Periodic homogenization for Levy-type processes

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PRIRODOSLOVNO–MATEMATIČKI FAKULTET

MATEMATIČKI ODSJEK

Ivana Valentić

Periodička homogenizacija za procese Lévyjevog tipa

DOKTORSKI RAD

Mentor:

Izv. prof. dr. sc. Nikola Sandrić

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ZAHVALE

Željela bih se zahvaliti svom mentoru Izv. prof. dr. sc. Nikoli Sandriću koji me uveo u ovo područje i vodio kroz nastajenje ovog doktorata. Uvijek je imao vremena za moja pitanja, sve što sam napisala čitao je u najkraćem roku i imao vrlo konkretan plan za moj rad. Zahvalna sam svojim roditeljima koji su kod mene od malena razvijali ljubav prema matematici i poticali me kroz moje obrazovanje. Također se zahvaljujem svojim kolegama koji su mi bili podrška u teškim razdobiljima i društvo u veselim. Zahvaljujem se Teni i Anti, koji su moja velika motivacija i najveća radost, a i svima onima koji su ih čuvali dok je ova dizertacija nastajala. Najviše sam zahvalna Grguru koji je uvijek vjerovao u mene i poticao me u mom radu, koji je volio kritički rasporavljati o problemima iz radnje i pružao mi podršku u problemima oko radnje.

SUMMARY

The main goal of this thesis is to discuss periodic homogenization of a Lévy-type pseudo-differential operator. Our approach to this problem is based on probabilistic techniques. More precisely, as the main result we show that the appropriately centered and scaled Lévy-type process (LTP) generated by this operator converges weakly to a Brownian motion with covariance matrix given in terms of the operator coefficients. We specially focus on a class of Levy-type processes admitting "small jumps" only and a class of diffusion processes having degenerate diffusion term. These results generalize and refine the classical and well-known results related to periodic homogenization of diffusion process and of Lévy-type process in balanced form. In order to resolve these problems, it is necessary to combine both probabilistic and analytical approaches and tools, such as theory of semimartingales, stochastic stability theory and theory of integro-differential equations.

Keywords: Brownian motion, Markov processes, Lévy-type processes, Feller processes, semimartingales, ergodicity, central limit theorem, homogenization, stochastic differential equations, partial differential equations, Itô's formula, Feynman-Kac formula, Poisson equation

SAŽETAK

Glavni cilj ove disertacije je diskutirati periodičku homogenizaciju pseudo-diferencijalnog operatora Lévyjevog tipa. Naš pristup ovom problemu bazira se na vjerojatnosnim metodama. Preciznije, kao glavni rezultat dokazujemo da odgovarajuće centriran i skaliran proces Lévyjevog tipa generiran takvim operatorom slabo konvergira prema Brownovom gibanju s kovarijacijskom matricom danom u terminima koeficijenata operatora. Posebno se koncentriramo na klasu procesa Lévyjevog tipa koji dozvoljavaju samo "male skokove" i na klasu procesa difuzija s degeneriranim difuzijskim koeficijentom. Ti rezultati generaliziraju i produbljuju klasične i dobro poznate rezultate vezane uz periodičku homogenizaciju difuzije i procesa Lévyjevog tipa u balansiranom obliku. Kako bismo razriješili ove probleme nužno je kombinirati vjerojatnosni i analitički pristup i metode, kao što su teorija semimartingala, teorija stohastičke stabilnosti i teorija integro-diferencijalnih jednadžbi.

Ključne riječi: Brownovo gibanje, Markovljevi procesi, Procesi Lévyjevog tipa, Fellerovi procesi, semimartingali, ergodičnost, centralni granični teorem, homogenizacija, stohastičke diferencijalne jednadžbe, parcijalne diferencijalne jednadžbe, Itôva formula, Poissonova jednadžba, Feynman-Kacova formula

Contents

1. Introduction

1.1. Motivation

Many phenomena arising in nature, engineering and social sciences involve heterogeneous media, such as problems related to diffusion of population, composite materials and large financial market movements. Because of heterogeneity, mathematical models used in describing these phenomena (typically stochastic processes or integro-differential equations) are characterized by heterogeneous coefficients, and as such are very complicated and hard to analyse. However, on the macroscopic scales, they often show an effective scale structure. More precisely, in many cases when the coefficients (rapidly) vary on small scales it is possible to use the fine microscopic structure of the media to derive an effective (homogenized) model which is a valid approximation of the initial model and, in general, it is of much simpler form (typically it is characterized by constant coefficients).

The problem of homogenization of a local (second-order elliptic) operator is a very well-studied topic and there is a vast amount of literature on this subject, especially from the analytical point of view. In this thesis we approach the problem of homogenization using probabilistic methods, which were first introduced by A. Bensoussan, J-L. Lions and G. C. Papanicolaou, see [12]. These methods rely on the well-known connection between convergence in distribution of Markov processes (central limit theorems) and the convergence of the corresponding infinitesimal generators. Accordingly, the central limit theorem arises as an appropriate tool in addressing the homogenization problem, which results in deep connection between the theory of Markov processes and the theory of differential equations. More precisely, main steps in the probabilistic approach to the homogenization of second-order elliptic operator with periodic coefficients are the following: projection of the corresponding diffusion process on the cell of periodicity, discussion of the stochastic stability property of the projection (which is possible since the projection is also a Markov process due to periodicity of the coefficients) and finally, by employing concluded stochastic stability property of the projection and central limit theorem, homogenization of the operator.

Discussion of stochastic stability of the projected process includes detecting the equilibrium (stationary distribution) of the Markov process as well as determining the rate at which it converges (with respect to the total variation distance) to the equilibria. If this rate is exponential, we call the process geometrically ergodic. Geometric ergodicity of Markov processes is very well studied in the literature, see [29], [69], [70] and [106].

The classical approach in using central limit theorem is through the so-called martingale problem, which is a rather technically demanding and restrictive approach (especially in the case of Markov processes with jumps). Our approach will rely on the characteristics of semimartingales method, see [50].

In the case of a local operator the corresponding Markov process is a diffusion process. The general theory of diffusion processes is very well developed and understood, see [86]. The central limit theorem for such processes has been studied in [14] under the assumption that the diffusion coefficient is uniformly elliptic. In this thesis we expand these results to the case of a singular diffusion coefficient, see also [43].

The problem of homogenization of non-local operator (for example, generated by a stochastic differential equation with jumps) is a largely uninvestigated problem both from analytical and probabilistic point of view. Results in this area were obtained in [35], [37] and [42], where authors focus on so-called stable-like operators, that is, on the case when Lévy kernel admits "large jumps" of power-type only. In [87] periodic homogenization of such an operator in balanced form (that is, when the drift term vanishes and the Lévy kernel is symmetric) and with small jumps only (precisely, Lévy kernel having finite second moment) is discussed. However, in a large number of situations (such as homogenization problems related to porous media) the balanced form assumption is a serious restriction. We generalize this result to the case with non-vanishing drift term and non-symmetric Lévy kernel.

1.2. LITERATURE OVERVIEW

Our work contributes to the classical theory of periodic homogenization. Most of the existing literature on this subject focuses on the problem of homogenization of non-degenerate differential operators, mostly based on PDE methods. We refer the interested readers to the classical monographs [2], [12], [17], [24], [51], [102] and the references therein.

In this thesis we extend and refine this classical theory of periodic homogenization. We do this in two ways. Firstly, our work in Chapter 3 relates to the active research on homogenization of integro-differential operators, and Markov processes with jumps. Secondly, our work in Chapter 4 was motivated by developments of the recent years in understanding the homogenization of degenerate PDEs.

Our work in Chapter 3 is highly motivated by the results in [12], [13] and [87] where, by employing probabilistic techniques, the authors considered periodic homogenization of second-order elliptic operator in non-divergence form and integro-differential operator in the balanced form, respectively. In this thesis, we generalize both results by including the non-local part of the operator, as well as non-symmetries caused by the drift term and the Lévy kernel. In a closely related work [81], by using analytic techniques (the corrector method), the authors discuss periodic homogenization of the operator with a convolution-type Lévy kernel. The homogenized operator is again a second-order elliptic operator with constant coefficients. This case is not covered in this thesis since finiteness of Lévy kernel excludes regularity properties of the corresponding semigroup assumed in Section 3.2.

Results related to the problem of periodic homogenization of non-local operators, based on probabilistic techniques, were obtained in [35], [36], [37], [40], [44], [45], [46] and [103]. In all this works the focus is on the so-called stable-like operators (possibly with variable order), that is, on the case when Lévy kernel admits “large jumps” of power-type only. In this case, by using subdiffusive scaling, the homogenized operator is the infinitesimal generator of a stable Lévy process with the index of stability being equal to the power of the scaling factor. The problem of stochastic homogenization (that is, homogenization of operators with random coefficients) of this type of operators has been considered in [85]. PDE and other analytical approaches to the problem of periodic homogenization and stochastic homogenization of stable-like operators can be found in [3], [4], [5], [9], [8], [33], [34], [54], [93], [95], [96].

Let us also remark that the class of processes considered in Chapter 3 od this thesis constitute of both diffusion and pure-jump part, and the behaviour of the homogenized process depends on both of them. This makes the approach to this problem more subtle since we need to take care of diffusion processes, diffusion processes with jumps and pure jump processes, simultaneously.

In the recent years homogenization of degenerate PDEs has attracted much attention due to its significance both in theory and applications. This was a motivation for our work in Chapter 4. We refer the readers to [26], [78], [79] and [80] for a PDE approach to this problem, and [22], [27], [83] and [84] for a probabilistic approach.

However, in all these works the major limitation is that the diffusion term can fully degenerate (vanish) on a "small" part of the domain only. In the first five references it is allowed that it vanishes on a set of Lebesgue measure zero only and in the rest of the domain it must have a full rank. While in [27], [83] and [84] it is allowed that it degenerates everywhere, but its rank must be greater than or equal to one except maybe on a set of Lebesgue measure zero.

In this thesis we partly fill this gap and focus on the case when the diffusion part vanishes on a set of positive Lebesgue measure. In the closely related article [43] (see also [99] and [77] in the context of semilinear elliptic and parabolic PDEs), by also employing probabilistic methods, the authors are concerned with the same questions we discuss in this article. However, unfortunately, there seems to be a doubt about their proof of the functional CLT in [43, Theorem 3.1]. In the Chapter 3 of this thesis, under slightly weaker assumptions (and by employing different techniques) we resolve this issue, or at least suggest an alternative approach to the problem.

Throughout the thesis we use some of the general methods in probability theory, theory of stochastic processes and real analysis which can be found in [1, 30, 68, 86]. For obtaining the stochastic stability results we use [62, 69, 70, 105, 107]. For general facts and specific results about the semimartigale theory we will use [16, 50, 91]. For general theory of Feller processes and generating examples we use [15, 19, 20, 21, 42, 55, 56].

1.3. Chapter overview

This thesis is divided into five chapters. In Chapter 2 we give most of the definitions and general results necessary for understanding the material in the rest of the thesis. Chapters 3 and 4 is where main results, central limit theorems, are stated and proven. In Chapter 3 this is done for class of processes which we refer to as Lévy-type processes with "small jumps" and in Chapter 4 for class of processes we refer to as "degenerate" diffusions. Chapter 5 is dedicated to applying results proven in previous chapters to homogenization of operators corresponding to those processes. In the case of "degenerate" diffusions in this chapter we also discuss homogenization of the associated elliptic boundary-value problem and parabolic initial-value problem.

Chapter 2 is divided into seven sections. In Section 2.1 we introduce notation which is used throughout the thesis. Most importantly we define projection onto torus which is crucial in proving ergodicity results and general Hölder spaces which are used in conditions implying central limit theorem in the Chapter 3. In Section 2.2 we define Markov processes, the associated semigroup, Feller processes and state results related to ergodic property of the process. In Section 2.3 we define and state some basic properties of the infinitesimal generator, define Lévy-type processes and formally state what it means for such process to have "small jumps" only. In Section 2.4 we introduce the periodic structure and state state some properties this condition implies. In Section 2.5 we define the resolvent and show its connection to Poisson equation. In Section 2.6 we define semimartingales, state Itô's formula in this general setting, define characteristics of the semimartingale and state results connecting the convergence of a semimartingale with convergence of its characteristics. In Section 2.7 we take a closer look at a special class of continuous LTPs, which will play a central role in Chapter 4. We show how they connect to the previously stated theory but also acknowledge some of the specificities.

Chapter 3 is divided into four sections. Each of the first three sections adds additional assumption necessary for the proof in the Section 3.4. The assumption in Section 3.1 is strong Feller property and irreducibility and in this section we also provide examples of processes satisfying this condition and prove that this assumption implies geometric ergodicity of the process. The assumption in Section 3.2 implies the regularity of the solution of Poisson equation and again in this section we provide examples. In Section 3.3 we show three results, every time assuming less regularity of the function for which we apply Itô's formula. At the end of this section we give one last condition which combines these regularity assumptions. Section 3.4 is entirely dedicated to stating and proving the Central limit theorem, which is one of the two biggest contributions of this thesis.

Chapter 4 is divided into five sections. In Section 4.1 we explain what it means for a diffusion to be "degenerate". In Section 4.2 we give a condition which compensates for the condition in the previous section and show that this implies Geometric ergodicity of the process. In Section 4.3 we concentrate on the special case when these conditions are enough to prove the Central limit theorem, which we also do in this section. This result is important because it is the first result in the literature showing the CLT in the case of "degenerate" diffusions and the assumptions imposed on the process are very mild. In the Section 4.4 we proceed with the general case and add additional conditions which guarantee regularity needed for the use of generalization of Itô's formula, which is also proven in this section. Section 4.5 provides the proof of the second of two biggest contributions of this thesis under the assumptions made in Sections 2.7, 4.1, 4.2 and 4.4. In this section we also provide an example of the process satisfying these conditions.

