, one obtains

$$\sum_{i\in\mathbb{Z}}\sum_{j=1}^{L_i}\mathbb{1}_{\{|T_i|\leq K\}}\delta_{T_i+T_{ij}} \xrightarrow{d} \sum_{i\in\mathbb{Z}}\sum_{j=1}^{L_i}\delta_{T_i+T_{ij}}\,, \tag{4.14}$$

in $M_p(\mathbb{R})$. Indeed, let $f:\mathbb{R}\to\mathbb{R}_+$ be an arbitrary continuous bounded function with a bounded support. Then, by the monotone convergence theorem,

$$\sum_{i\in\mathbb{Z}}\sum_{j=1}^{L_i}\mathbb{1}_{\{|T_i|\leq K\}}f(T_i+T_{ij}) \xrightarrow{a.s.} \sum_{i\in\mathbb{Z}}\sum_{j=1}^{L_i}f(T_i+T_{ij})\,,$$

as $K \to \infty$. Since almost sure convergence implies convergence in distribution, Theorem 2.4.2 yields the equation (4.14).

To complete the proof, by the variant of the Slutsky's lemma (see Theorem 4.2 in [Bil68]) and Theorem 2.4.2, it suffices to show that for every $u > 0$ and every function f as above

$$\lim_{K\to\infty}\limsup_{t\to\infty}\mathbb{P}\left(\left|\sum_{i\geq 0}\sum_{j=1}^{L_i'}\mathbb{1}_{\{|T_i'-t|\leq K\}}f(T_i'-t+T_{ij}') - \sum_{i\geq 0}\sum_{j=1}^{L_i'}f(T_i'-t+T_{ij}')\right| > u\right) = 0\,.$$

Assume that the support of f is contained in $\langle -C, C\rangle$, for some $C>0$. Observe that for $K>C$

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{i\geq 0}\sum_{j=1}^{L_i'}\mathbb{1}_{\{|T_i'-t|\leq K\}}f(T_i'-t+T_{ij}') - \sum_{i\geq 0}\sum_{j=1}^{L_i'} f(T_i'-t+T_{ij}')\right| > u\right)\\
&\qquad = \mathbb{P}\left(\left|\sum_{i\geq 0}\sum_{j=1}^{L_i'}\mathbb{1}_{\{|T_i'-t|> K\}}f(T_i'-t+T_{ij}')\right| > u\right)\\
&\qquad \leq \mathbb{P}\big(\exists i, j\leq L_i' : |T_i'-t|>K, |T_i'-t+T_{ij}'|<C\big)\,.
\end{aligned}$$

Denote

$$\sigma(t) = \inf\{i\geq 0 : T_i' > t\}$$

and observe that the last expression above is bounded by

$$\begin{aligned}
&\mathbb{P}\big(\exists i : T_i'-t<-K, T_i'-t+R_i'>-C\big) + \mathbb{P}\big(\exists i : T_i'-t>K, T_i'-t-R_i'<C\big)\\
&\quad \leq \mathbb{P}\big(T_0'-t<-K, T_0'-t+R_0'>-C\big)\\
&\quad + \sum_{i\geq 1}\mathbb{P}\big(T_{i-1}'-t\leq -K, T_{i-1}'-t+X_i'+R_i'>-C\big)\\
&\quad + \mathbb{P}\big(T_{\sigma(t+K)}'-(t+K)-R_{\sigma(t+K)}'<C-K\big)\\
&\quad + \sum_{i\geq 1}\mathbb{P}\big(T_{i-1}'-t\geq K, T_{i-1}'+X_i'-t-R_i'<C\big)\\
&\quad =: I_0^+(t,K)+I_1^+(t,K)+I_0^-(t,K)+I_1^-(t,K)\,.
\end{aligned}$$

It is easy to see that

$$\lim_{K\to\infty}\limsup_{t\to\infty} I_0^+(t,K) \leq \lim_{K\to\infty}\mathbb{P}(R_0'>K-C)=0\,.$$

Denote by U' the renewal function and recall from Section 3.1 that $U'=\sum_{i\geq 0}F_{T_i'}$, where $F_{T_i'}$ is the distribution function of the arrival time T_i'. Then, using the independence between T_{i-1}' and X_i', R_i', for all $i\geq 1$, we obtain

$$\begin{aligned}
I_1^+(t,K) &= \sum_{i\geq 1}\int_{[0,t-K]}\mathbb{P}(y-t+X_i'+R_i'>-C)\mathrm{d}F_{T_{i-1}'}(y)\\
&= \int_{[0,t-K]}\mathbb{P}(X_1'+R_1'+C-K>t-K-y)\mathrm{d}U'(y)\,.
\end{aligned}$$

Observe that the function $y\mapsto \mathbb{P}(X_1'+R_1'+C-K>y)$ is nonnegative, nonincreasing and Riemann integrable on $\mathbb{R}_+$, since

$$\int_0^\infty \mathbb{P}(X_1'+R_1'+C-K>y)\mathrm{d}y = \int_{K-C}^\infty \mathbb{P}(X_1'+R_1'>y)\mathrm{d}y \leq \mathbb{E}[X_1'+R_1']<\infty\,,$$

by integrability assumptions on X_1' and R_1'. By Lemma 3.4.4, it is also a directly Riemann integrable function on $\mathbb{R}_+$, hence by the key renewal theorem (Theorem 3.4.6) and integrability assumptions on X_1' and R_1'

$$\lim_{K\to\infty}\limsup_{t\to\infty} I_1^+(t,K) = \lim_{K\to\infty}\frac{1}{\mu}\mathbb{E}(X_1'+R_1'+C-K)_+ = 0\,.$$

One can argue similarly as in Example 2.4.3 and conclude that Proposition 4.1.2 and Theorem 2.4.2 yield

$$(T'_{\sigma(t)}-t, R'_{\sigma(t)}) \xrightarrow{d} (T_0, R_0)\,,$$

as $t\to\infty$, where $R_0 = \sup\limits_{j\le L_0}|T_{0j}|$. Hence, by the continuous mapping argument it follows that

$$\lim_{K\to\infty}\limsup_{t\to\infty} I_0^-(t,K) = \lim_{K\to\infty}\mathbb{P}\left(T_0 - R_0 < C-K\right) \le \lim_{K\to\infty}\mathbb{P}(R_0 > K-C) = 0\,.$$

As before, using independence assumption between T'_{i-1} and X_i', R_i', for all $i\ge 1$, we obtain

$$\begin{aligned} I_1^-(t,K) &= \int_{[t+K,\infty\rangle}\mathbb{P}(X_1'-R_1'-C+K<t+K-y)\mathrm{d}U'(y) \\ &\le \int_{[t+K,\infty\rangle}\mathbb{P}(-R_1'-C+K<t+K-y)\mathrm{d}U'(y)\,. \end{aligned}$$

Note that the function $y\mapsto\mathbb{P}(-R_1'-C+K<y)$ is nonnegative, nondecreasing and Riemann integrable on $\mathbb{R}_-$, since

$$\int_{-\infty}^0\mathbb{P}(-R_1'-C+K<y)\mathrm{d}y = \int_{K-C}^\infty\mathbb{P}(R_1'>y)\mathrm{d}y \le \mathbb{E}R_1' < \infty\,,$$

by integrability assumption on R_1'. Another application of Lemma 3.4.4 now yields that $y\mapsto\mathbb{P}(-R_1'-C+K<y)$ is a directly Riemann integrable function on $\mathbb{R}_-$. Hence, by the key renewal theorem on a negative half line (Theorem 3.4.7 (i)) and integrability assumptions on X_1' and R_1'

$$\lim_{K\to\infty}\limsup_{t\to\infty} I_1^-(t,K) = \lim_{K\to\infty}\frac{1}{\mu}\mathbb{E}(-R_1'-C+K)_- = 0\,,$$

which completes the proof. □

Remark 4.3.2. Let us arrange the points of the renewal cluster point process ξ' in a nondecreasing order

$$\ldots \le S'_{-2} \le S'_{-1} \le 0 \le S'_0 \le S'_1 \le S'_2 \le \ldots\,.$$

Generally, $S'_k = T'_i + T'_{ij}$, for some i, j. For an arbitrary $t > 0$, set

$$\sigma(t) = \inf\{k \in \mathbb{N} : S'_k > t\} ,$$

then

$$R(t) = S'_{\sigma(t)} - t$$

is a forward recurrence time. In Sections 4.6 and 4.7, for some specific cluster point processes, we will obtain closed formulae for the limiting distribution of the forward recurrence time as a consequence of Theorem 4.3.1.

4.4 Blackwell's renewal theorem

To deduce Blackwell's renewal theorem for renewal cluster point processes from the extended renewal theorem, we need a technical result involving uniform integrability. Recall that the random variables X_t, $t \in T$, are said to be uniformly integrable if

$$\lim_{x\to\infty} \left(\sup_{t\in T} \mathbb{E}\left[|X_t| \mathbb{1}_{\{|X_t|>x\}} \right] \right) = 0 ,$$

where T is a nonempty index set.

Lemma 4.4.1. *Assume that ξ' is the renewal cluster point process introduced in Section 4.2. Suppose that $L'_0, L'_1, L'_1 X'_1$ and $L'_1 R'_1$ are integrable random variables. Then for all $x > 0$,*

$$\xi'\langle t, t+x] , \quad t \geq 0 ,$$

are uniformly integrable random variables.

Proof. Based on the position of the cluster centers T'_i, we obtain the following decomposition

$$\xi'\langle t, t+x] = \xi'_1\langle t, t+x] + \xi'_2\langle t, t+x] + \xi'_3\langle t, t+x] ,$$

where

$$\xi'_1\langle t, t+x] = \sum_{i\geq 0} \sum_{j=1}^{L'_i} \mathbb{1}_{\{T'_i + T'_{ij} \in \langle t, t+x]\}} \mathbb{1}_{\{T'_i \leq t\}} ,$$

$$\xi'_2\langle t, t+x] = \sum_{i\geq 0} \sum_{j=1}^{L'_i} \mathbb{1}_{\{T'_i + T'_{ij} \in \langle t, t+x]\}} \mathbb{1}_{\{t < T'_i \leq t+x\}}$$

and

$$\xi_3'\langle t, t+x] = \sum_{i\geq 0}\sum_{j=1}^{L_i'} \mathbb{1}_{\{T_i'+T_{ij}'\in\langle t,t+x]\}}\mathbb{1}_{\{T_i'>t+x\}}\,.$$

Note that if T_{ij}' take only negative values, $\xi_1'\langle t, t+x]$ is identically 0. Hence, when dealing with this term, one needs to take care only of $T_{ij}' \geq 0$, for all $i, j \leq L_i'$. Similarly, when dealing with $\xi_3'\langle t, t+x]$ one can ignore positive T_{ij}' and assume that $T_{ij}' \leq 0$, for all $i, j \leq L_i'$. These simple observations will be useful in the sequel. Once the uniform integrability of all three terms is proved, statement of the lemma follows by Theorem 4.6 in Chapter 5 of [Gut05]. We now show uniform integrability of each term separately.

Term ξ_1'. Observe that

$$\begin{aligned}\xi_1'\langle t, t+x] &\leq \sum_{i\geq 0} L_i'\mathbb{1}_{\{T_i'+R_i'>t\}}\mathbb{1}_{\{T_i'\leq t\}}\\ &\leq L_0' + \sum_{i\geq 1} L_i'\mathbb{1}_{\{T_i'-t+R_i'>0\}}\mathbb{1}_{\{T_i'\leq t\}} =: L_0' + I_1'(t)\,. \qquad (4.15)\end{aligned}$$

Trivially, an integrable random variable L_0' is uniformly integrable. For the other term in the above relation, using the independence between T_{i-1}' and X_i', L_i', R_i', for all $i \geq 1$, we get

$$\begin{aligned}\mathbb{E}I_1'(t) &= \mathbb{E}\left[\sum_{i\geq 1} L_i'\mathbb{1}_{\{T_i'-t+R_i'>0\}}\mathbb{1}_{\{T_i'\leq t\}}\right]\\ &= \mathbb{E}\left[\sum_{i\geq 1}\mathbb{E}\left[L_i'\mathbb{1}_{\{T_{i-1}'+X_i'-t+R_i'>0\}}\mathbb{1}_{\{T_{i-1}'+X_i'\leq t\}}\mathbb{1}_{\{T_{i-1}'\leq t\}}|T_{i-1}'\right]\right]\\ &= \sum_{i\geq 1}\int_{[0,t]}\mathbb{E}\left[L_i'\mathbb{1}_{\{X_i'+R_i'>t-y\}}\mathbb{1}_{\{X_i'\leq t-y\}}\right]\mathrm{d}F_{T_{i-1}'}(y)\\ &= \int_{[0,t]}\mathbb{E}\left[L_1'\mathbb{1}_{\{X_1'+R_1'>t-y\}}\mathbb{1}_{\{X_1'\leq t-y\}}\right]\mathrm{d}U'(y)\,.\end{aligned}$$

By the key renewal theorem (Theorem 3.4.6), as $t \to \infty$, the above expression converges to

$$\frac{1}{\mu}\int_0^\infty \mathbb{E}\left[L_1'\mathbb{1}_{\{X_1'+R_1'>y\}}\mathbb{1}_{\{X_1'\leq y\}}\right]\mathrm{d}y = \frac{1}{\mu}\mathbb{E}\left[L_1'\int_{X_1'}^{X_1'+R_1'}\mathrm{d}y\right] = \frac{1}{\mu}\mathbb{E}\left[L_1'R_1'\right] =: c_1\,,$$

if $y \mapsto \mathbb{E}\left[L_1'\mathbb{1}_{\{X_1'+R_1'>y\}}\mathbb{1}_{\{X_1'\leq y\}}\right]$ is a directly Riemann integrable function on $\mathbb{R}_+$. This follows by Lemma 3.4.5, since

$$\int_0^\infty \mathbb{E}\left[L_1'\mathbb{1}_{\{X_1'+R_1'>y\}}\mathbb{1}_{\{X_1'\leq y\}}\right]\mathrm{d}y = \mathbb{E}\left[L_1'R_1'\right] < \infty$$

by the assumption,

$$\mathbb{E}\left[L_1' \mathbb{1}_{\{X_1'+R_1'>y\}} \mathbb{1}_{\{X_1'\leq y\}}\right] \leq \mathbb{E}\left[L_1' \mathbb{1}_{\{X_1'+R_1'>y\}}\right] , \tag{4.16}$$

for all $y \geq 0$ and the majorant in (4.16) is directly Riemann integrable on $\mathbb{R}_+$ by Lemma 3.4.4. Indeed, the majorant is nonnegative, nonincreasing and

$$\int_0^\infty \mathbb{E}\left[L_1' \mathbb{1}_{\{X_1'+R_1'>y\}}\right] \mathrm{d}y = \mathbb{E}\left[L_1'(X_1'+R_1')\right] < \infty ,$$

by the assumptions of the theorem.

Let $R_i = \sup_{j\leq L_i} |T_{ij}|$, for all $i \in \mathbb{Z}$. We now show that, as $t \to \infty$,

$$I_1'(t) \xrightarrow{d} \sum_{i\leq -1} L_i \mathbb{1}_{\{T_i+R_i>0\}} =: I_1' ,$$

by checking the conditions of Theorem 4.2 from [Bil68]. Simple modifications in the proof of Theorem 4.3.1 give

$$\sum_{i\geq 1} L_i' \delta_{T_i'-t+R_i'} \mathbb{1}_{\{T_i'\leq t\}} \xrightarrow{d} \sum_{i\leq -1} L_i \delta_{T_i+R_i} ,$$

hence Theorem 2.4.2 yields

$$\sum_{i\geq 1} L_i' \mathbb{1}_{\{0<T_i'-t+R_i'\leq M\}} \mathbb{1}_{\{T_i'\leq t\}} \xrightarrow{d} \sum_{i\leq -1} L_i \mathbb{1}_{\{0<T_i+R_i\leq M\}} ,$$

as $t \to \infty$, for an arbitrary $M > 0$. Considering the random variable on the right hand side, letting $M \to \infty$ and applying the monotone convergence theorem, we get

$$\sum_{i\leq -1} L_i \mathbb{1}_{\{0<T_i+R_i\leq M\}} \xrightarrow{a.s.} \sum_{i\leq -1} L_i \mathbb{1}_{\{T_i+R_i>0\}} .$$

It now remains to show that for every $u > 0$

$$\lim_{M\to\infty} \limsup_{t\to\infty} \mathbb{P}\left(\left|\sum_{i\geq 1} L_i' \mathbb{1}_{\{0<T_i'-t+R_i'\leq M\}} \mathbb{1}_{\{T_i'\leq t\}} - I_1'(t)\right| > u\right) = 0 .$$

Indeed,

$$\begin{aligned}
\mathbb{P}&\left(\left|\sum_{i\geq 1} L_i'\mathbb{1}_{\{0<T_i'-t+R_i'\leq M\}}\mathbb{1}_{\{T_i'\leq t\}}-\sum_{i\geq 1} L_i'\mathbb{1}_{\{T_i'-t+R_i'>0\}}\mathbb{1}_{\{T_i'\leq t\}}\right|>u\right)\\
&=\mathbb{P}\left(\left|\sum_{i\geq 1} L_i'\mathbb{1}_{\{T_i'-t+R_i'>M\}}\mathbb{1}_{\{T_i'\leq t\}}\right|>u\right)\\
&\leq \mathbb{P}(\exists i\geq 1: T_i'-t+R_i'>M, T_{i-1}'\leq t)\\
&\leq \sum_{i\geq 1}\mathbb{P}\left(T_{i-1}'+X_i'-t+R_i'>M, T_{i-1}'\leq t\right)\\
&=\sum_{i\geq 1}\int_{[0,t]}\mathbb{P}(y+X_i'-t+R_i'>M)\mathrm{d}F_{T_{i-1}'}(y)\\
&=\int_{[0,t]}\mathbb{P}(X_1'+R_1'-M>t-y)\mathrm{d}U'(y)\,,
\end{aligned}$$

which converges by the key renewal theorem (Theorem 3.4.6) to

$$\frac{1}{\mu}\int_0^\infty \mathbb{P}(X_1'+R_1'-M>y)\mathrm{d}y=\frac{1}{\mu}\mathbb{E}(X_1'+R_1'-M)_+\,,$$

as $t\to\infty$, if $y\mapsto \mathbb{P}(X_1'+R_1'-M>y)$ is a directly Riemann integrable function on $\mathbb{R}_+$. This condition is easily proved to be true as an immediate consequence of Lemma 3.4.4 and integrability assumptions on X_1' and R_1'. Under the same integrability assumptions, we conclude that the above expression converges to 0, as $M\to\infty$.

By construction, T_i is independent of R_i, L_i, and $T_i \stackrel{d}{=} -T_{-i-1}$, $X_1\stackrel{d}{=}X_1'$, $R_i\stackrel{d}{=}R_1'$ and $L_i\stackrel{d}{=}L_1'$, for all $i\leq -1$. Hence, we obtain

$$\begin{aligned}
\mathbb{E}I_1'&=\mathbb{E}\left[\sum_{i\leq -1}L_i\mathbb{1}_{\{T_i+R_i>0\}}\right]=\mathbb{E}\left[\sum_{i=-\infty}^{-1}\mathbb{E}\left[L_i\mathbb{1}_{\{T_i+R_i>0\}}|T_i\right]\right]\\
&=\int_{\langle-\infty,0]}\sum_{i=-\infty}^{-1}\mathbb{E}\left[L_i\mathbb{1}_{\{y+R_i>0\}}\right]\mathrm{d}F_{T_i}(y)=\int_{\langle-\infty,0]}\sum_{i=0}^{\infty}\mathbb{E}\left[L_1'\mathbb{1}_{\{y+R_1'>0\}}\right]\mathrm{d}F_{T_i}(-y)\\
&=\int_{[0,\infty\rangle}\mathbb{E}\left[L_1'\mathbb{1}_{\{R_1'>y\}}\right]\mathrm{d}U(y)=\frac{1}{\mathbb{E}X_1}\int_0^\infty\mathbb{E}\left[L_1'\mathbb{1}_{\{R_1'>y\}}\right]\mathrm{d}y=\frac{1}{\mu}\mathbb{E}[L_1'R_1']=c_1\,,
\end{aligned}$$

which is finite by the assumptions.