Chapter 5 is divided into five sections. In Section 5.1 we show how convergence of processes implies convergence of corresponding infinitesimal generators. In Section 5.2 we present motivation for Feynman-Kac formula which provides connection between stochastic processes and partial differential equations. We state this result formally in Section 5.3 for viscosity solution of PDEs, which we also define in this section. In Section 5.4 we show that the solution to elliptic boundary-value problem converges to the solution of homogenized equation and in Section 5.5 we do the same for initial-value parabolic problem.

2. Preparatory Material

2.1. Function Spaces

We use $\mathbb{R}^{\mathrm{d}}$, $\mathrm{d} \in \mathbb{N}$, to denote real-valued d-dimensional vectors, and write $\mathbb{R}$ for $\mathrm{d} = 1$. All vectors will be column vectors. The Euclidean norm on $\mathbb{R}^{\mathrm{d}}$ is denoted by $|\cdot|$. By M^{T} and $\|M\|_{\mathrm{HS}} := (\mathrm{Tr}\, MM^{\mathrm{T}})^{1/2}$ we denote the transpose and the Hilbert-Schmidt norm of a $n \times m$-matrix M, respectively. For a square matrix M, $\mathrm{Tr}\, M$ stands for its trace. We use $\mathbb{S}^{\mathrm{d}}$, to denote the space of symmetric $\mathrm{d} \times \mathrm{d}$ matrices. For a set $A \subseteq \mathbb{R}^{\mathrm{d}}$, the symbols A^c, $\mathbb{1}_A$, $\overline{A}$ and ∂A stand for the complement, indicator function, (topological) closure and (topological) boundary of A, respectively. $B_r(x)$ denotes the ball of radius r around $x \in \mathbb{R}^{\mathrm{d}}$. For $x, y \in \mathbb{R}$ by $x \wedge y$ we denote the minimum.

Let $\tau = (\tau_1, \ldots, \tau_{\mathrm{d}}) \in (0, \infty)^{\mathrm{d}}$ be fixed, and let $\tau\mathbb{Z}^{\mathrm{d}} := \tau_1\mathbb{Z} \times \ldots \times \tau_{\mathrm{d}}\mathbb{Z}$. For $x \in \mathbb{R}^{\mathrm{d}}$ define

$$x_\tau := \{y \in \mathbb{R}^{\mathrm{d}} : x - y \in \tau\mathbb{Z}^{\mathrm{d}}\} \qquad \text{and} \qquad \mathbb{R}^{\mathrm{d}}/\tau\mathbb{Z}^{\mathrm{d}} := \{x_\tau : x \in \mathbb{R}^{\mathrm{d}}\}.$$

In the sequel, we denote $\mathbb{T}^{\mathrm{d}}_\tau = \mathbb{R}^{\mathrm{d}}/\tau\mathbb{Z}^{\mathrm{d}}$. Clearly, $\mathbb{T}^{\mathrm{d}}_\tau$ is obtained by identifying the opposite faces of $[0, \tau] := [0, \tau_1] \times \ldots \times [0, \tau_{\mathrm{d}}]$. Let $\Pi_\tau : \mathbb{R}^{\mathrm{d}} \to \mathbb{T}^{\mathrm{d}}_\tau$, $\Pi_\tau(x) := x_\tau$, be the covering map. A function $f : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ is called *τ-periodic* if

$$f(x + \tau) = f(x), \qquad x \in \mathbb{R}^d.$$

Clearly, every τ-periodic function $f(x)$ is completely and uniquely determined by its restriction $f|_{[0,\tau]}(x)$ to $[0, \tau]$, and since $f|_{[0,\tau]}(x)$ assumes the same value on opposite faces of $[0, \tau]$, it can be identified by a function $f_\tau : \mathbb{T}^{\mathrm{d}}_\tau \to \mathbb{R}$ given with $f_\tau(x_\tau) = f(x)$. For notational convenience, we will often omit the subscript τ and simply write x instead of x_τ, and f instead of f_τ when there is no chance of confusion.

We let $\mathscr{B}(\mathbb{R}^{\mathrm{d}})$ and $B(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$ denote the Borel σ-algebra on $\mathbb{R}^{\mathrm{d}}$ and the space of $\mathscr{B}(\mathbb{R}^{\mathrm{d}})/\mathscr{B}(\mathbb{R}^{\mathrm{n}})$-measurable function, respectively. For $A\subseteq\mathbb{R}^{\mathrm{d}}$, $\mathscr{B}(A)$ stands for $\{A\cap B : B\in\mathscr{B}(\mathbb{R}^{\mathrm{d}})\}$. For a Borel measure $\mu(\mathrm{d}x)$ on $\mathscr{B}(\mathbb{R}^{\mathrm{d}})$ and $f\in B(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, we often use the convenient notation $\mu(f)=\int_{\mathbb{R}^{\mathrm{d}}}f(x)\,\mu(\mathrm{d}x)$. For $f\in B(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$ we let $\|f\|_\infty := \sup_{x\in\mathbb{R}^{\mathrm{d}}}|f(x)|$ denote its supremum norm, and $B_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ stands for $\{f\in B(\mathbb{R}^{\mathrm{d}},\mathbb{R}) : \|f\|_\infty<\infty\}$. We say that $f=(f_1,\dots,f_{\mathrm{n}})^{\mathrm{T}}\in B_b(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$ if $f_k\in B_b(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$ for each $k=1,\dots\mathrm{n}$. With Id we denote the identity operator on the space $B_b(\mathbb{R}^{d},\mathbb{R}^{\mathrm{n}})$. We use $C_b^k(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $C_{u,b}^k(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $C_\infty^k(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$ and $C_c^k(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $k\in\mathbb{N}_0\cup\{\infty\}$, to denote the subspaces of $B_b(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})\cap C^k(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$ of all k times differentiable functions such that all derivatives up to order k are bounded, uniformly continuous and bounded, vanish at infinity, and have compact support, respectively. Gradient of $f\in C^1(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ is denoted by $\nabla f(x)=(\partial_1 f(x),\dots,\partial_{\mathrm{d}}f(x))'$, and for $f=(f_1,\dots,f_{\mathrm{n}})^{\mathrm{T}}\in C^1(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$ we write $\mathrm{D}f(x)=(\nabla f_1(x),\dots,\nabla f_{\mathrm{n}}(x))^{\mathrm{T}}$ for the corresponding Jacobian. Space $C_c^k(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ is a Banach space endowed with the norm $\|f\|_k:=\sum_{m:\,|m|\le k}\|D^m f\|_\infty$, where $m=(m_1,\dots,m_{\mathrm{d}})^{\mathrm{T}}\in\mathbb{N}_0^{\mathrm{d}}$, $|m|=m_1+\cdots+m_{\mathrm{d}}$, and $D^m f(x)=\partial^{m_1}\dots\partial^{m_{\mathrm{d}}}f(x)$. By $\hat f(\xi):=(2\pi)^{-\mathrm{d}}\int_{\mathbb{R}^{\mathrm{d}}}e^{-i\langle\xi,x\rangle}f(x)\,\mathrm{d}x$ we denote the Fourier transform of the function $f(x)$.

Function $f:\mathbb{R}^d\to\mathbb{R}$ is said to be *lower semi-continuous* if

$$\liminf_{y\to x} f(y)\,\ge\, f(x)\,,\qquad x\in\mathbb{R}^{\mathrm{d}}$$

and *upper semi-continuous* if

$$\limsup_{y\to x} f(y)\,\le\, f(x)\,,\qquad x\in\mathbb{R}^{\mathrm{d}}\,.$$

Function $\phi:(0,1]\to(0,\infty)$ is said to be *almost increasing* if there exists a constant $\kappa\in(0,1]$ such that $\kappa\phi(r)\le\phi(R)$ for all $0<r\le R\le 1$.

Let now $\psi:(0,1]\to[0,\infty)$ be such that $\psi(1)=1$ and $\lim_{r\to0}\psi(r)=0$. For $f\in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ and $j\in\mathbb{N}_0$, define

$$[f]_{-j,\psi}:=\sup_{x\in\mathbb{R}^{\mathrm{d}}}\ \sup_{h\in\bar B_1(0)\setminus\{0\}}\frac{|f(x+h)-f(x)|}{\psi(|h|)|h|^{-j}}\,.$$

Also, let

$$A_\psi:=\{\gamma\in\mathbb{R} : r\mapsto\psi(r)r^{-\gamma}\text{ is almost increasing in }(0,1]\}\,.$$

If $\gamma_1 < \gamma_2$ and $\gamma_2 \in A_\psi$ we know that there exists $\kappa \in (0,1]$ such that for all $0 < r \le R \le 1$ $\kappa\,\psi(r) \le \psi(R)$ and therefore

$$\begin{aligned}&\kappa\psi(r)r^{-\gamma_1} = \kappa\psi(r)r^{-\gamma_2}r^{\gamma_2-\gamma_1} \le \psi(R)R^{-\gamma_2}r^{\gamma_2-\gamma_1} = \\ &\psi(R)R^{-\gamma_1}\left(\frac{r}{R}\right)^{\gamma_2-\gamma_1} \le \psi(R)R^{-\gamma_1},\end{aligned}$$

which proves that also $\gamma_1 \in A_\psi$. Set

$$m_\psi := \sup A_\psi .$$

If $m_\psi > 0$, we call $\psi(r)$ the *Hölder exponent*. Observe that the product of two Hölder exponents is a Hölder exponent. Indeed if $\psi(r)$ and $\varphi(r)$ are Hölder exponents this means that there are $\gamma_\psi, \gamma_\varphi > 0$ and $\kappa_\psi, \kappa_\varphi \in (0,1]$ such that

$$\kappa_\psi \psi(r)r^{-\gamma_\psi} \le \psi(R)R^{-\gamma_\psi} \text{ and } \kappa_\varphi \varphi(r)r^{-\gamma_\varphi} \le \varphi(R)R^{-\gamma_\varphi} .$$

From this we conclude

$$\left(\kappa_\psi\kappa_\varphi\right)\psi(r)\varphi(r)r^{-\left(\gamma_\psi+\gamma_\varphi\right)} \le \psi(R)\varphi(R)R^{-\left(\gamma_\psi+\gamma_\varphi\right)}$$

and therefore $m_{\psi\varphi} > 0$.

If $\psi(r)$ is the Hölder exponent let $k \in \mathbb{N}_0$ be such that $m_\psi \in (k, k+1]$. Note that this implies that the function $r \mapsto \psi(r)r^{-k}$ is almost increasing in $(0,1]$ and function $r \mapsto \psi(r)r^{-(k+1)}$ is not. Define

$$C_b^\psi(\mathbb{R}^{\mathrm{d}},\mathbb{R}) := \{f \in C_b^k(\mathbb{R}^{\mathrm{d}}) : [D^m f]_{-k,\psi} < \infty \text{ for all } m \in \mathbb{N}_0^{\mathrm{d}} \text{ such that } |m| = k\} .$$

This space is called a *generalized Hölder space*, and it is a normed vector space with the norm

$$\|f\|_\psi := \|f\|_k + \sum_{m:\,|m|=k} [D^m f]_{-k,\psi} ,$$

(see [7]). Observe that if $m_\psi \in (k, k+1]$ for some $k \in \mathbb{N}_0$ then $C_b^{k+1}(\mathbb{R}^{\mathrm{d}},\mathbb{R}) \subsetneq C_b^\psi(\mathbb{R}^{\mathrm{d}},\mathbb{R}) \subsetneq C_b^k(\mathbb{R}^{\mathrm{d}},\mathbb{R})$.

Example 2.1.1. The name generalized Hölder space suggests the previous definition generalises Hölder spaces. Let us prove that this is so. Recall that we say $f : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ is Hölder continuous if there exist constants $C, \alpha \ge 0$ such that $|f(x) - f(y)| \le C|x-y|^\alpha$ for all $x,y\in\mathbb{R}^{\mathrm{d}}$. For $k\in N_0$, $\alpha\in(0,1)$ we denote by $C^{k,\alpha}(\mathbb{R}^{\mathrm{d}},\mathbb{R})\subseteq C^k(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ the set of functions f such that its kth partial derivatives are Hölder continuous with exponent α. We will write $C^{\alpha}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ instead of $C^{0,\alpha}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$. Notice that in the definition of Hölder exponent we can take $\psi(r)=r^{k+\alpha}$ since $1^{k+\alpha}=1$ and $\lim_{r\to 0}r^{k+\alpha}=0$. For $f\in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ and $j\in\mathbb{N}_0$, we have

$$[f]_{-j,\psi}=\sup_{x\in\mathbb{R}^{\mathrm{d}}}\sup_{h\in\bar{B}_1(0)\setminus\{0\}}\frac{|f(x+h)-f(x)|}{|h|^{-j+k+\alpha}},$$

specially for $k=j$

$$[f]_{-k,\psi}=\sup_{x\neq y\in\mathbb{R}^{\mathrm{d}}}\frac{|f(x)-f(y)|}{|x-y|^{\alpha}}.$$

Since constant function is almost increasing and $r\mapsto r^{-\varepsilon}$ is not, for any $\varepsilon>0$, we can conclude that

$$A^{k,\alpha}:=\{\gamma\in\mathbb{R}:\, r\mapsto r^{-\gamma+k+\alpha}\text{ is almost increasing in }(0,1]\}=(-\infty,k+\alpha],$$

This implies that $m^{k,\alpha}:=\sup A^{k,\alpha}=k+\alpha$ and since $k+\alpha\in(k,k+1]$ we get

$$C_b^{\psi}(\mathbb{R}^{\mathrm{d}},\mathbb{R})=C_b^{k,\alpha}(\mathbb{R}^{\mathrm{d}},\mathbb{R}):=C^{k,\alpha}(\mathbb{R}^{\mathrm{d}},\mathbb{R})\cap C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$$

and

$$\|f\|_{\psi}=\|f\|_k+\sum_{m:\,|m|=k}\sup_{x\neq y\in\mathbb{R}^{\mathrm{d}}}\frac{|D^m f(x)-D^m f(y)|}{|x-y|^{\alpha}}=\|f\|_{k,\alpha}.$$

□

Let $\mathscr{B}(\mathbb{T}_\tau^{\mathrm{d}})$ denote the Borel σ-algebra on $\mathbb{T}_\tau^{\mathrm{d}}$ (with respect to the standard quotient topology). Since $f\leftrightarrow f_\tau$ gives a one-to-one correspondence between $\{f:\mathbb{R}^{\mathrm{d}}\to\mathbb{R}:$ f is τ-periodic$\}$ and $\{f_\tau:\mathbb{T}_\tau^{\mathrm{d}}\to\mathbb{R}\}$, in an analogous way we define $B(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $B_b(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $C_b^k(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $C_{u,b}^k(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $C_\infty^k(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $C_c^k(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $C^{\psi}(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R})$ and $C^{k,\alpha}(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R})$. On the space of signed measures on $B(\mathbb{T}_\tau^{d})$ we denote by $\|\cdot\|_{\mathrm{TV}}$ the total variation norm, that is $\|\mu\|_{\mathrm{TV}}=\sup_{|f|\leq 1}|\mu(f)|$.