We have shown that

$$\mathbb{E}I_1'(t)\to c_1\,,\quad I_1'(t)\xrightarrow{d}I_1'\quad\text{and}\quad \mathbb{E}I_1'=c_1<\infty\,,$$

hence, by Lemma 5.11 in [Kal21], $I_1'(t)$, $t\geq 0$ are uniformly integrable random variables. Applying first Theorem 4.6 and then Theorem 4.5 in Chapter 5 of [Gut05] to (4.15), we conclude that $\xi_1'\langle t,t+x]$, $t\geq 0$ are also uniformly integrable random variables.

Term ξ_2'. Observe that

$$\tau(t) = \inf\{i \geq 0 : T_i' \in \langle t, t+x]\}$$

is a stopping time with respect to the natural filtration of $(T_i')_{i\geq 0}$. By the definition of $\xi_2'\langle t, t+x]$ it only makes sense to restrict our analysis to the event $\{\tau(t) < \infty\}$. Let $(W_i'')_{i\geq 1}$ be an independent copy of $(W_i')_{i\geq 1}$ and set $T_0'' = 0$, $T_i'' = X_1'' + \ldots + X_i''$, for all $i \geq 1$. Using the strong Markov property, we get

$$\begin{aligned}
\xi_2'\langle t, t+x] &\leq \sum_{i\geq 0} L_i' \mathbb{1}_{\{T_i' \in \langle t, t+x]\}} \\
&\leq L_{\tau(t)}' + \sum_{i\geq 0} L_{\tau(t)+i+1}' \mathbb{1}_{\{T_{\tau(t)+i}' \in \langle t, t+x]\}} \\
&\stackrel{d}{=} L_{\tau(t)}' + \sum_{i\geq 0} L_{i+1}'' \mathbb{1}_{\{T_{\tau(t)}' + T_i'' \in \langle t, t+x]\}} \\
&= L_{\tau(t)}' + \sum_{i\geq 0} L_{i+1}'' \mathbb{1}_{\{T_i'' \in \langle t - T_{\tau(t)}', t+x-T_{\tau(t)}']\}} \\
&\leq L_{\tau(t)}' + \sum_{i\geq 0} L_{i+1}'' \mathbb{1}_{\{T_i'' \in \langle -x, x\rangle\}} \,.
\end{aligned} \tag{4.17}$$

As in the proof of Theorem 4.3.1, we denote

$$\sigma(t) = \inf\{i \geq 0 : T_i' > t\} \,.$$

Observe that on event $\{\tau(t) < \infty\}$, stopping times $\tau(t)$ and $\sigma(t)$ coincide, hence

$$\{\sigma(t) < \infty\} \text{ and } L_{\tau(t)}' \stackrel{d}{=} L_{\sigma(t)}' \,.$$

Using the independence between T_{i-1}' and X_i', L_i', for all $i \geq 1$, we get

$$\begin{aligned}
\mathbb{E}L_{\tau(t)}' = \mathbb{E}L_{\sigma(t)}' &= \mathbb{E}\left[\sum_{i=0}^{\infty} L_i' \mathbb{1}_{\{\sigma(t)=i\}}\right] = \mathbb{E}\left[L_0' \mathbb{1}_{\{T_0'>t\}} + \sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_{i-1}' \leq t < T_i'\}}\right] \\
&= \mathbb{E}\left[L_0' \mathbb{1}_{\{T_0'>t\}}\right] + \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbb{E}\left[L_i' \mathbb{1}_{\{T_{i-1}' \leq t < T_{i-1}' + X_i'\}} | T_{i-1}'\right]\right] \\
&= \mathbb{E}\left[L_0' \mathbb{1}_{\{T_0'>t\}}\right] + \int_{[0,t]} \sum_{i=1}^{\infty} \mathbb{E}\left[L_i' \mathbb{1}_{\{X_i' > t-y\}}\right] \mathrm{d}F_{T_{i-1}}'(y) \\
&= \mathbb{E}\left[L_0' \mathbb{1}_{\{T_0'>t\}}\right] + \int_{[0,t]} \mathbb{E}\left[L_1' \mathbb{1}_{\{X_1' > t-y\}}\right] \mathrm{d}U'(y) \,.
\end{aligned} \tag{4.18}$$

Since $\mathbb{E}L_0' < \infty$ by the assumption, monotone convergence theorem for decreasing functions yields

$$\mathbb{E}\left[L_0' \mathbb{1}_{\{T_0'>t\}}\right] \to 0 \,,$$

as $t \to \infty$. Furthermore, by Lemma 3.4.4, $y \mapsto \mathbb{E}\left[L'_1 \mathbb{1}_{\{X'_1 > y\}}\right]$ is a directly Riemann integrable function on $\mathbb{R}_+$, since it is nonnegative, nonincreasing and

$$\int_0^\infty \mathbb{E}\left[L'_1 \mathbb{1}_{\{X'_1 > y\}}\right] \mathrm{d}y = \mathbb{E}[L'_1 X'_1] < \infty\,,$$

by the assumption. Hence, by the key renewal theorem (Theorem 3.4.6)

$$\int_{[0,t]} \mathbb{E}\left[L'_1 \mathbb{1}_{\{X'_1 > t-y\}}\right] \mathrm{d}U'(y) \to \frac{1}{\mu}\mathbb{E}[L'_1 X'_1]\,,$$

thus

$$\mathbb{E}L'_{\tau(t)} \to \frac{1}{\mu}\mathbb{E}[L'_1 X'_1]\,,$$

as $t \to \infty$. Furthermore, Proposition 4.1.2 yields

$$L'_{\tau(t)} \overset{d}{=} L'_{\sigma(t)} \xrightarrow{d} L_0\,,$$

as $t \to \infty$. By construction

$$\mathbb{E}L_0 = \frac{1}{\mu}\mathbb{E}[L'_1 X'_1]$$

and since we assumed strictly positive mean of X'_1 and integrability of $L'_1 X'_1$, it follows that $\mathbb{E}L_0 < \infty$. Lemma 5.11 in [Kal21] finishes the proof of uniform integrability of $L'_{\tau(t)}$, $t \geq 0$.

We now focus on the other term in relation (4.17). Independence between T''_i and L''_{i+1}, for all $i \geq 0$, integrability assumption on L'_1 and Lemma 3.1.5 yield

$$\mathbb{E}\left[\sum_{i\geq 0} L''_{i+1}\mathbb{1}_{\{T''_i \in \langle -x,x\rangle\}}\right] = \mathbb{E}L'_1 \mathbb{E}\left[\sum_{i\geq 0} \mathbb{1}_{\{T''_i \in \langle -x,x\rangle\}}\right] = \mathbb{E}L'_1 U''(x) < \infty\,,$$

thus $\sum_{i\geq 0} L''_{i+1}\mathbb{1}_{\{T''_i \in \langle -x,x\rangle\}}$ are uniformly integrable random variables. It now only remains to apply first Theorem 4.6 and then Theorem 4.5 in Chapter 5 of [Gut05] to (4.17) and conclude that $\xi'_2\langle t, t+x]$, $t \geq 0$ are also uniformly integrable random variables.

Term ξ'_3. We proceed similarly as in the first part of the proof and observe that

$$\begin{aligned} \xi'_3\langle t, t+x] &\leq \sum_{i\geq 0} L'_i \mathbb{1}_{\{T'_i - R'_i \leq t+x\}} \mathbb{1}_{\{T'_i > t+x\}} \\ &\leq L'_0 + \sum_{i\geq 1} L'_i \mathbb{1}_{\{T'_i - (t+x) - R'_i \leq 0\}} \mathbb{1}_{\{T'_i > t+x\}} := L'_0 + I'_3(t)\,. \end{aligned} \tag{4.19}$$

Since L'_0 is trivially uniformly integrable, we deal only with the second term in the above relation. Observe

$$\begin{aligned}\mathbb{E}I'_3(t) &= \mathbb{E}\left[\sum_{i\geq 1} L'_i \mathbb{1}_{\{T'_i-(t+x)-R'_i\leq 0\}}\mathbb{1}_{\{T'_i>t+x\}}\mathbb{1}_{\{T'_{i-1}\leq t+x\}}\right] \\ &\quad + \mathbb{E}\left[\sum_{i\geq 1} L'_i \mathbb{1}_{\{T'_i-(t+x)-R'_i\leq 0\}}\mathbb{1}_{\{T'_{i-1}>t+x\}}\right] \\ &=: E'_1(t)+E'_2(t)\,. \end{aligned}\tag{4.20}$$

Using the independence between T'_{i-1} and X'_i, L'_i, R'_i, for all $i\geq 1$, we get

$$\begin{aligned}E'_1(t) &= \mathbb{E}\left[\sum_{i\geq 1}\mathbb{E}\left[L'_i \mathbb{1}_{\{T'_{i-1}+X'_i-(t+x)-R'_i\leq 0\}}\mathbb{1}_{\{T'_{i-1}+X'_i>t+x\}}\mathbb{1}_{\{T'_{i-1}\leq t+x\}}|T'_{i-1}\right]\right] \\ &= \sum_{i\geq 1}\int_{[0,t+x]}\mathbb{E}\left[L'_i\mathbb{1}_{\{X'_i-R'_i\leq t+x-y\}}\mathbb{1}_{\{X'_i>t+x-y\}}\right]\mathrm{d}F_{T'_{i-1}}(y) \\ &= \int_{[0,t+x]}\mathbb{E}\left[L'_1\mathbb{1}_{\{X'_1-R'_1\leq t+x-y\}}\mathbb{1}_{\{X'_1>t+x-y\}}\right]\mathrm{d}U'(y)\,.\end{aligned}$$

It now follows by the key renewal theorem (Theorem 3.4.6) that the above expression converges to

$$\begin{aligned}\frac{1}{\mu}\int_0^\infty \mathbb{E}\left[L'_1\mathbb{1}_{\{X'_1-R'_1\leq y\}}\mathbb{1}_{\{X'_1>y\}}\right]\mathrm{d}y &= \frac{1}{\mu}\mathbb{E}\left[L'_1\int_{(X'_1-R'_1)\vee 0}^{X'_1}\mathrm{d}y\right] \\ &= \frac{1}{\mu}\mathbb{E}\left[L'_1\left(X'_1-(X'_1-R'_1)_+\right)\right]\,,\end{aligned}\tag{4.21}$$

as $t\to\infty$, since $y\mapsto\mathbb{E}\left[L'_1\mathbb{1}_{\{X'_1-R'_1\leq y\}}\mathbb{1}_{\{X'_1>y\}}\right]$ is dominated by a nonnegative, nonincreasing function $y\mapsto\mathbb{E}\left[L'_1\mathbb{1}_{\{X'_1>y\}}\right]$ which is Riemann integrable on $\mathbb{R}_+$ by the assumption. Hence, conditions of Lemmas 3.4.4 and 3.4.5 are satisfied which yields direct Riemann integrability of $y\mapsto\mathbb{E}\left[L'_1\mathbb{1}_{\{X'_1-R'_1\leq y\}}\mathbb{1}_{\{X'_1>y\}}\right]$ on $\mathbb{R}_+$ and justifies the use of the key renewal theorem. Note that, since μ is positive and $\mathbb{E}[L'_1X'_1]$ is finite by the assumption, (4.21) is finite. Similarly as above, one can show that

$$E'_2(t)=\int_{\langle t+x,\infty\rangle}\mathbb{E}\left[L'_1\mathbb{1}_{\{X'_1-R'_1\leq t+x-y\}}\right]\mathrm{d}U'(y)$$

and converges by the key renewal theorem on the negative half line (Theorem 3.4.7 (i)), see also the proof of Lemma 4.3 in [BIMR22], to

$$\frac{1}{\mu}\int_{-\infty}^0\mathbb{E}\left[L'_1\mathbb{1}_{\{X'_1-R'_1\leq y\}}\right]\mathrm{d}y=\frac{1}{\mu}\mathbb{E}\left[L'_1(R'_1-X'_1)_+\right]\,,\tag{4.22}$$

as $t \to \infty$. Again, one has to justify the use of the key renewal theorem by showing that $y \mapsto \mathbb{E}\left[L_1' \mathbb{1}_{\{X_1' - R_1' \le y\}}\right]$ is a directly Riemann integrable function on $\langle -\infty, 0\rangle$. This follows by Lemma 3.4.4, since $y \mapsto \mathbb{E}\left[L_1' \mathbb{1}_{\{X_1' - R_1' \le y\}}\right]$ is nonnegative, nondecreasing and

$$\int_{-\infty}^{0} \mathbb{E}\left[L_1' \mathbb{1}_{\{X_1' - R_1' \le y\}}\right] \mathrm{d}y \le \int_{-\infty}^{0} \mathbb{E}[L_1' \mathbb{1}_{\{-R_1' \le y\}}] \mathrm{d}y = \mathbb{E}[L_1' R_1'] < \infty ,$$

by the assumption. Since μ is positive, this also shows that (4.22) is finite. Hence, equations (4.20) - (4.22) yield

$$\mathbb{E} I_3'(t) \to \frac{1}{\mu} \mathbb{E}\left[L_1' \left(X_1' - (X_1' - R_1')_+\right)\right] + \frac{1}{\mu} \mathbb{E}\left[L_1' (R_1' - X_1')_+\right] := c_3 < \infty . \tag{4.23}$$

Next, we check the conditions of Theorem 4.2 from [Bil68] to conclude that

$$I_3'(t) = \sum_{i \ge 1} L_i' \mathbb{1}_{\{T_i' - (t+x) - R_i' \le 0\}} \mathbb{1}_{\{T_i' > t+x\}} \xrightarrow{d} \sum_{i \ge 0} L_i \mathbb{1}_{\{T_i - R_i \le 0\}} =: I_3' ,$$

as $t \to \infty$. From the proof of Theorem 4.3.1 one obtains

$$\sum_{i \ge 1} L_i' \delta_{T_i' - (t+x) - R_i'} \mathbb{1}_{\{T_i' > t+x\}} \xrightarrow{d} \sum_{i \ge 0} L_i \delta_{T_i - R_i} ,$$

hence for an arbitrary $M > 0$

$$\sum_{i \ge 1} L_i' \mathbb{1}_{\{-M \le T_i' - (t+x) - R_i' \le 0\}} \mathbb{1}_{\{T_i' > t+x\}} \xrightarrow{d} \sum_{i \ge 0} L_i \mathbb{1}_{\{-M \le T_i - R_i \le 0\}} ,$$

as $t \to \infty$, by Theorem 2.4.2. It easily follows as a consequence of the monotone convergence theorem that

$$\sum_{i \ge 0} L_i \mathbb{1}_{\{-M \le T_i - R_i \le 0\}} \xrightarrow{a.s.} \sum_{i \ge 0} L_i \mathbb{1}_{\{T_i - R_i \le 0\}} ,$$

as $M \to \infty$. Furthermore, observe that for all $u > 0$

$$\begin{aligned}
&\mathbb{P}\left(\left| \sum_{i \ge 1} L_i' \mathbb{1}_{\{-M \le T_i' - (t+x) - R_i' \le 0\}} \mathbb{1}_{\{T_i' > t+x\}} - \sum_{i \ge 1} L_i' \mathbb{1}_{\{T_i' - (t+x) - R_i' \le 0\}} \mathbb{1}_{\{T_i' > t+x\}} \right| > u \right) \\
&\quad = \mathbb{P}\left(\left| \sum_{i \ge 1} L_i' \mathbb{1}_{\{T_i' - (t+x) - R_i' < -M\}} \mathbb{1}_{\{T_i' > t+x\}} \right| > u \right) \\
&\quad \le \mathbb{P}(\exists i \ge 1 : T_i' - (t+x) - R_i' < -M, T_i' > t+x) \\
&\quad \le \mathbb{P}(T'_{\sigma(t+x)} - (t+x) - R'_{\sigma(t+x)} < -M) + \sum_{i \ge 1} \mathbb{P}\left(T_i' - (t+x) - R_i' < -M, T_{i-1}' \ge t+x\right) ,
\end{aligned}$$

where one shows that the term on the left hand side converges to 0, letting first $t \to \infty$ and then $M \to \infty$, just like in the proof of Theorem 4.3.1 (see Step 3, term $I_0^-(t, K)$).

For the term on the right hand side, we once again exploit the independence between T'_{i-1} and X'_i, R'_i, for all $i \geq 1$ and use the key renewal theorem on the negative half line (Theorem 3.4.7 (i)). Hence

$$
\begin{aligned}
&\sum_{i\geq 1} \mathbb{P}\left(T'_{i-1} + X'_i - (t+x) - R'_i < -M, T'_{i-1} \geq t+x\right) \\
&\quad = \sum_{i\geq 1} \int_{[t+x,\infty\rangle} \mathbb{P}(y + X'_i - (t+x) - R'_i < -M) \mathrm{d}F_{T'_{i-1}}(y) \\
&\quad = \int_{[t+x,\infty\rangle} \mathbb{P}(X'_1 - R'_1 + M < t+x-y) \mathrm{d}U'(y) \\
&\quad \to \frac{1}{\mu} \int_{-\infty}^{0} \mathbb{P}(X'_1 - R'_1 + M < y) \mathrm{d}y \\
&\quad \leq \frac{1}{\mu} \int_{0}^{\infty} \mathbb{P}(R'_1 - M > y) \mathrm{d}y = \frac{1}{\mu} \mathbb{E}(R'_1 - M)_+ \,,
\end{aligned}
$$

as $t \to \infty$, if $y \mapsto \mathbb{P}(X'_1 - R'_1 + M < y)$ is a directly Riemann integrable function on $\mathbb{R}_-$. Letting $M \to \infty$, we conclude that the above expression converges to 0, by the integrability assumption on R'_1. It remains to prove direct Riemann integrability, however this follows easily from Lemma 3.4.4 and the same integrability assumption. Hence, we have just shown that for every $u > 0$

$$
\lim_{M\to\infty} \limsup_{t\to\infty} \mathbb{P}\left(\left|\sum_{i\geq 1} L'_i \mathbb{1}_{\{-M \leq T'_i - (t+x) - R'_i \leq 0\}} \mathbb{1}_{\{T'_i > t+x\}} - I'_3(t)\right|\right) = 0 \,,
$$

which completes the checking of the assumptions of Theorem 4.2 from [Bil68].

Next, we show that

$$
\mathbb{E} I'_3 = \mathbb{E}\left[L_0 \mathbb{1}_{\{T_0 - R_0 \leq 0\}}\right] + \mathbb{E}\left[\sum_{i\geq 1} L_i \mathbb{1}_{\{T_i - R_i \leq 0\}}\right] \tag{4.24}
$$

equals c_3. The first term can be easily calculated using inversion formula (2.4)

$$
\begin{aligned}
\mathbb{E}\left[L_0 \mathbb{1}_{\{T_0 - R_0 \leq 0\}}\right] &= \frac{1}{\mu} \mathbb{E}\left[\int_0^{X'_1} L'_1 \mathbb{1}_{\{X'_1 - y - R'_1 \leq 0\}} \mathrm{d}y\right] \\
&= \frac{1}{\mu} \mathbb{E}\left[L'_1 \int_{(X'_1 - R'_1)\vee 0}^{X'_1} \mathrm{d}y\right] = \frac{1}{\mu} \mathbb{E}\left[L'_1 (X'_1 - (X'_1 - R'_1)_+)\right] .
\end{aligned}
$$

For the second term in (4.24), as before, using the independence assumption between T_{i-1}

and X_i, R_i, L_i, for all $i \geq 1$, and since $X_1' \overset{d}{=} X_1$, $R_1' \overset{d}{=} R_1$ and $L_1' \overset{d}{=} L_1$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sum_{i\geq 1} L_i \mathbb{1}_{\{T_i - R_i \leq 0\}}\right] &= \mathbb{E}\left[\sum_{i=1}^{\infty} L_i \mathbb{1}_{\{T_{i-1}+X_i-R_i\leq 0\}}\right] = \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbb{E}\left[L_i \mathbb{1}_{\{T_{i-1}+X_i-R_i\leq 0\}} | T_{i-1}\right]\right] \\
&= \int_{[0,\infty\rangle} \sum_{i=1}^{\infty} \mathbb{E}\left[L_i \mathbb{1}_{\{y+X_i-R_i\leq 0\}}\right] \mathrm{d}F_{T_{i-1}}(y) = \int_{[0,\infty\rangle} \mathbb{E}\left[L_1 \mathbb{1}_{\{X_1-R_1\leq -y\}}\right] \mathrm{d}U(y) \\
&= \frac{1}{\mathbb{E}X_1} \int_0^{\infty} \mathbb{E}\left[L_1 \mathbb{1}_{\{X_1-R_1\leq -y\}}\right] \mathrm{d}y = \frac{1}{\mu}\int_{-\infty}^0 \mathbb{E}\left[L_1 \mathbb{1}_{\{X_1-R_1\leq y\}}\right] \mathrm{d}y \\
&= \frac{1}{\mu}\mathbb{E}\left[L_1 \int_{(X_1-R_1)\wedge 0}^0 \mathrm{d}y\right] = \frac{1}{\mu}\mathbb{E}[L_1'(R_1'-X_1')_+]\,.
\end{aligned}$$