We say that a function $f:[0,\infty)\to\mathbb{R}^{\mathrm{d}}$ is *càdlàg* (from French ”continue à droite, limite à gauche”) if it is right-continuous with left limits. Special case of càdlàg functions are continuous functions. Denote by $\mathbb{D}\left([0,\infty),\mathbb{R}^{\mathrm{d}}\right)$ the space of all càdlàg functions and by $\mathbb{C}\left([0,\infty),\mathbb{R}^{\mathrm{d}}\right)$ the space of all continuous functions. On the space $\mathbb{C}\left([0,\infty),\mathbb{R}^{\mathrm{d}}\right)$ we introduce tho following metric

$$d(\alpha,\beta)=\sum_{n=0}^{\infty}2^{-n}(1\wedge\sup_{s\le n}|\alpha(s)-\beta(s)|)$$

for $\alpha,\beta\in\mathbb{C}\left([0,\infty),\mathbb{R}^{\mathrm{d}}\right)$. Topology associated with this metric is called *local uniform topology*.

Intuitively, one might say that this topology allows us to ”wiggle space a bit”. We next define topology on the set of càdlàg functions which can, intuitively allow us to ”wiggle space and time a bit”.

Let Λ be the set of all continuous strictly increasing bijections $\lambda:[0,\infty)\to[0,\infty)$ (we say that such a function λ is a change of time).

On the space $\mathbb{D}\left([0,\infty),\mathbb{R}^{\mathrm{d}}\right)$ there is a metrizable topology, called the Skorokhod J_1-topology such that a sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges to α if and only if there is a sequence $(\lambda_n)_{n\in\mathbb{N}}\subset\Lambda$ such that

$$\sup_{s\ge 0}|\lambda_n(s)-s|\to 0,\ \text{and}$$

$$\sup_{s\le N}|\alpha_n(\lambda_n(s))-\alpha(s)|\to 0 \text{ for all } N\in\mathbb{N}.$$

Next results can be found in [50, VI.1b.]

Proposition 2.1.2. *Let $\alpha_n,\beta_n,\alpha,\beta\in\mathbb{D}\left([0,\infty),\mathbb{R}^{\mathrm{d}}\right)$, for $n\in\mathbb{N}$. Then*

(i) if a sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges to α locally uniformly then it converges to α in the Skorokhod J_1*-topology.*

(ii) if α is a continuous function, a sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges to α in the Skorokhod J_1*-topology if and only if it converges to α locally uniformly.*

(iii) if a sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges to α and a sequence $(\beta_n)_{n\in\mathbb{N}}$ converges to β in the Skorokhod J_1*-topology, where β is continuous, then sequence $(\alpha_n+\beta_n)_{n\in\mathbb{N}}$ converges to $\alpha+\beta$ in the Skorokhod* J_1*-topology (this is not generally true if β is not continuous).*

2.2. STABILITY OF MARKOV PROCESSES

Let $(\Omega, \mathscr{F})$ and $(S, \mathscr{S})$ be two measurable spaces and let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be a sequence of σ-algebras such that $\mathscr{F}_n \subseteq \mathscr{F}_{n+1} \subseteq \mathscr{F}$ for each $n \in \mathbb{N}$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence such that X_n is $\mathscr{F}_n/\mathscr{S}$-measurable. For each $x \in S$ let $\mathbb{P}_x$ be a probability measure on $(\Omega, \mathscr{F})$ such that the map $x \mapsto \mathbb{P}_x(X_n \in B)$ is Borel measurable for each $n \in \mathbb{N}$ and all $B \in \mathscr{S}$ and that $\mathbb{P}_x(X_0 = x) = 1$. Recall that we say $\{X_n\}_{n\in\mathbb{N}}$ is a time homogeneous Markov chain if it satisfies the Markov property, that is

$$\mathbb{P}_x(X_{n+k} \in B \mid \mathscr{F}_n) = \mathbb{P}_{X_n}(X_k \in B),$$

for each $n, k \in \mathbb{N}$ and all $x \in S$, $B \in \mathscr{S}$.

For the purpose of this thesis we will need to study a somewhat more complicated objects called Markov processes which posses similar properties to Markov chains but are defined for time $t \in [0, \infty)$, that is, they are a continuous-time equivalent to discrete time Markov chains. In this chapter we investigate how properties of Markov chains translate for processes.

Let E be locally compact and separable metric space and $\mathscr{B}(E)$ a Borel field on E. Throughout this thesis space $(E, \mathscr{B}(E))$ will be either $(\mathbb{R}^{\mathrm{d}}, \mathscr{B}(\mathbb{R}^{\mathrm{d}}))$ or $(\mathbb{T}^{\mathrm{d}}_\tau, \mathscr{B}(\mathbb{T}^{\mathrm{d}}_\tau))$. Let $(\Omega, \mathscr{F})$ be a measurable space and let $\{\mathscr{F}_t\}_{t\geq 0}$ be a family of σ-algebras such that $\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F}$ for each $0 \leq s \leq t$. Let $\{X_t\}_{t\geq 0}$ be a process such that X_t is $\mathscr{F}_t/\mathscr{B}(E)$-measurable for each $t \geq 0$. For each $x \in E$ let $\mathbb{P}_x$ be a probability measure on $(\Omega, \mathscr{F})$ such that the map $x \mapsto \mathbb{P}_x(X_t \in B)$ is Borel measurable for all $t \geq 0$ and all $B \in \mathscr{B}(E)$ and that $\mathbb{P}_x(X_0 = x) = 1$. We say $\{X_t\}_{t\geq 0}$ is a *time homogeneous Markov process* if it satisfies the *Markov property*, that is

$$\mathbb{P}_x(X_{t+s} \in B \mid \mathscr{F}_t) = \mathbb{P}_{X_t}(X_s \in B),$$

for all $t, s \geq 0$ and all $x \in E$, $B \in \mathscr{B}(E)$.

Notice that if we take a Markov process $\{X_t\}_{t\geq 0}$ and a parameter $h > 0$ than by setting $Y_n := X_{nh}$ for $n \in \mathbb{N}$ we have constructed a Markov chain $\{Y_n\}_{n\in\mathbb{N}}$. We call this chain a *skeleton chain* of Markov process $\{X_t\}_{t\geq 0}$.

Denote by $\mathbb{E}_x$ expectation with respect to $\mathbb{P}_x(\mathrm{d}\omega)$, $x \in S$ in a discrete and $x \in E$ in a continuous-time setting. Then $\{X_n\}_{n\in\mathbb{N}}$ satisfies Markov property if and only if for all

$f \in B_b(S,\mathbb{R})$ and all $x \in S$, $n,k \in \mathbb{N}$ we have

$$\mathbb{E}_x\left[f(X_{n+k}) \mid \mathscr{F}_n\right] = \mathbb{E}_{X_n}\left[f(X_k)\right].$$

The same holds for a processes, that is $\{X_t\}_{t\geq 0}$ satisfies Markov property if and only if for all $f \in B_b(E,\mathbb{R})$ and all $x \in E$, $t,s \geq 0$ we have

$$\mathbb{E}_x\left[f(X_{t+s}) \mid \mathscr{F}_t\right] = \mathbb{E}_{X_t}\left[f(X_s)\right].$$

Let N be a stopping time on $(\Omega, \{\mathscr{F}_n\}_{n\in\mathbb{N}})$ and let T be a stopping time on $(\Omega, \{\mathscr{F}_t\}_{t\geq 0})$. We define $\mathscr{F}'_N := \{A \in \mathscr{F} : A \subseteq \{N < \infty\}, \{N \leq n\} \cap A \in \mathscr{F}_n, \forall n \in \mathbb{N}\}$ and $\mathscr{F}'_T := \{A \in \mathscr{F} : A \subseteq \{T < \infty\}, \{T \leq t\} \cap A \in \mathscr{F}_t, \forall t \geq 0\}$. Recall that a Markov chain $\{X_n\}_{n\in\mathbb{N}}$ satisfies a strong Markov property if for each stopping time N, $k \in \mathbb{N}$ and all $x \in S$, $B \in \mathscr{S}$,

$$\mathbb{P}_x\left(X_{N+k} \in B \mid \mathscr{F}'_N\right) = \mathbb{P}_{X_N}\left(X_k \in B\right), \quad \text{on the set } \{N < \infty\}.$$

We say that a Markov process $\{X_t\}_{t\geq 0}$ satisfies a *strong Markov property* if for each stopping time T, $s \geq 0$ and all $x \in E$, $B \in \mathscr{B}(E)$,

$$\mathbb{P}_x\left(X_{T+s} \in B \mid \mathscr{F}'_T\right) = \mathbb{P}_{X_T}\left(X_s \in B\right), \quad \text{on the set } \{T < \infty\}.$$

Strong Markov property clearly implies the Markov property and in the case of Markov chains the opposite is also true, but in the case of Markov processes this is not so. For an example see [61, pp. 215]. A class of continuous-time processes which do possess a strong Markov property (or a modification which does) are Feller processes. They are clearly a subclass of Markov processes and in order to define Feller processes let us first define and discuss properties of a semigroup of a Markov process $\{X_t\}_{t\geq 0}$

Analysis of a Markov chain $\{X_n\}_{n\in\mathbb{N}}$ on a discrete state space S is often done by analysing transition probabilities $p(n,x,y) := \mathbb{P}_x(X_n = y)$. When analysing a Markov process $\{X_t\}_{t\geq 0}$ on a state space E, which is not discrete, we introduce transition kernels $p(t,x,\mathrm{d}y) = \mathbb{P}_x\left(X_t \in \mathrm{d}y\right)$, $t \geq 0$, $x \in E$. The associated family of linear operators $\{P_t\}_{t\geq 0}$ defined by

$$P_t f(x) := \mathbb{E}_x\left[f(X_t)\right] = \int_{\mathbb{R}^d} f(y)p(t,x,\mathrm{d}y), \qquad t \geq 0, f \in B_b(E,\mathbb{R}),$$

forms a *semigroup* on the Banach space $(B_b(E,\mathbb{R}), \|\cdot\|_\infty)$. That is

$P_0 = \mathrm{Id}$, since $P_0 f(x) = \mathbb{E}_x\big[f(X_0)\big] = f(x)$, and

$P_s \circ P_t = P_{s+t}$ for all $s,t \geq 0$, since due to the Markov property we have $P_s \circ P_t f(x) = \mathbb{E}_x[P_t f(X_s)] = \mathbb{E}_x[\mathbb{E}_{X_s}[f(X_t)]] = \mathbb{E}_x[\mathbb{E}_x[f(X_{s+t}) \mid \mathscr{F}_s]] = \mathbb{E}_x[f(X_{s+t})] = P_{s+t} f(x)$.

Moreover, the semigroup $\{P_t\}_{t\geq 0}$ is

contractive, that is $\|P_t f\|_\infty \leq \|f\|_\infty$ for all $t \geq 0$ and $f \in B_b(E)$, since $\|P_t f\|_\infty = \sup_{x\in E} |\mathbb{E}_x[f(X_t)]| \leq \sup_{x\in E} \mathbb{E}_x[|f(X_t)|] \leq \|f\|_\infty$,

conservative, that is $P_t \mathbb{1}_E(x) = \mathbb{P}_x(X_t \in E) = 1 = \mathbb{1}_E(x)$ for all $t \geq 0$, and

positivity preserving, that is $P_t f \geq 0$ for all $t \geq 0$ and $f \in B_b(E)$ satisfying $f \geq 0$.

Definition 2.2.1. A Markov process $\{X_t\}_{t\geq 0}$ is said to be a *Feller process* if its corresponding semigroup $\{P_t\}_{t\geq 0}$ satisfies the following two properties

(i) $\{P_t\}_{t\geq 0}$ is *strongly continuous*, that is, $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$ for all $f \in C_\infty(E,\mathbb{R})$,

(ii) $\{P_t\}_{t\geq 0}$ enjoys the *Feller property*, that is, $P_t(C_\infty(E)) \subseteq C_\infty(E,\mathbb{R})$ for all $t \geq 0$.

In that case we say that $\{P_t\}_{t\geq 0}$ is a *Feller semigroup*.

As mentioned before every Feller process (admits a modification that) has càdlàg sample paths and possesses the strong Markov property (see [49, Theorems 3.4.19 and 3.5.14]).

A Markov process $\{X_t\}_{t\geq 0}$ is said to be a C_b-*Feller process* if its semigroup $\{P_t\}_{t\geq 0}$ satisfies $P_t(C_b(E,\mathbb{R})) \subseteq C_b(E,\mathbb{R})$ for all $t \geq 0$. It is said to be a *strong Feller* if we have $P_t(B_b(E,\mathbb{R})) \subseteq C_b(E,\mathbb{R})$ for all $t \geq 0$. Clearly, if $\{X_t\}_{t\geq 0}$ is strong Feller then it is also C_b-Feller. According to [90, Corollary 3.4] (since $\{P_t\}_{t\geq 0}$ is conservative) a Feller process $\{X_t\}_{t\geq 0}$ is also a C_b-Feller.