Hence,

$$\mathbb{E}I_3' = \frac{1}{\mu}\mathbb{E}\left[L_1'(X_1' - (X_1'-R_1')_+)\right] + \frac{1}{\mu}\mathbb{E}[L_1'(R_1'-X_1')_+] = c_3\,.$$

By Lemma 5.11 in [Kal21], $I_3'(t)$, $t \geq 0$ are uniformly integrable random variables. Application of Theorem 4.6 and then Theorem 4.5 in Chapter 5 of [Gut05] to (4.19) proves the uniform integrability of $\xi_3'\langle t, t+x]$, $t \geq 0$. □

Theorem 4.4.2 (Blackwell's renewal theorem for cluster point processes)**.** *Under the assumptions of Lemma 4.4.1*

$$\mathbb{E}\xi'\langle t, t+x] \to \frac{1}{\mu}\mathbb{E}L_1' x\,,$$

as $t \to \infty$, for all $x > 0$.

Proof. Theorem 4.3.1 and Theorem 2.4.2 yield

$$\theta_t \xi'(B) \xrightarrow{d} \xi(B)\,,$$

as $t \to \infty$, for all $B \in \mathcal{B}_b(\mathbb{R})$ such that $\xi(\partial B) = 0$ a.s. Let $B = \langle 0, x]$, for an arbitrary $x > 0$. Then,

$$\theta_t \xi'\langle 0, x] \xrightarrow{d} \xi\langle 0, x]\,,$$

as $t \to \infty$, because $\mathbb{P}(\xi\{0, x\} = 0) = 1$, e.g. by Proposition 4.2.2. Note that we have already shown in Lemma 4.4.1 that $\theta_t\xi'\langle 0, x]$, $t \geq 0$ are uniformly integrable random variables. Therefore, by Lemma 5.11 in [Kal21]

$$\mathbb{E}\xi'\langle t, t+x] = \mathbb{E}\theta_t\xi'\langle 0, x] \to \mathbb{E}\xi\langle 0, x] = \frac{1}{\mu}\mathbb{E}L_1' x\,,$$

as $t \to \infty$, where equality on the right follows from Proposition 4.2.2 and distributional equality of L_1' and L_1, as well as X_1' and X_1. □

In the cluster point process framework, just like in the standard setting, Blackwell's renewal theorem implies the elementary renewal theorem.

Theorem 4.4.3 (Elementary renewal theorem for cluster point processes)**.** *Under the assumptions of Lemma 4.4.1*

$$\frac{\mathbb{E}\xi'\langle 0,t]}{t} \to \frac{1}{\mu}\mathbb{E}L_1',$$

as $t \to \infty$.

Proof. The proof is based on the same arguments as in the standard renewal theory, see for instance Section 3.7 in [Res92]. First, we observe that Theorem 4.4.2 implies

$$\mathbb{E}\xi'\langle k-1,k] \to \frac{1}{\mu}\mathbb{E}L_1',$$

as $k \to \infty$ and since convergent sequences are Cesàro convergent to the same limit

$$\frac{1}{k}\mathbb{E}\xi'\langle 0,k] = \frac{1}{k}\sum_{j=1}^{k}\mathbb{E}\xi'\langle j-1,j] \to \frac{1}{\mu}\mathbb{E}L_1',$$

as $k \to \infty$. Let $\lfloor t \rfloor$ denote the largest integer not exceeding t and observe

$$\frac{\mathbb{E}\xi'\left\langle 0,\lfloor t \rfloor\right]}{\lfloor t \rfloor}\frac{\lfloor t \rfloor}{t} \le \frac{\mathbb{E}\xi'\langle 0,t]}{t} \le \frac{\mathbb{E}\xi'\langle 0,\lfloor t \rfloor+1]}{\lfloor t \rfloor+1}\frac{\lfloor t \rfloor+1}{t}.$$

Letting $t \to \infty$ completes the proof. $\square$

4.5 Key renewal theorem

Since Blackwell's renewal theorem and the key renewal theorem in the classical renewal process framework are equivalent, it is natural to ask if the equivalence can also be established in our setting. The answer is affirmative and it is fairly easy to check that the two versions of renewal theorem for cluster point processes are equivalent, since the proof essentially follows the classical argument. However, certain preparatory work must be done. Analogously as in Chapter 3, we define the renewal function $U_{\xi'} : \mathbb{R} \to [0,\infty]$ by

$$U_{\xi'}(t) = \mathbb{E}\left[\sum_{i=0}^{\infty}\sum_{j=1}^{L_i'}\mathbb{1}_{\{T_i'+T_{ij}'\le t\}}\right].$$

Lemma 4.5.1. *Suppose that $L_0', L_1', L_1'X_1'$ and $L_1'R_1'$ are integrable random variables, then the renewal function $U_{\xi'}$ is finite, for all $t \in \mathbb{R}$.*

Proof. For all $t \geq 0$

$$U_{\xi'}(t) \leq \mathbb{E}L_0' + \mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'-R_i'\leq t\}} \mathbb{1}_{\{T_i'\leq t\}}\right] + \mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'-R_i'\leq t\}} \mathbb{1}_{\{T_i'> t\}}\right]. \tag{4.25}$$

The first term in (4.25) is finite by the assumption. For the second term, observe that

$$\mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'-R_i'\leq t\}} \mathbb{1}_{\{T_i'\leq t\}}\right] = \mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'\leq t\}}\right] \leq \mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_{i-1}'\leq t\}}\right] = \mathbb{E}L_1'U'(t) < \infty\,,$$

by the integrability assumption on L_1' and Lemma 3.1.5. Just like in the proof of Lemma 4.4.1 (see term $I_3'(t)$), one can show that as $t \to \infty$

$$\mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'-R_i'\leq t\}} \mathbb{1}_{\{T_i'> t\}}\right] \to c_3 < \infty\,,$$

where c_3 is defined in (4.23). Hence, for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'-R_i'\leq t\}} \mathbb{1}_{\{T_i'> t\}}\right] \\
&\quad= \mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'-R_i'\leq t\}} \mathbb{1}_{\{T_i'> t\}}\mathbb{1}_{\{t>t_\varepsilon\}}\right] + \mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'-R_i'\leq t\}} \mathbb{1}_{\{T_i'> t\}}\mathbb{1}_{\{t\leq t_\varepsilon\}}\right] \\
&\quad< c_3 + \varepsilon + \mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'-R_i'\leq t_\varepsilon+1\}} \mathbb{1}_{\{t<T_i'\leq t_\varepsilon+1\}}\right] + \mathbb{E}\left[\sum_{i=1}^{\infty} L_i' \mathbb{1}_{\{T_i'-R_i'\leq t_\varepsilon+1\}} \mathbb{1}_{\{T_i'> t_\varepsilon+1\}}\right] \\
&\quad\leq c_3 + \varepsilon + \mathbb{E}L_1'U'(t_\varepsilon+1) + c_3 + \varepsilon < \infty\,.
\end{aligned}$$

Obviously, $U_{\xi'}(-t) \leq U_{\xi'}(t)$, for all $t > 0$ which completes the proof. □

Arguing just like in Section 3.1 after Lemma 3.1.5, we associate with renewal function an unique measure on $\mathbb{R}$ denoted again by $U_{\xi'}$.

Let G denote distribution of X_0' and F common distribution of $X_i', i \geq 1$. We first recall the renewal equation (3.11)

$$U'(t) = G(t) + F * U'(t), \quad t \geq 0\,,$$

where $U'(t) = \mathbb{E}\left[\sum_{i=0}^{\infty} \mathbb{1}_{\{T_i'\leq t\}}\right] = \mathbb{E}\left[\sum_{i=0}^{\infty} \delta_{T_i'}[0,t]\right]$. Using the independence assumption between T'_{i-1} and $L'_i, (T'_{ij})_j$, for all $i \geq 1$, we get that

$$\begin{aligned}
U_{\xi'}(t) &= \mathbb{E}\left[\sum_{i=0}^{\infty}\sum_{j=1}^{L'_i} \delta_{T'_i+T'_{ij}}\langle -\infty, t]\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{L'_0} \mathbb{1}_{\{T'_0+T'_{0j}\leq t\}}\right] + \mathbb{E}\left[\sum_{i=1}^{\infty}\sum_{j=1}^{L'_i} \mathbb{1}_{\{T'_{i-1}+X'_i+T'_{ij}\leq t\}}\right] \\
&= \phi(t) + \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbb{E}\left[\sum_{j=1}^{L'_i} \mathbb{1}_{\{T'_{i-1}+X'_i+T'_{ij}\leq t\}} | T'_{i-1}\right]\right] \\
&= \phi(t) + \int_{[0,\infty\rangle} \sum_{i=1}^{\infty} \mathbb{E}\left[\sum_{j=1}^{L'_i} \mathbb{1}_{\{X'_i+T'_{ij}\leq t-y\}}\right] \mathrm{d}F'_{T_{i-1}}(y) \\
&= \phi(t) + \int_{[0,\infty\rangle} \mathbb{E}\left[\sum_{j=1}^{L'_1} \mathbb{1}_{\{X'_1+T'_{1j}\leq t-y\}}\right] \mathrm{d}U'(y) \\
&= \phi(t) + \int_{[0,\infty\rangle} \psi(t-y)\mathrm{d}U'(y)\,,
\end{aligned}$$

where $\phi(t) = \mathbb{E}\left[\sum_{j=1}^{L'_0} \mathbb{1}_{\{T'_0+T'_{0j}\leq t\}}\right]$ and $\psi(t) = \mathbb{E}\left[\sum_{j=1}^{L'_1} \mathbb{1}_{\{X'_1+T'_{1j}\leq t\}}\right]$, $t \in \mathbb{R}$. One now easily obtains a convolution equation

$$\begin{aligned}
U_{\xi'}(t) &= \phi(t) + U' * \psi(t) \\
&= \phi(t) + (G + F * U') * \psi(t) \\
&= \phi(t) + G * \psi(t) + F * U' * \psi(t) \\
&= (1-F) * \phi(t) + G * \psi(t) + F * \phi(t) + F * U' * \psi(t) \\
&= (1-F) * \phi(t) + G * \psi(t) + F * U_{\xi'}(t)\,,
\end{aligned} \tag{4.26}$$

for all $t \geq 0$.