Next, we introduce some properties crucial for studying stability of Markov processes. We will mainly follow the definitions and cite theorems from [29], [69] and [105].

Recall that, for a Markov chain $\{X_n\}_{n\in\mathbb{N}}$ on a discrete state space S, an invariant probability measure $\pi(\mathrm{d}x)$ on space $(S,\mathscr{S})$ satisfies

$$\pi(y) = \sum_{x\in S} p(1,x,y)\pi(x)\,.$$

In the case of Markov processes we have a similar definition.

Definition 2.2.2. *Invariant probability measure* $\pi(\mathrm{d}x)$ of a Markov process $\{X_t\}_{t\geq 0}$ is a measure on space $(E,\mathscr{B}(E))$ satisfying

$$\int_E p\,(t,x,B)\;\pi(\mathrm{d}x)\;=\;\pi(B)\,,\qquad \text{for all } t\geq 0\,,\; B\in\mathscr{B}(E)\,.$$

It will be useful to prove the existence and uniqueness of an invariant probability measure. Here we discuss one way of doing so. For the proof of existence we introduce the following notion of boundedness.

Definition 2.2.3. A Markov process $\{X_t\}_{t\geq 0}$ is *bounded in probability on average* if for each initial condition $x\in E$ and each $\varepsilon>0$, there exists a compact subset $C\subseteq E$ such that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t p\,(s,x,C)\;\mathrm{d}s\geq 1-\varepsilon\,.$$

Next proposition is a direct consequence of [69, Theorem 3.1] and the fact that every Markov process $\{X_t\}_{t\geq 0}$ on a compact state space is bounded in probability on average.

Proposition 2.2.4. *If Markov process $\{X_t\}_{t\geq 0}$ with a compact state space E is C_b-Feller then an invariant probability measure $\pi(\mathrm{d}x)$ exists for $\{X_t\}_{t\geq 0}$.*

To prove the uniqueness we need to define the notion of irreducibility and state what it means for a Markov process to be transient or recurrent.

Recall that a Markov chain $\{X_n\}_{n\in\mathbb{N}}$ on a discrete state space S is irreducible if for every $x,y\in S$ there is $n\in\mathbb{N}$ such that $p(n,x,y)>0$ which is equivalent to saying that $\sum_{n=1}^{\infty}p(n,x,y)>0$ for every $x,y\in S$. Next definition generalises this definition for both Markov process and Markov chain on a state space $(E,\mathscr{B}(E))$.

Definition 2.2.5. If there is a σ-finite measure ψ such that for every $B\in\mathscr{B}(E)$,

- $\psi(B)>0$ implies that $\int_0^\infty p(t,x,B)\,\mathrm{d}t>0$ for all $x\in E$ we say that a Markov process $\{X_t\}_{t\geq 0}$ is ψ-*irreducible*.
- $\psi(B)>0$ implies that $\sum_{n=1}^{\infty}p(n,x,B)\,\mathrm{d}t>0$ for all $x\in E$ we say that a Markov chain $\{X_n\}_{n\in\mathbb{N}}$ is ψ-*irreducible*.

Recall that, for a Markov chain $\{X_n\}_{n\in\mathbb{N}}$ on a discrete state space S, a state $y\in S$ is recurrent if $\sum_{n=1}^{\infty}p(n,y,y)=\infty$ and if on top of that $\{X_n\}_{n\in\mathbb{N}}$ is also irreducible then for every $x\in S$ we have $\sum_{n=1}^{\infty}p(n,x,y)=\infty$. In a continuous-time setting we have a following definition.

Definition 2.2.6. A ψ-irreducible Markov process $\{X_t\}_{t\geq 0}$ is *recurrent* if $\psi(A) > 0$ implies that $\int_0^\infty p(t,x,A)\,\mathrm{d}t = \infty$ for every $x \in E$.

Definition 2.2.7. A Markov process $\{X_t\}_{t\geq 0}$ is *transient* if there exist $\{A_n\}_{n\in\mathbb{N}}$ such that $\cup_{n\in\mathbb{N}} A_n = E$ and $\{M_n\}_{n\in\mathbb{N}}$, $M_n < \infty$ such that $\int_0^\infty p(t,x,A_n)\,\mathrm{d}t \leq M_n$ for every $x \in E$ and every $n \in \mathbb{N}$.

Recall that a irreducible recurrent Markov chain $\{X_n\}_{n\in\mathbb{N}}$ on a discrete state space S admits a unique (up to constant multiplies) invariant measure. The equivalent of this argument in a continuous-time setting will play a crucial role in the next proposition.

Proposition 2.2.8. *If $\{X_t\}_{t\geq 0}$ is a ψ-irreducible Markov process such that an invariant probability measure $\pi(\mathrm{d}x)$ exists then $\pi(\mathrm{d}x)$ is unique.*

Proof. According to [105, Theorem 2.3] every ψ-irreducible Markov process is either transient or recurrent. Due to the fact that $\{X_t\}_{t\geq 0}$ admits at least one invariant probability measure it clearly cannot be transient. Indeed, suppose $\{X_t\}_{t\geq 0}$ is transient i.e. there exist $\{A_n\}_{n\in\mathbb{N}}$ such that $\cup_{n\in\mathbb{N}} A_n = E$ and $\{M_n\}_{n\in\mathbb{N}}$, $M_n < \infty$ and for $x \in E$ $\int_0^\infty p(t,x,A_n)\,\mathrm{d}t \leq M_n$ holds. Then

$$M_n = \int_E M_n \pi(\mathrm{d}x) \geq \int_E \int_0^\infty p(t,x,A_n)\,\mathrm{d}t\,\pi(dx) = \int_0^\infty \pi(A_n)\,\mathrm{d}t$$

implies $\pi(A_n) = 0$ for each $n \in \mathbb{N}$ which is not possible since $\cup_{n\in\mathbb{N}} A_n = E$. The assertion now follows from [105, Theorem 2.6] which states that every recurrent Markov process admits a unique (up to constant multiplies) invariant measure. ■

If an invariant measure exist we can discuss whether and how transition kernels of $\{X_t\}_{t\geq 0}$ converge to it. It is well known that when $\pi(\mathrm{d}x)$ is an invariant probability then $\|p(t,x,dy) - \pi(\mathrm{d}y)\|_{\mathrm{TV}}$ is a decreasing function of $t \geq 0$, see [104]. We will be interested in cases when we can deduce more then that and it will prove of a special interest to have an exponential speed of convergence, the so called geometric ergodicity.

Definition 2.2.9. Markov process $\{X_t\}_{t\geq 0}$ such that an invariant probability measure $\pi(\mathrm{d}x)$ exists is said to be

- *ergodic* if

$$\lim_{t\to\infty} \|p(t,x,dy) - \pi(\mathrm{d}y)\|_{\mathrm{TV}} = 0, \qquad \text{for all } x \in E.$$

- *geometrically ergodic* if there exists $\gamma > 0$ such that

$$\lim_{t\to\infty} \mathrm{e}^{\gamma t} \|p(t,x,\mathrm{d}y) - \pi(\mathrm{d}y)\|_{\mathrm{TV}} = 0, \qquad \text{for all } x \in E.$$

Useful consequence of geometric ergodicity is given in the following corollary

Corollary 2.2.10. *If a Markov process $\{X_t\}_{t\geq 0}$ is geometrically ergodic then there are constants $\gamma, \Gamma > 0$ such that*

$$\|P_t f\|_\infty \leq \Gamma e^{-\gamma t} \|f\|_\infty, \qquad t \geq 0,$$

for any $f \in B_b(E,\mathbb{R})$ with $\int_E f(x)\,\pi(\mathrm{d}x) = 0$.

Proof. Clearly geometric ergodicity is equivalent to the existence of constants $\gamma, \Gamma > 0$ such that

$$\sup_{|f|\leq 1} |P_t f(x) - \pi(f)| \leq \Gamma \mathrm{e}^{\gamma}, \qquad \text{for all } x \in E,$$

therefore the statement trivially follows. ■

Irreducibility of the Markov process will again play a crucial role in proving the geometric ergodicity of a process. Another important concept are petite sets.

Definition 2.2.11. A set $C \subseteq E$ is *petite* if there is a probability distribution a on $\mathscr{B}([0,\infty))$ and a nontrivial measure ν on $\mathscr{B}(E)$ such that

$$\int_0^\infty p(t,x,A)\,a(\mathrm{d}t) \geq \nu(A), \qquad \text{for all } x \in C \text{ and } A \in \mathscr{B}(E). \tag{2.1}$$

A special case of a petite set is if we take $a = \delta_{t_0}$, a Dirac delta measure at time t_0. Then condition (2.1) becomes $p(t_0,x,A) \geq \nu(A)$ for all $x \in C$ and $A \in \mathscr{B}(E)$. In this case we say that a set C is *small*. It will be of interest for us to study a situation when petite and small sets coincide.

Proposition 2.2.12. *Suppose $\{X_t\}_{t\geq 0}$ is ψ-irreducible C_b-Feller process on a compact state space E, where supp ψ has non empty interior and its skeleton chain $\{X_n\}_{n\in\mathbb{N}}$ is ϕ-irreducible then all petite sets are small.*

Proof. The statement follows from [105, Theorem 7.1.], [69, Theorem 3.2. (ii)], the fact that every Markov process $\{X_t\}_{t\geq 0}$ on a compact state space is bounded in probability on average and [69, Proposition 6.1]. ■

Recall that a irreducible Markov chain $\{X_n\}_{n\in\mathbb{N}}$ on a discrete state space S is aperiodic if there is $n_0 \in \mathbb{N}$ and $x \in S$ such that $p(n,x,x) > 0$ for all $n \geq n_0$.

Definition 2.2.13. A ψ-irreducible Markov process $\{X_t\}_{t\geq 0}$ is *aperiodic* if for some small set C such that $\psi(C) > 0$ there exists $t_0 > 0$ such that $p(t,x,C) > 0$ for all $x \in C$ and $t \geq t_0$.

Recall that for an irreducible, aperiodic Markov chain $\{X_n\}_{n\in\mathbb{N}}$ on a discrete state space S with invariant probability measure $\pi(\mathrm{d}x)$ we have $\lim_{n\to\infty} p(n,x,y) = \pi(y)$ for every $x,y \in S$. The following theorem follows directly from [29, Theorem 5.2.(b)] by taking $V_T \equiv 1, \lambda \equiv 0, b = 1$ and $C = E$.

Theorem 2.2.14. *Let $\{X_t\}_{t\geq 0}$ be a ψ-irreducible, aperiodic Markov process such that E is a petite set. Then $\{X_t\}_{t\geq 0}$ is geometrically ergodic.*

It might seem an unreasonable assumption for the space set E to be petite but the following lemma, which follows directly from [105, Theorem 5.1. and 7.1.], proves that this is not the case when $E = \mathbb{T}^{\mathrm{d}}_{\tau}$.

Lemma 2.2.15. *If $\{X_t\}_{t\geq 0}$ is a ψ-irreducible C_b-Feller process such that supp ψ has non-empty interior, then every compact set is petite.*

Next theorem also gives us sufficient conditions for a Markov process to be geometrically ergodic.

Theorem 2.2.16. *Suppose that $\{X_t\}_{t\geq 0}$ is ψ-irreducible strong Feller process on a compact state space E such that its skeleton chain $\{X_n\}_{n\in\mathbb{N}}$ is ϕ-irreducible. Then $\{X_t\}_{t\geq 0}$ is geometrically ergodic.*

Proof. Strong Feller property implies that [105, Condition T on page 177] is satisfied, to see this take $a := \delta_t$ a Dirac delta measure at time t and $A \in \mathscr{B}(E)$ then due to the strong Feller property function $P_t(\mathbb{1}_A)$ is continuous, that is condition T is satisfied. The fact that every Markov process with a compact state space is bounded in probability on average and [69, Proposition 3.1 (i)] imply that we can apply [69, Theorem 8.1(iii)]. From already mentioned we conclude that conditions of [69, Theorem 3.2] are met. This together with the fact that the skeleton chain $\{X_n\}_{n\in\mathbb{N}}$ is ϕ-irreducible implies [69, Theorem 6.1] from which we conclude that measure in Theorem 8.1(iii) does not depend on $x \in E$ which concludes the proof. ∎

For a Markov process $\{X_t\}_{t\geq 0}$ there are several different notions of irreducibility. We have already defined what it means for a Markov process to be ψ-irreducible. The following notion of irreducibility relies on the topology structure of state space E.

Definition 2.2.17. A Markov process $\{X_t\}_{t\geq 0}$ is *open-set irreducible* if for any $t > 0$, and any $x \in E$ and any non-empty open set $O \subseteq E$, $p(t,x,O) > 0$.

The following proposition gives us a connection between these two definitions of irreducibility.

Proposition 2.2.18. *If a strong Feller process $\{X_t\}_{t\geq 0}$ is open-set irreducible then the process $\{X_t\}_{t\geq 0}$ and its skeleton chain $\{X_n\}_{n\in\mathbb{N}}$ are ψ-irreducible.*

Proof. If $\{X_t\}_{t\geq 0}$ is open-set irreducible then clearly so is its skeleton chain $\{X_n\}_{n\in\mathbb{N}}$. The statement of this proposition is now a direct consequence of [105, Theorem 3.2.(ii)] and [105, Condition T on page 177], where in Condition T we take $a := \delta_t$ a Dirac delta measure at time t and use the fact that for any $A \in \mathscr{B}(E)$ function $P_t(\mathbb{1}_A)$ is continuous due to the strong Feller property. ■

That the property of ergodicity is important can be seen in the following continuous-time version of Birkhoff ergodic theorem (see [14, Proposition 2.5] and the note under it or [88])

Theorem 2.2.19. *Let $\{X_t\}_{t\geq 0}$ be ergodic Markov process with invariant measure $\pi(\mathrm{d}x)$. If $f \in L^p(E,\pi)$ then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s = \pi(f) \quad \textit{a.s. and in } L^p\,.$$

2.3. Lévy-Type Processes

In this section we will concentrate on a special class of Feller processes, the so called Lévy-type processes which are the central objects of the Chapter 3. We shall see that examples of Lévy-type processes include Lévy processes and diffusions, the latter of which are the central objects of the forth chapter. Throughout this section $\{X_t\}_{t\geq 0}$ denotes a Markov process with state space $\left(\mathbb{R}^{\mathrm{d}}, \mathscr{B}\left(\mathbb{R}^{\mathrm{d}}\right)\right)$.