Theorem 4.5.2 (Key renewal theorem for cluster point processes)**.** *Under the assumptions of Lemma 4.4.1 and additionally if $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a directly Riemann integrable function on $\mathbb{R}_+$, then*

$$\lim_{t\to\infty} \int_{[0,t]} g(t-y)\mathrm{d}U_{\xi'}(y) = \frac{1}{\mu}\mathbb{E}L'_1 \int_0^{\infty} g(y)\mathrm{d}y\,.$$

Proof. The proof is divided into three steps, with successively more complex g.

Step 1. Suppose

$$g(t) = \mathbb{1}_{[(k-1)h,kh\rangle}(t)\,, \quad t \geq 0\,,$$

for a fixed $k \in \mathbb{N}$ and $h > 0$. Then $g(t-y) = 1$ if and only if $t - kh < y \leq t - (k-1)h$, thus

$$\int_{[0,t]} g(t-y)\mathrm{d}U_{\xi'}(y) = U_{\xi'}\langle t-kh, t-(k-1)h]\mathbb{1}_{\{t \geq kh\}} + U_{\xi'}\langle 0, t-(k-1)h]\mathbb{1}_{\{(k-1)h \leq t < kh\}}\,.$$

Observe that for all $t \in [(k-1)h, kh\rangle$,

$$U_{\xi'}\langle 0, t-(k-1)h] \leq U_{\xi'}(h) < \infty\,, \tag{4.27}$$

hence Theorem 4.4.2 yields

$$\lim_{t\to\infty}\int_{[0,t]} g(t-y)\mathrm{d}U_{\xi'}(y) = \frac{1}{\mu}\mathbb{E}L_1'h + 0 = \frac{1}{\mu}\mathbb{E}L_1'\int_0^\infty g(y)\mathrm{d}y\,.$$

Step 2. Suppose

$$g(t) = \sum_{k\geq 1} c_k \mathbb{1}_{[(k-1)h,kh\rangle}(t)\,, \quad t \geq 0\,,$$

where $(c_k)_{k\geq 1}$ is a sequence of nonnegative numbers such that $\sum_{k\geq 1} c_k < \infty$ and h is chosen so that $F(h) < 1$. Then

$$\int_{[0,t]} g(t-y)\mathrm{d}U_{\xi'}(y) = \sum_{k=1}^{\infty} c_k U_{\xi'}\langle t-kh, t-(k-1)h]\mathbb{1}_{\{t\geq kh\}} + \sum_{k=1}^{\infty} c_k U_{\xi'}\langle 0, t-(k-1)h]\mathbb{1}_{\{(k-1)h\leq t<kh\}}\,.$$

For each $k \in \mathbb{N}$ and $t \geq kh$, we have

$$\lim_{t\to\infty} U_{\xi'}\langle t-kh, t-(k-1)h] = \frac{1}{\mu}\mathbb{E}L_1'h\,.$$

It follows from the equation (4.26), that

$$\phi(t) + G * \psi(t) \geq (1-F) * \phi(t) + G * \psi(t) = (1-F) * U_{\xi'}(t)\,,$$

hence

$$\begin{aligned}
\mathbb{E}L_0' + \mathbb{E}L_1' &= \mathbb{E}L_0' + \int_{[0,\infty\rangle} \mathbb{E}L_1'\mathrm{d}G(y) \\
&\geq \phi(t-(k-1)h) + \int_{[0,\infty\rangle} \psi(t-(k-1)h-y)\mathrm{d}G(y) \\
&\geq \int_{[t-kh,t-(k-1)h]} (1-F(t-(k-1)h-y))\mathrm{d}U_{\xi'}(y) \\
&\geq (1-F(h))U_{\xi'}\langle t-kh, t-(k-1)h]\,.
\end{aligned}$$

Therefore

$$\sup_{t,k} U_{\xi'}\langle t-kh, t-(k-1)h] \leq \frac{\mathbb{E}L_0' + \mathbb{E}L_1'}{1-F(h)}\,,$$

which is finite by integrability assumptions on L_0', L_1' and choice of h. For an arbitrary k and $t \in [(k-1)h, kh\rangle$, recall (4.27). Hence, by the dominated convergence theorem

$$\lim_{t\to\infty} \int_{[0,t]} g(t-y)\mathrm{d}U_{\xi'}(y) = \frac{1}{\mu}\mathbb{E}L_1' h \sum_{k\geq 1} c_k + 0 = \frac{1}{\mu}\mathbb{E}L_1' \int_0^\infty g(y)\mathrm{d}y\,.$$

Step 3. Let g be an arbitrary directly Riemann integrable function on $\mathbb{R}_+$ and define

$$\begin{aligned}\overline{g}(t) &= \sum_{k=1}^{\infty} \sup_{(k-1)h\leq y<kh} g(y)\mathbb{1}_{[(k-1)h,kh\rangle}(t)\\ \underline{g}(t) &= \sum_{k=1}^{\infty} \inf_{(k-1)h\leq y<kh} g(y)\mathbb{1}_{[(k-1)h,kh\rangle}(t)\,,\end{aligned}$$

for all $t \geq 0$. By the definition of direct Riemann integrability on $\mathbb{R}_+$

$$\sum_{k=1}^{\infty} \inf_{(k-1)h\leq y<kh} g(y) \leq \sum_{k=1}^{\infty} \sup_{(k-1)h\leq y<kh} g(y) < \infty\,,$$

thus $\overline{g}$ and $\underline{g}$ have the same structure as the functions considered in Step 2. As in (3.8), we denote

$$\overline{\sigma}(h) = h\sum_{k=1}^{\infty} \sup_{(k-1)h\leq y<kh} g(y) \quad \text{and} \quad \underline{\sigma}(h) = h\sum_{k=1}^{\infty} \inf_{(k-1)h\leq y<kh} g(y)\,.$$

It follows that

$$\begin{aligned}\frac{1}{\mu}\mathbb{E}L_1'\underline{\sigma}(h) &= \liminf_{t\to\infty} \int_{[0,t]} \underline{g}(t-y)\mathrm{d}U_{\xi'}(y) \leq \liminf_{t\to\infty} \int_{[0,t]} g(t-y)\mathrm{d}U_{\xi'}(y)\\ &\leq \limsup_{t\to\infty} \int_{[0,t]} g(t-y)\mathrm{d}U_{\xi'}(y) \leq \limsup_{t\to\infty} \int_{[0,t]} \overline{g}(t-y)\mathrm{d}U_{\xi'}(y) = \frac{1}{\mu}\mathbb{E}L_1'\overline{\sigma}(h)\,,\end{aligned} \tag{4.28}$$

since $\underline{g} \leq g \leq \overline{g}$. By the definition of direct Riemann integrability on $\mathbb{R}_+$

$$\lim_{h\to 0+} (\overline{\sigma}(h) - \underline{\sigma}(h)) = 0$$

and by Lemma 3.4.3

$$\lim_{h\to 0+} \overline{\sigma}(h) = \int_0^\infty g(y)\mathrm{d}y\,.$$

Hence, letting $h \to 0+$ in (4.28) the result follows. $\square$

The proof above shows that Blackwell's renewal theorem implies the key renewal theorem. The reverse implication is straightforward if we set $g = \mathbb{1}_{[0,x\rangle}$, for all $x \geq 0$. This proves that these two renewal theorems are equivalent, as suspected.

4.6 Bartlett-Lewis process

In this and the following section, we will study two Poisson cluster processes, i.e. cluster point processes with a Poissonian parent process. First of them is a point process which is nowadays best known as a Bartlett-Lewis process.

In 1963, Bartlett [Bar63] derived a model to describe a clustering effect in motor traffic and called it a Poisson clustering process. Lewis in [Lew64] used the same model, which he calls a branching Poisson process, in the analysis of computer failure patterns, see also [Lew69]. More modern approach to the analysis of the same process can be found in [FGAMS06], where it is used to model teletraffic arrivals. However, results in all of the references above heavily rely on the special properties of the observed process and thus can not be directly generalized to renewal cluster point processes.

Suppose that $T_i', i \geq 1$ are the points of a homogeneous Poisson process on $\mathbb{R}_+$ with intensity λ and $Y_{ij}', i \geq 1, j \geq 1$ are i.i.d. nonnegative random variables with distribution F independent of T_i', for all i. Let $L_i', i \geq 1$ be i.i.d. nonnegative integer-valued random variables independent of T_i' and Y_{ij}', for all i, j. Define $T_{ik}' = \sum_{j=1}^{k} Y_{ij}'$ and set $T_{i0}' = 0$, for every $i \geq 1$. Bartlett-Lewis point process is given by

$$\sum_{i \geq 1} \sum_{j=0}^{L_i'} \delta_{T_i' + T_{ij}'} .$$

Note that this model coincides with a a homogeneous Poisson process on $\mathbb{R}_+$ if $L_1' = 0$ almost surely.