In the previous section we have defined a semigroup $\{P_t\}_{t\geq 0}$ of a Markov process $\{X_t\}_{t\geq 0}$. To better understand the next definition let us first consider a function $p : [0,\infty) \to \mathbb{R}$ which satisfies properties analogue to those of the semigroup. That is, we are interested in what can be said about a solution to a functional equation

$$
\begin{aligned}
p(0) &= 1 \\
p(s)\cdot p(t) &= p(s+t)\,.
\end{aligned}
$$

If we additionally require p to be continuous then the unique solution to this equation is $p(t) = \mathrm{e}^{at}$. We see that the properties of p are determined by $a \in \mathbb{R}$ which can be obtained from p by following formula

$$
a = \lim_{t\to 0} \frac{p(t)-1}{t}\,.
$$

In the case of a semigroup $\{P_t\}_{t\geq 0}$ we will try to proceed in a similar meaner, but we need to make sure this limit is well defined. The *infinitesimal generator* $(\mathscr{A}^b, \mathscr{D}_{\mathscr{A}^b})$ of the semigroup $\{P_t\}_{t\geq 0}$ (or of a Markov process $\{X_t\}_{t\geq 0}$) is a linear operator $\mathscr{A}^b : \mathscr{D}_{\mathscr{A}^b} \to B_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ defined by

$$
\mathscr{A}^b f := \lim_{t\to 0} \frac{P_t f - f}{t}, \qquad f \in \mathscr{D}_{\mathscr{A}^b} := \left\{ f \in B_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R}) : \lim_{t\to 0} \frac{P_t f - f}{t} \text{ exists in } \|\cdot\|_\infty \right\}.
$$

We call $(\mathscr{A}^b, \mathscr{D}_{\mathscr{A}^b})$ the B_b-*generator* for short. Further, in the case of Feller processes, we call $(\mathscr{A}^\infty, \mathscr{D}_{\mathscr{A}^\infty}) := (\mathscr{A}^b, \mathscr{D}_{\mathscr{A}^b} \cap C_\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R}))$ the *Feller generator* for short. Observe that in this case $\mathscr{D}_{\mathscr{A}^\infty} \subseteq C_\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ and $\mathscr{A}^\infty(\mathscr{D}_{\mathscr{A}^\infty}) \subseteq C_\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$. Following propositions give some basic properties of the generator in the relation to semigroup and their proofs can be found in [32, 1.1.5 Proposition] and [32, 4.1.7 Proposition] respectively

Proposition 2.3.1. *(i) If $f \in C_\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ and $t \geq 0$, then $\int_0^t P_t f \,\mathrm{d}s \in \mathscr{D}_{\mathscr{A}^\infty}$ and*

$$
P_t f - f = \mathscr{A}^\infty \int_0^t P_s f \,\mathrm{d}s\,.
$$

(ii) *If $f \in \mathscr{D}_{\mathscr{A}^\infty}$ and $t \geq 0$, then $P_t f \in \mathscr{D}_{\mathscr{A}^\infty}$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t f = \mathscr{A}^\infty P_t f = P_t \mathscr{A}^\infty f .$$

(iii) *If $f \in \mathscr{D}_{\mathscr{A}^\infty}$ and $t \geq 0$, then*

$$P_t f - f = \int_0^t \mathscr{A}^\infty P_s f \,\mathrm{d}s = \int_0^t P_s \mathscr{A}^\infty f \,\mathrm{d}s .$$

Proposition 2.3.2. *If $f \in \mathscr{D}_{\mathscr{A}^b}$ and $g = \mathscr{A}^b f$ then the process $\{M_t\}_{t\geq 0}$ such that $M_t := f(X_t) - \int_0^t g(X_s)\,\mathrm{d}s$ is an $\{\mathscr{F}_t\}_{t\geq 0}$-martingale.*

Notice that properties from Proposition 2.3.1 match the following properties of function $p(t) = \mathrm{e}^{at}$

(i) $p(t) - 1 = \mathrm{e}^{at} - 1 = a(\frac{1}{a}\mathrm{e}^{at} - \frac{1}{a}) = a \int_0^t p(s)\,\mathrm{d}s$,

(ii) $\frac{\mathrm{d}}{\mathrm{d}t} p(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{at} = a\mathrm{e}^{at} = ap(t)$.

For the proof of the following lemma se [52, Lemma 19.26]

Lemma 2.3.3. *Let $\{X_t\}_{t\geq 0}$ and $\{\tilde{X}_t\}_{t\geq 0}$ be two Feller processes with the corresponding C_∞-generators $(\mathscr{A}^\infty, \mathscr{D}_{\mathscr{A}^\infty})$ and $(\tilde{\mathscr{A}}^\infty, \mathscr{D}_{\tilde{\mathscr{A}}^\infty})$, respectively. Then*

$$P_t f = \tilde{P}_t f + \int_0^t \tilde{P}_s (\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty) P_{t-s} f \,\mathrm{d}s , \qquad f \in \mathscr{D}_{\mathscr{A}^\infty} \cap \mathscr{D}_{\tilde{\mathscr{A}}^\infty} . \tag{2.2}$$

Again notice that the equivalent property holds for functions $p(t) = \mathrm{e}^{at}$ and $\tilde{p}(t) = \mathrm{e}^{\tilde{a}t}$, indeed

$$\mathrm{e}^{\tilde{a}t} + \int_0^t \mathrm{e}^{\tilde{a}s}(a-\tilde{a})\mathrm{e}^{a(t-s)}\,\mathrm{d}s = \mathrm{e}^{\tilde{a}t} + (a-\tilde{a})\mathrm{e}^{at}\int_0^t \mathrm{e}^{(\tilde{a}-a)s}\,\mathrm{d}s = \mathrm{e}^{\tilde{a}t} - \mathrm{e}^{at}(\mathrm{e}^{(\tilde{a}-a)t} - 1) = \mathrm{e}^{at} .$$

Corollary 2.3.4. *The equation (2.2) holds for any $f \in B_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ if $\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty$ is bounded operator on $(B_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R}), \|\cdot\|_\infty)$ and $C_c^\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R}) \subseteq \mathscr{D}_{\mathscr{A}^\infty} \cap \mathscr{D}_{\tilde{\mathscr{A}}^\infty}$.*

Proof. Take any open set $O \subseteq \mathbb{R}^d$ and a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that $f_n \nearrow \mathbb{1}_O$.Due to boundedness of $\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty$ and the dominated convergence theorem we see that equation (2.2) also holds for $f(x) = \mathbb{1}_O(x)$. The claim now follows from Dynkin's monotone class theorem. ■

Let us give an example of infinitesimal generators for specific class of Markov processes.

Example 2.3.5. Recall that the process $\{L_t\}_{t\geq 0}$ is a Lévy process if

$L_0 = 0$,

it has *stationary increments*, that is for all $0 \leq s \leq t$ $L_t - L_s \stackrel{d}{=} L_{t-s}$,

it has *independent increments*, that is for all $0 \leq s \leq t$ $L_t - L_s$ is independent of $\sigma(L_r, r \leq s)$ and

it is *continuous in probability*, that is for every $\varepsilon > 0$ $\lim_{t\to\infty} \mathbb{P}(|L_t - L_0| > \varepsilon) = 0$.

Lévy process is uniquely and completely characterized through its characteristic exponent $q : \mathbb{R}^{\mathrm{d}} \to \mathbb{C}$

$$\mathbb{E}\left[\mathrm{e}^{i\langle \xi, L_t\rangle}\right] = \mathrm{e}^{-tq(\xi)}, \quad t \geq 0, \xi \in \mathbb{R}^{\mathrm{d}}.$$

Lévy processes are Feller processes and the domain of its generator contains $C_c^\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ functions. If $\mathscr{A}^\infty$ is a generator of $\{L_t\}_{t\geq 0}$ then for any $f \in C_c^\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$

$$\mathscr{A}^\infty f = \langle b, \nabla f\rangle + 2^{-1}\mathrm{Tr}\left(c\nabla\nabla^{\mathrm{T}} f\right) + \int_{\mathbb{R}^d} \left(f(\cdot + y) - f - \langle y, \nabla f\rangle \mathbb{1}_{B_1(0)}(y)\right) \nu(\mathrm{d}y), \tag{2.3}$$

where $b \in \mathbb{R}^{\mathrm{d}}$, $c \in \mathbb{R}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}}$ is non-negative definite symmetric matrix and ν is a σ-finite measure on $\mathscr{B}(\mathbb{R}^{\mathrm{d}})$ satisfying

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}^d} \left(1 \wedge |y|^2\right) \nu(\mathrm{d}y) < \infty.$$

The triplet $(b, c, \nu(\mathrm{d}y))$ is called the Lévy triplet of $\{L_t\}_{t\geq 0}$. Equivalently, $\mathscr{A}^\infty$ is a pseudo differential operator, that is it can be written in the form

$$\mathscr{A}^\infty|_{C_c^\infty(\mathbb{R}^{\mathrm{d}})} f(x) = -\int_{\mathbb{R}^{\mathrm{d}}} q(x,\xi) e^{i\langle \xi, x\rangle} \hat{f}(\xi)\, \mathrm{d}\xi,$$

where the symbol $q(x, \xi)$ does not depend on x and is given by the Lévy-Khintchine formula

$$q(\xi) = -i\langle \xi, b\rangle + \frac{1}{2}\langle \xi, c\xi\rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, y\rangle} + i\langle \xi, y\rangle 1_{B_1(0)}(y)\right) \nu(\mathrm{d}y).$$

$\square$

As already mentioned the important question when thinking about the infinitesimal generator is whether a function is in its domain. What makes it easier for one to work with generators is to have a large class of "nice" functions in this domain. This is what makes the definition of Lévy-type process important.

Definition 2.3.6. Feller process $\{X_t\}_{t\geq 0}$ such that its Feller generator $(\mathscr{A}^\infty, \mathscr{D}_{\mathscr{A}^\infty})$ satisfies

(LTP) $C_c^\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R}) \subseteq \mathscr{D}_{\mathscr{A}^\infty}$,

is called a *Lévy-type process* (LTP). Throughout this thesis, the symbol $\{X_t\}_{t\geq 0}$ denotes a Lévy-type process.

The name Lévy-type processes suggests a strong connection to Lévy processes. Let us see that this is so. According to [25, Theorem 3.4], $\mathscr{A}^\infty|_{C_c^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R})}$ is a *pseudo-differential operator*, that is, it can be written in the form

$$\mathscr{A}^\infty|_{C_c^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R})} f(x) = -\int_{\mathbb{R}^{\mathrm{d}}} q(x,\xi) e^{i\langle \xi, x\rangle} \hat{f}(\xi)\, \mathrm{d}\xi\,. \tag{2.4}$$

The function $q : \mathbb{R}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \to \mathbb{C}$ is called the *symbol* of the pseudo-differential operator. It is measurable and locally bounded in (x,ξ), and is continuous and negative definite as a function of ξ. Hence, by [48, Theorem 3.7.7], the function $\xi \mapsto q(x,\xi)$ has for each $x \in \mathbb{R}^{\mathrm{d}}$ the following Lévy-Khintchine representation

$$q(x,\xi) = q(x,0) - i\langle \xi, b(x)\rangle + \frac{1}{2}\langle \xi, c(x)\xi\rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, y\rangle} + i\langle \xi, y\rangle 1_{B_1(0)}(y)\right) \nu(x,\mathrm{d}y) \tag{2.5}$$

where $q(x,0) : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ is non-negative Borel measurable function, $b : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}}$ is Borel measurable function, $c := (c_{ij})_{1\leq i,j\leq d} : \mathbb{R}^{\mathrm{d}} \to \mathbb{S}^{\mathrm{d}}$ is a symmetric non-negative definite $\mathrm{d}\times\mathrm{d}$ matrix-valued Borel measurable function and $\nu : \mathbb{R}^{\mathrm{d}} \times \mathscr{B}(\mathbb{R}^{\mathrm{d}}) \to [0,+\infty\rangle$ is a Borel kernel called the *Lévy kernel*, satisfying

$$\nu(x,\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}^d} \left(1 \wedge |y|^2\right) \nu(x,\mathrm{d}y) < \infty, \qquad x \in \mathbb{R}^d\,.$$

In the sequel we additionally assume

(C) $q(x,0) \equiv 0$,

The triplet $(b(x), c(x), \nu(x,\mathrm{d}y))$ is called the *Lévy triplet* of $\mathscr{A}^\infty|_{C_c^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R})}$ (or of $q(x,\xi)$).

Property (C) is closely related to the *conservativeness* property of LTP $\{X_t\}_{t\geq 0}$, that is, $\mathbb{P}_x(X_t \in \mathbb{R}^d) = 1$ for all $t \geq 0$ and $x \in \mathbb{R}^d$. Namely, under the assumption that the x-coefficients of $q(x,\xi)$ are uniformly bounded (which is certainly the case in the periodic setting), $q(x,0) = 0$ implies that $\{X_t\}_{t\geq 0}$ is conservative. Conversely, if process $\{X_t\}_{t\geq 0}$ is conservative and $q(x,0)$ is continuous then $q(x,0) \equiv 0$ (see [90, Theorem 5.2]). Let us also remark that local boundedness of $q(x,\xi)$ implies local boundedness of the corresponding x-coefficients, and vice versa (see [91, Lemma 2.1 and Remark 2.2]).

Note that by combining (2.4) and (2.5) $\mathscr{A}^\infty|_{C_c^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R})}$ takes the form

$$\begin{aligned}\mathscr{L}f(x) &= \langle b(x), \nabla f(x)\rangle + \frac{1}{2}\mathrm{Tr}\, c(x)\nabla^2 f(x)\\ &\quad + \int_{\mathbb{R}^d}\Big(f(x+y)-f(x)-\langle y,\nabla f(x)\rangle 1_{B_1(0)}(y)\Big)\,\nu(x,\mathrm{d}y)\,,\end{aligned} \tag{2.6}$$

Conversely, if $\mathscr{L}: C_c^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R}) \to C_\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ is a linear operator of the form (2.6) satisfying the so-called *positive maximum principle*

$$f(x_0) = \sup_{x\in\mathbb{R}^{\mathrm{d}}} f(x) \geq 0 \Longrightarrow \mathscr{L}f(x_0) \leq 0 \text{ for any } f \in C_c^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R}) \tag{2.7}$$

and such that $(\lambda-\mathscr{L})(C_c^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R}))$ is dense in $C_\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ for some (or all) $\lambda>0$, then, according to the Hille-Yosida-Ray theorem, $\mathscr{L}$ is closable and the closure is the generator of a Feller semigroup. In particular, the corresponding Feller process is a LTP.

Notice that the class of processes we consider in this thesis contains diffusion processes and Lévy processes.

Example 2.3.7. A typical example of a LTP is a solution to the following SDE

$$\mathrm{d}X_t = \Phi(X_{t-})\,\mathrm{d}L_t\,, \qquad X_0 = x \in \mathbb{R}^{\mathrm{d}}\,, \tag{2.8}$$

where $\Phi:\mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}\times\mathrm{n}}$ is locally Lipschitz continuous and bounded (which is not a restriction in the periodic setting), and $\{L_t\}_{t\geq 0}$ is an n-dimensional Lévy process with characteristic exponent $q_L(\xi)$ and Lévy triplet $(b,c,\nu(\mathrm{d}y))$. Equation (2.8) is a shorthand for the system of stochastic integral equations

$$X_t^i = x^i + \sum_{j=1}^n \int_0^t \Phi(X_{s-})^{ij}\,\mathrm{d}L_s^j\,, \qquad i = 1,\dots,\mathrm{d}\,,$$

where integration with respect to Lévy process should be interpreted as follows. Define by $\mu(\omega,\mathrm{d}y,\mathrm{d}t) := \sum_{t:\,\Delta L_t(\omega)\neq 0}\delta_{(\Delta L_t(\omega),t)}(\mathrm{d}y,\mathrm{d}t)$ the jump measure of $\{L_t\}_{t\geq 0}$ then for any $f \in B\left(\mathbb{R}^{\mathrm{d}},\mathbb{R}\right)$ such that $0 \notin \operatorname{supp} f$ we have $\int f(x)\mu(\omega,\mathrm{d}y,[0,t]) = \sum_{s\leq t} f(\Delta L_s(\omega))$. We call the measure $\nu(\mathrm{d}y)\mathrm{d}t$ the compensator measure and it satisfies the following property

$\nu(\mathrm{d}y)\mathrm{d}t = \mathbb{E}[\mu(\cdot,\mathrm{d}y,\mathrm{d}t)]$. Then the Lévy-Ito decomposition of process $\{L_t\}_{t\geq 0}$ implies

$$\begin{aligned} X_t &= x+\int_0^t \Phi(X_{s-})\,\mathrm{d}L_s = \int_0^t \Phi(X_{s-})b\,\mathrm{d}s+\int_0^t \Phi(X_{s-})\sigma\,\mathrm{d}W_s \\ &\qquad + \int_0^t\int_{\mathbb{R}^\mathrm{d}} \Phi(X_{s-})y\mathbb{1}_{B_1^c(0)}(y)\,\mu(\cdot,\mathrm{d}y,\mathrm{d}s) \\ &\qquad + \int_0^t\int_{\mathbb{R}^\mathrm{d}} \Phi(X_{s-})y\mathbb{1}_{B_1(0)}(y)\,(\mu(\cdot,\mathrm{d}y,\mathrm{d}s)-\nu(\mathrm{d}y)\mathrm{d}s)\,, \end{aligned} \tag{2.9}$$

where σ is a $\mathrm{n}\times\mathrm{n}$ matrix such that $\sigma^\mathrm{T}\sigma = c$. In [92, Theorems 3.1 and 3.5 and Corollary 3.3] it has been shown that the unique solution $\{X_t\}_{t\geq 0}$ to the SDE in (2.8) (which exists by standard arguments) is a LTP with symbol of the form $q(x,\xi) = q_L(\Phi^\mathrm{T}(x)\xi)$. Observe that the following SDE is a special case of (2.8),

$$\mathrm{d}X_t = \Phi_1(X_t)\,\mathrm{d}t+\Phi_2(X_t)\,\mathrm{d}W_t+\Phi_3(X_{t-})\mathrm{d}Z_t\,, \qquad X_0 = x\in\mathbb{R}^\mathrm{d}\,, \tag{2.10}$$

where $\Phi_1:\mathbb{R}^\mathrm{d}\to\mathbb{R}^\mathrm{d}$, $\Phi_2:\mathbb{R}^\mathrm{d}\to\mathbb{R}^{\mathrm{d}\times\mathrm{p}}$ and $\Phi_3:\mathbb{R}^\mathrm{d}\to\mathbb{R}^{\mathrm{d}\times\mathrm{q}}$, with $\mathrm{p}+\mathrm{q} = \mathrm{n}-1$, are locally Lipschitz continuous and, bounded, $\{W_t\}_{t\geq 0}$ is a p-dimensional Brownian motion, and $\{Z_t\}_{t\geq 0}$ is a q-dimensional pure-jump Lévy process (that is, a Lévy process determined by a Lévy triplet of the form $(0,0,\nu_Z(\mathrm{d}y))$). Namely, set $\Phi(x) = (\Phi_1(x),\Phi_2(x),\Phi_3(x))$ for any $x\in\mathbb{R}^\mathrm{d}$, and $L_t = (t,W_t,Z_t)^\mathrm{T}$ for $t\geq 0$.

□

In what follows we see that a situation in Example 2.3.7 is not that uncommon in a sense that every LTP can be represented in the manner similar to (2.9). We define the jump measure of a LTP $\{X_t\}_{t\geq 0}$ in the same way as for a Lévy process in Example 2.3.7

$$\mu(\omega,\mathrm{d}y,\mathrm{d}s) := \sum_{s:\,\Delta X_s(\omega)\neq 0} \delta_{(\Delta X_s(\omega),s)}(\mathrm{d}y,\mathrm{d}s)\,.$$

The important difference now is that Lévy kernel $\nu(x,\mathrm{d}y)$ depends on $x\in\mathbb{R}^\mathrm{d}$ and this problem is addressed as follows. By [16, Theorem 3.33], there exist a suitable enlargement of the stochastic basis $(\Omega,\mathscr{F},\{\mathbb{P}_x\}_{x\in\mathbb{R}^d},\{\mathscr{F}_t\}_{t\geq 0})$, say $(\widetilde{\Omega},\widetilde{\mathscr{F}},\{\widetilde{\mathbb{P}}_x\}_{x\in\mathbb{R}^d},\{\widetilde{\mathscr{F}}_t\}_{t\geq 0})$, supporting a d-dimensional Brownian motion $\{\tilde{W}_t\}_{t\geq 0}$ and a Poisson random measure $\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)$ on $\mathscr{B}(\mathbb{R})\otimes\mathscr{B}([0,\infty))$ with compensator $\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s$, such that $\{X_t\}_{t\geq 0}$ is a solution to the following stochastic differential equation

$$\begin{aligned} X_t &= x+\int_0^t b(X_s)\,\mathrm{d}s+\int_0^t \tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s+\int_0^t\int_{\mathbb{R}} k(X_{s-},z)\mathbb{1}_{\{u:|k(X_{s-},u)|\geq 1\}}(z)\,\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s) \\ &\quad + \int_0^t\int_{\mathbb{R}} k(X_{s-},z)\mathbb{1}_{\{u:|k(X_{s-},u)|<1\}}(z)\,(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s)\,, \end{aligned} \tag{2.11}$$

where $\tilde{\sigma}(x)$ is a $\mathrm{d}\times\mathrm{d}$ matrix-valued Borel measurable function such that for any $x\in\mathbb{R}^{\mathrm{d}}$ $\tilde{\sigma}(x)^{\mathrm{T}}\tilde{\sigma}(x)=c(x)$, $\tilde{\nu}(\mathrm{d}z)$ is any given σ-finite non-finite and non-atomic measure on $\mathscr{B}(\mathbb{R})$, and $k:\mathbb{R}^{\mathrm{d}}\times\mathbb{R}\to\mathbb{R}^{\mathrm{d}}$ is a Borel measurable function satisfying

$$\mu(\cdot,\mathrm{d}y,\mathrm{d}s) = \tilde{\mu}\left(\cdot,\{(z,u)\in\mathbb{R}\times[0,\infty):(k(X_{u-},z),u)\in(\mathrm{d}y,\mathrm{d}s)\}\right), \tag{2.12}$$

and

$$\nu(x,\mathrm{d}y) = \tilde{\nu}\left(\{z\in\mathbb{R}:k(x,z)\in\mathrm{d}y\}\right). \tag{2.13}$$

Notice that a representation in (2.11) makes sense since from (2.12) and (2.13) for $f\in B\left(\mathbb{R}^{\mathrm{d}},\mathbb{R}\right)$ and $g\in B\left(\mathbb{R}^{\mathrm{d}}\times[0,\infty),\mathbb{R}\right)$ we have

$$\int_{\mathbb{R}^{\mathrm{d}}} f(y)\,\nu(x,\mathrm{d}y) = \int_{\mathbb{R}} f(k(x,y))\tilde{\nu}(\mathrm{d}y) \quad \text{and}$$
$$\int_0^t\int_{\mathbb{R}^{\mathrm{d}}} g(y,s)\,\mu(\cdot,\mathrm{d}y,\mathrm{d}s) = \int_0^t\int_{\mathbb{R}} g(k(X_{s-},y),s)\tilde{\mu}(\cdot,\mathrm{d}y,\mathrm{d}s),$$

in a sense that if one side converges that so does the other and they are equal.

For the rest of this thesis we assume that $\{X_t\}_{t\geq 0}$ admits "small jumps" only, that is,

(SJ) $\sup_{x\in\mathbb{R}^{\mathrm{d}}}\int_{\mathbb{R}^{\mathrm{d}}}|y|^2\nu(x,\mathrm{d}y)<\infty.$

Notice that when defining the Lévy triplet we mentioned that for all $x\in\mathbb{R}^{\mathrm{d}}$ a Lévy kernel satisfies $\int_{\mathbb{R}^{\mathrm{d}}}\left(1\wedge|y|^2\right)\nu(x,\mathrm{d}y)<\infty$ which implies that $\int_{B_1(0)}|y|^2\nu(x,\mathrm{d}y)<\infty$. The condition (SJ) is, clearly stronger and one obvious way it can be satisfied is if $\nu=0$.

Due to (SJ) we have that

$$\int_{\mathbb{R}}|k(X_{s-},z)|\,\mathbb{1}_{\{u:|k(X_{s-},u)|\geq 1\}}(z)\tilde{\nu}(\mathrm{d}z) = \int_{\mathbb{R}^{\mathrm{d}}}|y|\mathbb{1}_{B_1^c(0)}(y)\,\nu(X_{s-},\mathrm{d}y) \leq$$
$$\int_{B_1^c(0)}|y|^2\,\nu(X_{s-},\mathrm{d}y) \leq \sup_{x\in\mathbb{R}^{\mathrm{d}}}\int_{\mathbb{R}^{\mathrm{d}}}|y|^2\nu(x,\mathrm{d}y)<\infty,$$

which together with (2.11) implies

$$\begin{aligned} X_t &= x+\int_0^t b(X_s)\mathrm{d}s+\int_0^t\tilde{\sigma}(X_s)\mathrm{d}\tilde{W}_s+\int_0^t\int_{\mathbb{R}}k(X_s,z)\mathbb{1}_{\{u:|k(X_s,u)|\geq 1\}}(z)\,\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s \\ &\quad +\int_0^t\int_{\mathbb{R}}k(X_{s-},z)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right). \end{aligned} \tag{2.14}$$

For more on Lévy-type processes we refer the readers to the monograph [15].

2.4. LTPs with Periodic Coeffcients

As mentioned in the Introduction we wish to model phenomena characterized by heterogeneous coefficients which rapidly vary on small scale. We wish to use the fine microscopic structure of the media to derive a homogenized model which is a valid approximation of the initial model. We achieve this by assuming that coefficients of a LTP $\{X_t\}_{t\geq 0}$ with state space $\left(\mathbb{R}^{\mathrm{d}}, \mathscr{B}\left(\mathbb{R}^{\mathrm{d}}\right)\right)$ are periodic and then by having this period converge to zero we obtain a homogenized process.

Throughout this thesis we will assume that the symbol $q(x,\xi)$ of a LTP $\{X_t\}_{t\geq 0}$ satisfies the following condition

(P) $x \mapsto q(x,\xi)$ is τ-periodic for all $\xi \in \mathbb{R}^{\mathrm{d}}$.

Directly from the Lévy-Khintchine formula (2.5) it follows that **(P)** is equivalent to the τ-periodicity of the corresponding Lévy triplet $(b(x), c(x), \nu(x,\mathrm{d}y))$.