Let $x > 0$. As we earlier stressed out, the proofs in this chapter, as well as the statements of the theorems, are written without counting the points of the parent process which is now not the case. Hence, a simple modification of Theorem 4.4.2 is needed, which yields

$$\lim_{t \to \infty} \mathbb{E} \left[\sum_{i \geq 1} \sum_{j=0}^{L_i'} \delta_{T_i' + T_{ij}'} \right] \langle t, t+x] = \frac{1}{\mu} \mathbb{E}[L_1' + 1] x = \lambda \mathbb{E}[L_1' + 1] x \,, \tag{4.29}$$

under the assumptions of Theorem 4.4.2. Furthermore, if L_1' and Y_{11}' are integrable, then by Theorem 4.3.1 and Theorem 2.4.1

$$\lim_{t\to+\infty} \mathbb{P}\left(\sum_{i\geq 1}\sum_{j=0}^{L_i'} \delta_{T_i'+T_{ij}'}\langle t, t+x] = 0\right) = \mathbb{P}\left(\sum_{i\in\mathbb{Z}}\sum_{j=0}^{L_i} \delta_{T_i+T_{ij}}\langle 0, x] = 0\right)$$
$$= \exp\left\{-\lambda\left(x + \mathbb{E}L_1' \int_0^x \mathbb{P}(Y_{11}' > y)\mathrm{d}y\right)\right\}\,, \tag{4.30}$$

where the proof of the last equality can be found in Section 5 of [FGAMS06], see also Example 6.3 (b) in [DVJ03]. As before, let R denotes the forward recurrence time. Hence,

$$\lim_{t\to+\infty} \mathbb{P}(R(t) \leq x) = 1 - \exp\left\{-\lambda\left(x + \mathbb{E}L_1' \int_0^x \mathbb{P}(Y_{11}' > y)\mathrm{d}y\right)\right\}\,.$$

Remark 4.6.1. Equations (4.29) and (4.30) which we obtained as a consequence of Theorem 4.3.1 and 4.4.2 are not surprising and can be found already in [Lew69], see Theorem 2.3 and [Lew64], see equation (4.3.5).

4.7 Neyman-Scott process

Neyman-Scott process was first introduced in [NS58] as a model for the distribution of galaxies in the universe. However, we will study only the temporal version.

Suppose that $T_i', i \geq 1$ are the points of a homogeneous Poisson process on $\mathbb{R}_+$ with intensity λ and $T_{ij}', i \geq 1, j \geq 1$ are i.i.d. nonnegative random variables with distribution F independent of T_i' for all i. Let $L_i', i \geq 1$ be i.i.d. nonnegative integer-valued random variables independent of T_i' and T_{ij}', for all i, j. Neyman-Scott point process is given by

$$\sum_{i\geq 1}\sum_{j=1}^{L_i'} \delta_{T_i'+T_{ij}'}\,.$$

Let $x > 0$. If the assumptions of Theorem 4.4.2 are satisfied, then it is immediate that

$$\lim_{t\to\infty} \mathbb{E}\left[\sum_{i\geq 1}\sum_{j=1}^{L_i'} \delta_{T_i'+T_{ij}'}\right]\langle t, t+x] = \lambda \mathbb{E}L_1' x\,.$$

If L_1' and T_{11}' are integrable random variables, then by Theorem 4.3.1 and Theorem 2.4.1

$$\lim_{t\to+\infty} \mathbb{P}\left(\sum_{i\geq 1}\sum_{j=1}^{L_i'} \delta_{T_i'+T_{ij}'}\langle t, t+x] = 0\right) = \mathbb{P}\left(\sum_{i\in\mathbb{Z}}\sum_{j=1}^{L_i} \delta_{T_i+T_{ij}}\langle 0, x] = 0\right)$$
$$= \exp\left\{-\lambda \int_{-\infty}^{+\infty} \left[1 - \phi_{L_1'}\left(F(y) + \overline{F}(x+y)\right)\right]\mathrm{d}y\right\}\,, \tag{4.31}$$

where $\phi_{L_1'}$ denotes a probability generating function of a discrete random variable L_1'. The last equality is a known result, see for instance Example 6.3 (a) in [DVJ03] where it is obtained from the probability generating functional of the Neyman-Scott process. Interested reader can find below, see Lemma 4.7.1, a different proof which exploits the form of the Laplace functional of the marked Poisson process. Furthermore, it follows immediately from (4.31) that

$$\lim_{t\to+\infty} \mathbb{P}(R(t) \le x) = 1 - \exp\left\{-\lambda \int_{-\infty}^{+\infty} \left[1 - \phi_{L_1'}\left(F(y) + \overline{F}(x+y)\right)\right] \mathrm{d}y\right\} .$$

Lemma 4.7.1. *Suppose that* $\sum_{i\in\mathbb{Z}} \sum_{j=1}^{L_i} \delta_{T_i+T_{ij}}$ *is a stationary version of a Neyman-Scott process introduced above. Then, for all* $x > 0$*,*

$$\mathbb{P}\left(\sum_{i\in\mathbb{Z}} \sum_{j=1}^{L_i} \delta_{T_i+T_{ij}} \langle 0, x] = 0\right) = \exp\left\{-\lambda \int_{-\infty}^{+\infty} \left[1 - \phi_{L_1'}\left(F(y) + \overline{F}(x+y)\right)\right] \mathrm{d}y\right\} .$$

Proof. It is easy to see that, by construction, T_i, $i \in \mathbb{Z}$, are the points of the homogeneous Poisson process on $\mathbb{R}$ with intensity λ, T_{ij}, $i \in \mathbb{Z}$, $j \ge 1$ are i.i.d. nonnegative random variables with common distribution F and L_i, $i \in \mathbb{Z}$ are i.i.d. nonnegative integer-valued random variables distributed as L_1'. Furthermore, T_i, T_{ij} and L_i are mutually independent, for all i, j. Hence,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i\in\mathbb{Z}} \sum_{j=1}^{L_i} \delta_{T_i+T_{ij}} \langle 0, x] = 0\right) \\
&= \mathbb{P}\left(T_i + T_{ij} \le 0 \cup T_i + T_{ij} > x, \forall j \in \{1, 2, \ldots, L_i\}, \forall i \in \mathbb{Z}\right) \\
&= \mathbb{E}\left[\mathbb{1}_{\{T_i+T_{ij}\le 0\}\cup\{T_i+T_{ij}>x\}}, \forall j \in \{1, 2, \ldots, L_i\}, \forall i \in \mathbb{Z}\right] \\
&= \mathbb{E}\left[\prod_{i\in\mathbb{Z}} \prod_{j=1}^{L_i} \left(\mathbb{1}_{\{T_i+T_{ij}\le 0\}} + \mathbb{1}_{\{T_i+T_{ij}>x\}}\right)\right] \\
&= \mathbb{E}\left[\exp\left\{\sum_{i\in\mathbb{Z}} \log \prod_{j=1}^{L_i} (\mathbb{1}_{\{T_i+T_{ij}\le 0\}} + \mathbb{1}_{\{T_i+T_{ij}>x\}})\right\}\right] \\
&= \mathcal{L}_{\sum_{i\in\mathbb{Z}} \delta_{(T_i, W_i)}}(f) ,
\end{aligned} \tag{4.32}$$

where $\mathcal{L}_{\sum_{i\in\mathbb{Z}} \delta_{(T_i,W_i)}}$ is a Laplace functional of the point process $\sum_{i\in\mathbb{Z}} \delta_{(T_i,W_i)}$, where $W_i = (L_i, (T_{ij})_{j\ge 1})$. Further, define $f : \mathbb{R} \times \mathbb{N}_0 \times \mathbb{R}_+^{\mathbb{N}}$ by

$$f(y, w) = -\log \prod_{j=1}^{l} \left(\mathbb{1}_{\{y+y_j\le 0\}} + \mathbb{1}_{\{y+y_j>x\}}\right) ,$$

where $w = (l, (y_j)_j)$. By Proposition 3.8 in [Res87], $\sum_{i\in\mathbb{Z}} \delta_{(T_i,W_i)}$ is a Poisson process with mean measure $\Lambda = \lambda Leb \times \mathbb{Q}$, where $\mathbb{Q} = \mathbb{P}(L_1 \in \cdot) \times \prod_{i\in\mathbb{N}} \mathrm{d}F$. Note that Proposition

3.8 in [Res87] applies to point processes on locally compact spaces, but its proof is easily extended to a general Polish space. Therefore, (4.32) equals

$$
\begin{aligned}
&\exp\left\{-\int_{-\infty}^{\infty}\sum_{l=0}^{\infty}\mathbb{P}(L_1=l)\int_{[0,\infty\rangle}\cdots\int_{[0,\infty\rangle}\left(1-\prod_{j=1}^{l}\left(\mathbb{1}_{\{y+y_j\le 0\}}+\mathbb{1}_{\{y+y_j>x\}}\right)\right)\mathrm{d}F(y_1)\cdots\mathrm{d}F(y_l)\lambda\mathrm{d}y\right\}\\
&=\exp\left\{-\lambda\int_{-\infty}^{\infty}\sum_{l=0}^{\infty}\mathbb{P}(L_1=l)\left(1-\left(\int_{[0,\infty\rangle}\left(\mathbb{1}_{\{y+y_1\le 0\}}+\mathbb{1}_{\{y+y_1>x\}}\right)\mathrm{d}F(y_1)\right)^{l}\right)\mathrm{d}y\right\}\\
&=\exp\left\{-\lambda\int_{-\infty}^{\infty}\sum_{l=0}^{\infty}\mathbb{P}(L_1=l)\left(1-\left(\int_{[0,y]}\mathrm{d}F(y_1)+\int_{\langle x+y,\infty\rangle}\mathrm{d}F(y_1)\right)^{l}\right)\mathrm{d}y\right\}\\
&=\exp\left\{-\lambda\int_{-\infty}^{\infty}\sum_{l=0}^{\infty}\mathbb{P}(L_1=l)\left(1-\left(F(y)+1-F(x+y)\right)^{l}\right)\mathrm{d}y\right\}\\
&=\exp\left\{-\lambda\int_{-\infty}^{\infty}\left(1-\sum_{l=0}^{\infty}\mathbb{P}(L_1=l)\left(F(y)+1-F(x+y)\right)^{l}\right)\mathrm{d}y\right\}\\
&=\exp\left\{-\lambda\int_{-\infty}^{\infty}\left(1-\phi_{L_1}\left(F(y)+1-F(x+y)\right)\right)\mathrm{d}y\right\}\\
&=\exp\left\{-\lambda\int_{-\infty}^{\infty}\left(1-\phi_{L_1'}\left(F(y)+1-F(x+y)\right)\right)\mathrm{d}y\right\}\ ,
\end{aligned}
$$

where the first equality follows since T_{ij} are i.i.d. and the last one since L_1 and L_1' are equally distributed. □

4.8 Numerical results

In this section we study two renewal cluster point processes and check if the simulation data are in agreement with the theoretically obtained results.