Remark 2.4.1. Condition **(P)** implies that $(b(x/\varepsilon), c(x/\varepsilon), \nu(x/\varepsilon,\mathrm{d}y))$ is periodic with period $\varepsilon\tau$ where $\varepsilon > 0$ is a small parameter intended to tend to zero. Let $f \in C_c^\infty\left(\mathbb{R}^{\mathrm{d}}, \mathbb{R}\right)$, if $\mathscr{L}f$ if given in (2.29), where $\mathscr{L}$ is a Feller generator of $\{X_t\}_{t\geq 0}$ let's determine $\mathscr{L}_\varepsilon f$, where $\mathscr{L}_\varepsilon$ is a Feller generator of process $\{\varepsilon X_{\varepsilon^{-2}t}\}_{t\geq 0}$. Notice that $\varepsilon X_0 = x$ implies $X_0 = x/\varepsilon$, therefore

$$\begin{aligned}
\mathscr{L}_\varepsilon f(x) &= \lim_{t\to 0} \frac{\mathbb{E}_{x/\varepsilon}\left[f\left(\varepsilon X_{\varepsilon^{-2}t}\right)\right] - f(x)}{t} = \varepsilon^{-2} \lim_{t\to 0} \frac{\mathbb{E}_{x/\varepsilon}\left[\tilde{f}\left(X_{\varepsilon^{-2}t}\right)\right] - \tilde{f}(x/\varepsilon)}{\varepsilon^{-2}t} \\
&= \varepsilon^{-2}\mathscr{L}\tilde{f}(x/\varepsilon) = \varepsilon^{-2}\langle b(x/\varepsilon), \nabla\tilde{f}(x/\varepsilon)\rangle + \frac{\varepsilon^{-2}}{2}\mathrm{Tr}\, c(x/\varepsilon)\nabla^2\tilde{f}(x/\varepsilon) \\
&\quad + \varepsilon^{-2}\int_{\mathbb{R}^d}\left(\tilde{f}(x/\varepsilon + y) - \tilde{f}(x/\varepsilon) - \langle y, \nabla\tilde{f}(x/\varepsilon)\rangle 1_{B_1(0)}(y)\right)\nu(x/\varepsilon,\mathrm{d}y)
\end{aligned}$$

where $\tilde{f}(x) := f(\varepsilon x)$. This implies that $\nabla\tilde{f}(x/\varepsilon) = \varepsilon\nabla f(x)$ and $\nabla\nabla^{\mathrm{T}}\tilde{f}(x/\varepsilon) = \varepsilon^2\nabla\nabla^{\mathrm{T}} f(x)$ and therefore

$$\begin{aligned}
\mathscr{L}_\varepsilon f(x) =& \varepsilon^{-1}\langle b(x/\varepsilon), \nabla f(x)\rangle + 2^{-1}\mathrm{Tr}\left(c(x/\varepsilon)\,\nabla\nabla^{\mathrm{T}} f(x)\right) \\
&+ \varepsilon^{-2}\int_{\mathbb{R}^d}\left(f(x+\varepsilon y) - f(x) - \varepsilon\langle y, \nabla f(x)\rangle \mathbb{1}_{B_1(0)}(y)\right)\nu(x/\varepsilon,\mathrm{d}y),
\end{aligned} \tag{2.15}$$

□

Since τ-periodicity of the Lévy triplet $(b(x), c(x), \nu(x, \mathrm{d}y))$ is equivalent to the τ-periodicity of $x \mapsto \mathbb{P}_x(X_t - x \in \mathrm{d}y)$ (see [87, Section 4]) we can conclude that $\{P_t\}_{t\geq 0}$ preserves the class of all bounded Borel measurable τ-periodic functions, that is, the function $x \mapsto P_t f(x)$ is τ-periodic for all $t \geq 0$ and all τ-periodic $f \in B_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$. Now, together with this, a straightforward adaptation of [57, Proposition 3.8.3] entails that $\{\Pi_\tau(X_t)\}_{t\geq 0}$ is a Markov process on $(\mathbb{T}^{\mathrm{d}}_\tau, \mathscr{B}(\mathbb{T}^{\mathrm{d}}_\tau))$ which we will denote by $\{X^\tau_t\}_{t\geq 0}$. Its transition kernel is given by

$$p^\tau(t,x,B) = p\left(t, y_x, \Pi_\tau^{-1}(B)\right) = p(t, y_x, \bigcup_{k_\tau \in \mathbb{Z}^{\mathrm{d}}_\tau} B + k_\tau) \tag{2.16}$$

for $\varepsilon \geq 0$, $t \geq 0$, $x \in \mathbb{T}^{\mathrm{d}}_\tau$, $B \in \mathscr{B}(\mathbb{T}^{\mathrm{d}}_\tau)$ and $y_x \in \Pi_\tau^{-1}(\{x\})$. Positivity preserving contraction semigroup $\{P^\tau_t\}_{t\geq 0}$ of the process $\{X^\tau_t\}_{t\geq 0}$ (on the space $(B_b(\mathbb{T}^{\mathrm{d}}_\tau), \|\cdot\|_\infty)$) is given by

$$P^\tau_t f(x) := \mathbb{E}^\tau_x\left[f(X^\tau_t)\right] = \int_{\mathbb{T}^{\mathrm{d}}_\tau} f(y) p^\tau(t,x,\mathrm{d}y)\,,$$

for $t \geq 0$, $x \in \mathbb{T}^{\mathrm{d}}_\tau$ and $f \in B_b(\mathbb{T}^{\mathrm{d}}_\tau, \mathbb{R})$. Denote the corresponding infinitesimal generator by $(\mathscr{A}^b_\tau, \mathscr{D}_{\mathscr{A}^b_\tau})$.

Lemma 2.4.2. *Let $f \in B_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ be τ-periodic then the following*

$$\int_{\mathbb{R}^{\mathrm{d}}} f(y)\, p(t,x,\mathrm{d}y) = \int_{\mathbb{T}^{\mathrm{d}}_\tau} f(z_y)\, p^\tau(t, x_\tau, \mathrm{d}y) \tag{2.17}$$

holds for all $t \geq 0$ and $z_y \in \Pi_\tau^{-1}(\{y\}) \cap [0,\tau]$.

Proof. To see this first take $A \in \mathscr{B}(\mathbb{T}^{\mathrm{d}}_\tau)$ and $f := \sum_{k_\tau \in \mathbb{Z}^{\mathrm{d}}_\tau} \mathbb{1}_{A+k_\tau}$. We have

$$\int_{\mathbb{R}^{\mathrm{d}}} f(y)\, p(t,x,\mathrm{d}y) = \sum_{k_\tau \in \mathbb{Z}^{\mathrm{d}}_\tau} \int_{\mathbb{R}^{\mathrm{d}}} \mathbb{1}_{A+k_\tau}(y)\, p(t,x,\mathrm{d}y) = \sum_{k_\tau \in \mathbb{Z}^{\mathrm{d}}_\tau} p(t,x,A+k_\tau) =$$
$$p(t,x, \bigcup_{k_\tau \in \mathbb{Z}^{\mathrm{d}}_\tau} A + k_\tau) = p^\tau(t, x_\tau, A) = \int_{\mathbb{T}^{\mathrm{d}}_\tau} f(z_y)\, p^\tau(t, x_\tau, \mathrm{d}y).$$

Now the statement of lemma follows for τ-periodic $f \in B_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ by standard arguments. ■

Remark 2.4.3. In this remark we comment on how process $\{X^\tau_t\}_{t\geq 0}$ inherits certain properties from the original process $\{X_t\}_{t\geq 0}$

(i) If $\{X_t\}_{t\geq 0}$ if C_b-Feller then so is $\{X_t^\tau\}_{t\geq 0}$. To see this take $f_\tau \in C_b(\mathbb{T}_\tau^{\mathrm{d}}, \mathbb{R})$, then its τ-periodic extension is $f \in C_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$. Now from Lemma 2.4.2 we see that $P_t^\tau f_\tau(\Pi_\tau(x)) = P_t f(x)$ therefore if $P_t f \in C_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ then $P_t^\tau f_\tau \in C_b(\mathbb{T}_\tau^{\mathrm{d}}, \mathbb{R})$.

(ii) If $\{X_t\}_{t\geq 0}$ if strong Feller then so is $\{X_t^\tau\}_{t\geq 0}$. To see this take $f_\tau \in B_b(\mathbb{T}_\tau^{\mathrm{d}}, \mathbb{R})$, then its τ-periodic extension is $f \in B_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$. Now from Lemma 2.4.2 we see that $P_t^\tau f_\tau(\Pi_\tau(x)) = P_t f(x)$ therefore if $P_t f \in C_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ then $P_t^\tau f_\tau \in C_b(\mathbb{T}_\tau^{\mathrm{d}}, \mathbb{R})$.

(iii) If $\{X_t\}_{t\geq 0}$ if open-set irreducibile then so is $\{X_t^\tau\}_{t\geq 0}$. To see this just note that for open set $O \subseteq \mathbb{T}_\tau^{\mathrm{d}}$ set $\Pi_\tau^{-1}(O)$ is an open set in $\mathbb{R}^{\mathrm{d}}$ and therefore for any $x \in \mathbb{T}_\tau^{\mathrm{d}}$ and $y_x \in \Pi_\tau^{-1}(\{x\})$ we have $p^\tau(t, x, O) = p(t, y_x, \Pi_\tau^{-1}(O)) > 0$, for all $t > 0$.

□

Remark 2.4.4. Since $\mathbb{T}_\tau^{\mathrm{d}}$ is compact, $\{X_t^\tau\}_{t\geq 0}$ is a Feller process. Denote the corresponding Feller generator by $(\mathscr{A}_\tau^\infty, \mathscr{D}_{\mathscr{A}_\tau^\infty})$ we clearly have $(\mathscr{A}_\tau^\infty, \mathscr{D}_{\mathscr{A}_\tau^\infty}) = (\mathscr{A}_\tau^b, \mathscr{D}_{\mathscr{A}_\tau^b} \cap C_\infty(\mathbb{T}_\tau^{\mathrm{d}}, \mathbb{R})) = (\mathscr{A}_\tau^b, \mathscr{D}_{\mathscr{A}_\tau^b})$. For any $f_\tau \in \mathscr{D}_{\mathscr{A}_\tau^\infty}$ and its τ-periodic extension f, from Lemma 2.4.2 we have that

$$\frac{P_t^\tau f_\tau(x_\tau) - f_\tau(x_\tau)}{t} = \frac{\int_{\mathbb{T}_\tau^{\mathrm{d}}} f_\tau(y)\, p^\tau(t, x_\tau, \mathrm{d}y) - f_\tau(x_\tau)}{t} =$$
$$\frac{\int_{\mathbb{R}^{\mathrm{d}}} f(y)\, p(t, x, \mathrm{d}y) - f(x)}{t} = \frac{P_t f(x) - f(x)}{t}$$

and therefore $f \in \mathscr{D}_{\mathscr{A}^b}$ and $\mathscr{A}^b f$ is a τ-periodic extension of $\mathscr{A}_\tau^\infty f_\tau$.

□

2.5. Poisson Equation

In this section we investigate the existence of a solution ζ to the Poisson equation

$$\mathscr{A}^b \zeta = f\,. \tag{2.18}$$

Let us again consider the motivation for introducing the infinitesimal generator, function $p(t) = \mathrm{e}^{at}$. Notice that when $a < 0$ we have the following

$$a\left(-\int_0^\infty p(t)\,\mathrm{d}t\right) = a\left(-\int_0^\infty \mathrm{e}^{at}\,\mathrm{d}t\right) = a\left(-\frac{1}{a}(0-1)\right) = 1\,.$$

The problem with this equation is that it does not work for $a \geq 0$ and the similar issue will arise in the case of a Poisson equation as well. We will approach this problem by first considering $r_\lambda \coloneqq \int_0^\infty \mathrm{e}^{-\lambda t} p(t)\,\mathrm{d}t$ which is well defined for $a < \lambda$ and as before we have $(\lambda - a) r_\lambda = 1$. In the case of a semigroup $\{P_t\}_{t\geq 0}$ we proceed similarly.

For $\lambda > 0$ the *resolvent* R_λ^τ is defined on set $B_b(\mathbb{T}_\tau^d, \mathbb{R})$ as

$$R_\lambda^\tau f_\tau(x) \coloneqq \int_0^\infty e^{-\lambda t} P_t^\tau f_\tau(x)\,\mathrm{d}t\,, \qquad x \in \mathbb{T}_\tau^{\mathrm{d}}.$$

Since

$$\begin{aligned}
&\frac{P_t^\tau\left(R_\lambda^\tau f\right) - R_\lambda^\tau f}{t} = \frac{1}{t}\int_0^\infty e^{-\lambda s}\left(P_{t+s}^\tau f_\tau - P_s^\tau f_\tau\right)\mathrm{d}s = \\
&\frac{1}{t}\left(e^{\lambda t}\int_t^\infty e^{-\lambda s} P_s^\tau f_\tau\,\mathrm{d}s - \int_0^\infty e^{-\lambda s} P_s^\tau f_\tau\,\mathrm{d}s\right) = \\
&\frac{e^{\lambda t}-1}{t}\int_0^\infty e^{-\lambda s} P_s^\tau f_\tau\,\mathrm{d}s - \frac{e^{\lambda t}}{t}\int_0^t e^{-\lambda s} P_s^\tau f_\tau\,\mathrm{d}s \to \lambda R_\lambda^\tau f_\tau - f_\tau, \qquad \text{as } t \to 0,
\end{aligned} \tag{2.19}$$

we see that $\left(\lambda - \mathscr{A}_\tau^b\right)^{-1} = R_\lambda^\tau$. However it is not clear that the zero-resolvent

$$R^\tau f_\tau(x) \coloneqq \int_0^\infty P_t^\tau f_\tau(x)\,\mathrm{d}t\,, \qquad x \in \mathbb{T}_\tau^{\mathrm{d}},$$

is well defined and in general it is not. The next proposition provides us with sufficient conditions for zero-resolvent to be well defined.