Example 4.8.1. Suppose that $(T_i')_{i\ge 0}$ is a delayed renewal process with exponential interarrival distribution with parameter λ, L_i' is a Bernoulli random variable with success probability $p\in\langle 0,1\rangle$, $T_{i0}'\equiv 0$ and T_{i1}' are uniformly distributed on interval $\langle -a,a\rangle$, $a>0$, for all $i\ge 0$. Let

$$\xi'=\sum_{i\ge 0}\sum_{j=0}^{L_i'}\delta_{T_i'+T_{ij}'}\,.$$

It is easily seen that the conditions of Theorem 4.3.1 are satisfied, hence one can show similarly as in the previous section (see (4.31) and Lemma 4.7.1)

$$\lim_{t\to\infty}\mathbb{P}(R(t)\le x)=\begin{cases}1-\exp\left\{-\lambda\left(\frac{ap}{2}+x\right)\right\}, & x\ge a\,,\\ 1-\exp\left\{-\lambda\left(x(1+p)-x^2\frac{p}{2a}\right)\right\}, & x<a\,.\end{cases}$$

We simulated 10000 realisations of the above introduced process, with $T'_0 \sim \chi^2(3)$, $\lambda = 0.8$, $p = 0.2$ and $a = 1$. Each renewal sequence has 1500 points. In Figure 4.1 we see that, for large t, distribution of distances of the first point after t in the simulated renewal cluster point processes to t agrees with the theoretical distribution.

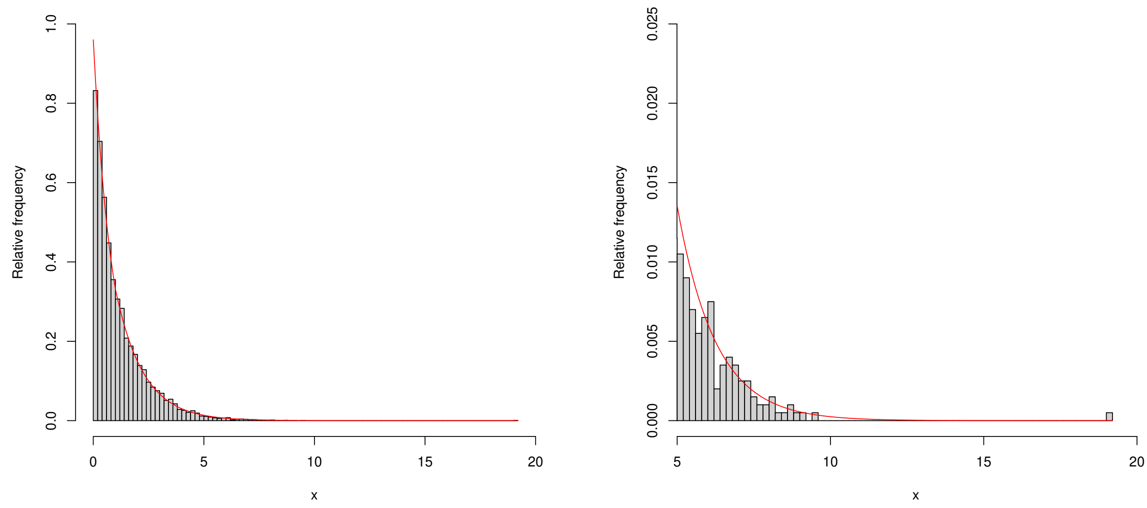

Figure 4.1: *Left:* Histogram of the time until the arrival of the first point after t, for $t = 1000$. Red line is an asymptotic probability density function of the overshoot $R(t)$. *Right:* Zoomed in tail of the histogram on the left, for $x \geq 5$.

Example 4.8.2. Suppose that $(T'_i)_{i\geq 0}$ is a pure renewal process with interarrivals distributed uniformly on $\langle 0, 5\rangle$. Let $L'_0 \equiv 0$ and suppose that both L'_i and T'_{ij} depend on the last interarrival X'_i, for all $i \geq 1$. Precisely, let L'_{i1} and L'_{i2} be i.i.d. Poisson random variables with parameters 0.5 and 5, respectively and let Y'_{ij} be i.i.d. random variables with standard normal distribution, for all $i, j \geq 1$. Then, we set

$$L'_i = L'_{i1}\mathbb{1}_{\{X'_i > 1\}} + L'_{i2}\mathbb{1}_{\{X'_i \leq 1\}} \quad \text{and} \quad T'_{ij} = X'_i + Y'_{ij}\,,$$

for all $i, j \geq 1$. Renewal cluster point process is given by

$$\xi' = \sum_{i\geq 0}\sum_{j=1}^{L'_i} \delta_{T'_i + T'_{ij}}\,.$$

By direct calculation one easily checks the conditions of Theorems 4.4.2 and 4.4.3, hence

$$\lim_{t\to\infty} \mathbb{E}\xi'\langle t, t+x] = \frac{1}{\mathbb{E}X'_1}\mathbb{E}L'_1 x = 0.4 \cdot 1.4x = 0.56x\,, \qquad \text{for all } x > 0$$

and

$$\lim_{t\to\infty} \frac{\mathbb{E}\xi'\langle 0,t]}{t} = \frac{1}{\mathbb{E}X'_1}\mathbb{E}L'_1 = 0.56\,.$$

Simulated data are in agreement with our theoretical results, see Figure 4.2. Furthermore,

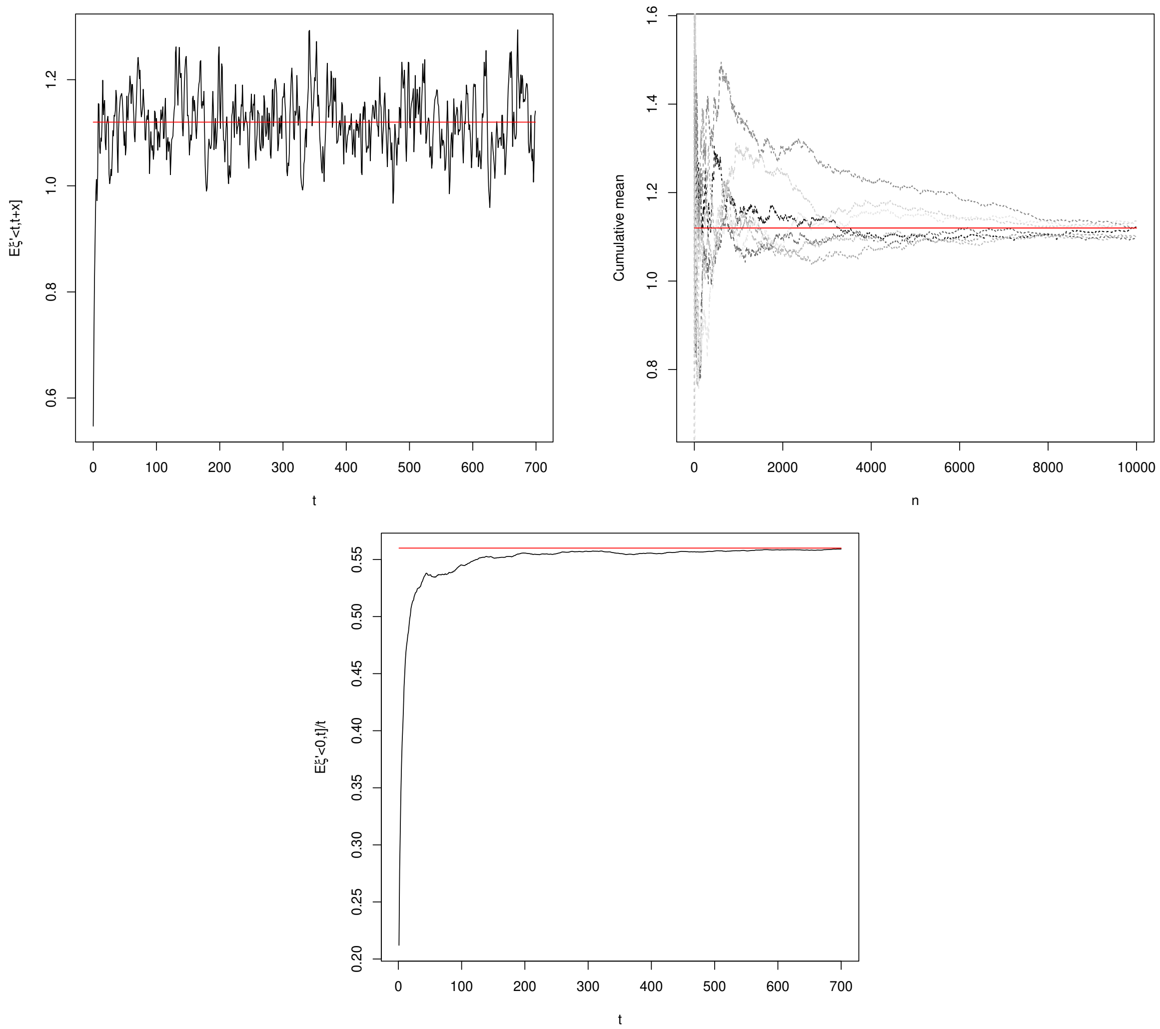

Figure 4.2: *Upper left:* The mean number of points of the process ξ' in the interval $\langle t, t+x]$, for $x = 2$ and 1000 simulations. Red line represents the theoretically obtained value 1.12. *Upper right:* Each of the 7 grey lines represents mean number of points of one simulation of the process ξ' in n consecutive intervals of length $x = 2$. Red line represents the theoretically obtained value 1.12. *Lower center:* The mean number of points of the process ξ' in the interval $\langle 0, t]$ divided by the length of the interval, for 1000 simulations. Red line represents the theoretically obtained value 0.56.

Figure 4.2 motivates an interesting question of the rate of convergence which we have not studied yet. It is the one of the possible topics for the future research.

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Marina Dajaković was born on 14 June 1992 in Zagreb. She finished primary and secondary school in Omiš. In 2010 she started her studies at the Department of Mathematics, Faculty of Science, University of Zagreb, where she obtained both her bachelor's and master's degree in 2013 and 2015 respectively. During her master studies she won two awards for achieving exceptional study results.

Between March 2016 and October 2016, she worked as a portfolio risk management associate in Zagrebačka banka d.d. Since October 2016, she has been working as a teaching assistant at the Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture of the University of Split. She was a teaching assistant for several courses including Probability and statistics.

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Potpis