Proposition 2.5.1. *If $\{X_t^\tau\}_{t\geq 0}$ is geometrically ergodic with invariant probability measure $\pi(\mathrm{d}x)$ then the zero-resolvent is well defined for any $f_\tau \in B_b(\mathbb{T}_\tau^{\mathrm{d}}, \mathbb{R})$, $\int_{\mathbb{T}_\tau^{\mathrm{d}}} f_\tau(x)\,\pi(\mathrm{d}x) = 0$.*

Proof. From Corollary 2.2.10 we have

$$\int_0^\infty \|P_t^\tau f_\tau\|_\infty \,\mathrm{d}t \le \int_0^\infty \Gamma e^{-\gamma t}\,\|f_\tau\|_\infty \,\mathrm{d}t = \frac{\Gamma}{\gamma}\|f_\tau\|_\infty < \infty .$$

Therefore $\int_0^\infty P_t^\tau f_\tau \,\mathrm{d}t$ is absolutely convergent and $\|R^\tau f_\tau\|_\infty \le \frac{\Gamma}{\gamma}\|f_\tau\|_\infty < \infty$. ∎

Similarly to (2.19) we see that $\left(-\mathscr{A}_\tau^b\right)^{-1} = R^\tau$. From this we can see that resolvent identity $R^\tau = R_\lambda^\tau\left(\mathbb{I} + \lambda R^\tau\right)$ holds true. Indeed,

$$R^\tau = \left(-\mathscr{A}_\tau^b\right)^{-1} = R_\lambda^\tau\left(\lambda - \mathscr{A}_\tau^b\right)\left(-\mathscr{A}_\tau^b\right)^{-1} = R_\lambda^\tau\left(\mathbb{I} + \lambda R^\tau\right).$$

If $\pi(\mathrm{d}x)$ is an invariant measure and $f_\tau \in B_b(\mathbb{T}_\tau^{\mathrm{d}}, \mathbb{R})$ such that $\int_{\mathbb{T}_\tau^{\mathrm{d}}} f_\tau(x)\,\pi(\mathrm{d}x) = 0$ we have

$$\int_{\mathbb{T}_\tau^{\mathrm{d}}} R^\tau f_\tau(x)\,\pi(\mathrm{d}x) = \int_{\mathbb{T}_\tau^{\mathrm{d}}}\int_0^\infty\int_{\mathbb{T}_\tau^{\mathrm{d}}} f_\tau(y) p^\tau(t,x,\mathrm{d}y)\,\mathrm{d}t\,\pi(\mathrm{d}x) = \int_0^\infty\int_{\mathbb{T}_\tau^{\mathrm{d}}} f_\tau(x)\,\pi(\mathrm{d}x)\,\mathrm{d}t = 0 .$$

Lemma 2.5.2. *Let $\{X_t^\tau\}_{t\ge 0}$ be geometrically ergodic C_b-Feller process with invariant probability measure $\pi(\mathrm{d}x)$ and let $f \in C_b(\mathbb{R}^d)$ be τ-periodic and such that $\int_{\mathbb{T}_\tau^{\mathrm{d}}} f_\tau(x)\,\pi(\mathrm{d}x) =$ 0. Denote by $\zeta_\tau(x) := -R^\tau f_\tau(x)$ for any $x \in \mathbb{T}_\tau^{\mathrm{d}}$. Then the τ-periodic extension ζ of ζ_τ is continuous and satisfies* (2.18)*. Moreover, ζ is the unique solution in the class of continuous and τ-periodic solutions to* (2.18) *satisfying $\int_{\mathbb{T}_\tau^{d}} \zeta_\tau(x)\,\pi(\mathrm{d}x) = 0$.*

Proof. Since $\{X_t^\tau\}_{t\ge 0}$ is C_b-Feller we know that ζ_τ is continuous from which we conclude that ζ is also continuous. That it satisfies (2.18) we conclude from Remark 2.4.4 and the fact that that ζ_τ is a solution to

$$\mathscr{A}_\tau^b \zeta_\tau = f_\tau .$$

To prove uniqueness let $\bar\zeta$ be another continuous and τ-periodic solution to (2.18) satisfying $\int_{\mathbb{T}_\tau^{\mathrm{d}}} \bar\zeta_\tau(x)\,\pi(\mathrm{d}x) = 0$. From Proposition 2.3.2 we know that $(\zeta - \bar\zeta)(X_t) - \int_0^t \mathscr{A}^b(\zeta - \bar\zeta)(X_s)\mathrm{d}s$ is a martingale and, since $\mathscr{A}^b(\zeta - \bar\zeta)(x) \equiv 0$, for $x \in \mathbb{R}^{\mathrm{d}}$ and $t \ge 0$ we have

$$(\zeta - \bar\zeta)(x) = \mathbb{E}_x\left[(\zeta - \bar\zeta)(X_t)\right] = \mathbb{E}_{x_\tau}^\tau\left[(\zeta_\tau - \bar\zeta_\tau)(X_t^\tau)\right] = P_t^\tau\left(\zeta_\tau - \bar\zeta_\tau\right)(x_\tau).$$

By letting now $t \to \infty$, it follows from Corollary 2.2.10 that $(\zeta - \bar\zeta)(x) \equiv 0$, which proves the uniqueness. ∎

2.6. Semimartingales

In this section we introduce the notion of semimartingales which is in a certain sense the biggest class of processes with respect to which stochastic integration is possible. We will se how Itô's formula works in this general setting and state some sufficient conditions for a semimartingale to converge in the space of càdlàg functions endowed with the Skorohod J_1-topology. For more on this topic see [50].

We give several definitions for processes on filtered space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$.

Definition 2.6.1. We say that a process $\{M_t\}_{t\geq 0}$ is

- a *martingale* if it is adapted, $\mathbb{E}\left[|M_t|\right] < \infty$ and $\mathbb{E}\left[M_t \mid \mathscr{F}_s\right] = M_s$ for all $0 \leq s \leq t$.
- a *local martingale* if there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times such that $\lim\limits_{n\to\infty} T_n = \infty$ a.s. and that each stopped process $\{M_t^{T_n}\}_{t\geq 0}$ is a martingale,
- a *locally square integrable martingale* if there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times such that $\lim\limits_{n\to\infty} T_n = \infty$ a.s. and that each stopped process $\{M_t^{T_n}\}_{t\geq 0}$ is a square integrable martingale, that is a martingale such that $\sup_{t\geq 0}\mathbb{E}\left[M_t^2\right] < \infty$.

From [50, Theorem I.4.18] we get the following decomposition of local martingales.

Proposition 2.6.2. *Any local martingale $\{M_t\}_{t\geq 0}$ admits a unique decomposition $M_t = M_0 + M_t^c + M_t^d$, where $M_0^c = M_0^d = 0$, $\{M_t^c\}_{t\geq 0}$ is a continuous local martingale, and $\{M_t^d\}_{t\geq 0}$ is a purely discontinuous local martingale.*

Definition 2.6.3. We say that a process $\{B_t\}_{t\geq 0}$ is an *adapted process with finite variation* if it is a real-valued process which is càdlàg, adapted, with $B_0 = 0$ and has a finite variation over each finite interval $[0,t]$.

Definition 2.6.4. We say that the process $\{P_t\}_{t\geq 0}$ is predictable if it is measurable with respect to σ-algebra generated by sets $\{A \times (t,\infty) : t \geq 0, A \in \mathscr{F}_t\} \cup \{A \times \{0\} : A \in \mathscr{F}_0\}$.

From [50, Theorem I.4.2 and I.4.1] we see that the following is well defined.

Definition 2.6.5. If $\{M_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ are d-dimensional locally square integrable martingales such that $M_0 = N_0 = 0$ then $\left\{\left(\langle M,N\rangle_t^{ij}\right)_{1\leq i,j\leq \mathrm{d}}\right\}_{t\geq 0}$ is a *predictable quadratic covariation* of the pair $(\{M_t\}_{t\geq 0}, \{N_t\}_{t\geq 0})$ if it is a predictable process with finite-variation such that the process $\{(M^iN^j - \langle M,N\rangle^{ij})_t\}_{t\geq 0}$ is a local martingale. If $\{M_t\}_{t\geq 0} = \{N_t\}_{t\geq 0}$ we write $\langle M\rangle$.

Example 2.6.6. Special case of the previous definition is when $M_t = \int_0^t F_s\,\mathrm{d}W_s$, where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion and $\{F_t\}_{t\geq 0}$ is such that $\mathbb{E}\left[\int_0^t F_s^2\,\mathrm{d}s\right] < \infty$, for each $t \geq 0$, and it is *progressively measurable*, that is for every $A \in \mathscr{B}(\mathbb{R}^{\mathrm{d}})$ we have $\{(s,\omega) \in [0,t]\times\Omega, F_s(\omega) \in A\} \in \mathscr{B}([0,t]) \times F_t$, then the process $\{M_t\}_{t\geq 0}$ is a square integrable martingale and we get a well known formula $\langle M\rangle_t = \int_0^t F_s^2\,\mathrm{d}s$.

□

Example 2.6.7. Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion and $\{M_t\}_{t\geq 0}$ a square integrable martingale such that $|M_s| \leq f(s)$, where $f \in B\left(\mathbb{R}^{\mathrm{d}}\right)$ is such that $\int_0^t f(s)\,\mathrm{d}s < \infty$ for each $t \geq 0$. Then process $\{E_t\}_{t\geq 0}$ defined by $E_t = \mathrm{e}^{M_t - \frac{1}{2}\langle M\rangle_t}$, $t \geq 0$ is a martingale. Special case of this is if we combine this result with the previous example to conclude that $E_t = \mathrm{e}^{\int_0^t F_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^t F_s^2\,\mathrm{d}s}$, $t \geq 0$ is a martingale.

□

Definition 2.6.8. (i) We say that a process $\{S_t\}_{t\geq 0}$ is a d-dimensional *semimatingale* if $S_t = S_0 + M_t + B_t$ where S_0 is finite-valued and $\mathscr{F}_0$-measurable, where $\{M_t\}_{t\geq 0}$ is a local martingale such that $M_0 = 0$ and where $\{B_t\}_{t\geq 0}$ is an adapted process with finite variation.

(ii) A *special semimartingale* is a semimartingale $\{S_t\}_{t\geq 0}$ which admits a decomposition $S_t = S_0 + M_t + B_t$ as above, with a process $\{B_t\}_{t\geq 0}$ that is predictable.

Decomposition of a semimartingale is not generally unique but it is in the case of special semimartingales. This is a direct consequence of [50, Corrolary I.3.16]

Proposition 2.6.9. *If $\{S_t\}_{t\geq 0}$ is a special semimartingale then its decomposition $S_t = S_0 + M_t + B_t$, where process $\{B_t\}_{t\geq 0}$ is predictable is unique.*

From [50, Proposition I.4.27] we conclude that the following is well defined for every semimartingale $\{S_t\}_{t\geq 0}$.

Definition 2.6.10. We say that the process $\{S_t^c\}_{t\geq 0}$ is a *continuous martingale part* of a semimartingale $\{S_t\}_{t\geq 0}$ if $S_0^c = 0$ and $\{M_t^c\}_{t\geq 0} = \{S_t^c\}_{t\geq 0}$ for any decomposition $S_t = S_0 + M_t + B_t$.

Example 2.6.11. Let $\{X_t\}_{t\geq 0}$ be a LTP given by equation (2.14). As a direct consequence of (**SJ**) and [50, Proposition II.2.29] we see that $\{X_t\}_{t\geq 0}$ is a special semimartingale. From equation (2.14) we read the unique special semimartingale decomposition of $\{X_t\}_{t\geq 0}$.

$$
\begin{aligned}
S_0 &= x \\
B_t &= \int_0^t b(X_s)\mathrm{d}s + \int_0^t \int_{\mathbb{R}} k(X_s,z)\mathbb{1}_{\{u:|k(X_s,u)|\geq 1\}}(z)\,\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s \\
&= \int_0^t \left(b(X_s) + \int_{B_1^c(0)} y\,\nu(X_s,\mathrm{d}y) \right)\mathrm{d}s \\
M_t &= \int_0^t \tilde{\sigma}(X_s)\mathrm{d}\tilde{W}_s + \int_0^t \int_{\mathbb{R}} k(X_{s-},z)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s) - \tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right).
\end{aligned}
$$

Also

$$
X_t^c = M_t^c = \int_0^t \tilde{\sigma}(X_s)\mathrm{d}\tilde{W}_s\,.
$$

□

As already mentioned the stochastic integral $\{H\cdot S_t\}_{t\geq 0}$ is well defined for semimartingale $\{S_t\}_{t\geq 0}$ and a locally bounded predictable process $\{H_t\}_{t\geq 0}$. This is proven in [50, Theorem I.4.31] and a fundamental result by Bichteler, Dellacherie and Mokobodzki states that semimartingales are the biggest class of integrators for which this integral is well defined. Next theorem, proven in [50, Theorem I.4.57], is Itô's formula for semimartingales.

Theorem 2.6.12. *Let $\{S_t\}_{t\geq 0}$ be a* d*-dimensional semimartingale such that $S = \left(S^1,\dots,S^{\mathrm{d}}\right)$ and $f\in C^2(\mathbb{R}^{\mathrm{d}})$. Then $\{f(S)_t\}_{t\geq 0}$ is a semimartingale and we have*

$$
\begin{aligned}
f(S_t) = f(S_0) &+ \sum_{i=1}^{\mathrm{d}} \frac{\partial f}{\partial x_i}(S_-)\cdot S_t^i + \frac{1}{2}\sum_{i,j=1}^{\mathrm{d}} \frac{\partial^2 f}{\partial x_i \partial x_j}(S_-)\cdot \langle S^{i,c}, S^{j,c}\rangle_t \\
&+ \sum_{s\leq t}\left(f(S_s) - f(S_{s-}) - \sum_{i=1}^{\mathrm{d}} \frac{\partial f}{\partial x_i}(S_{s-})\Delta S_s^i \right)
\end{aligned}
$$

Let us now introduce the notion of characteristics of a semimartingale. Let $h:\mathbb{R}^{\mathrm{d}}\to\mathbb{R}^{\mathrm{d}}$ be a truncation function, that is, a bounded Borel measurable function which satisfies $h(x)=x$ in a neighborhood of the origin. Define

$$
\check{S}(h)_t := \sum_{s\leq t}\left(\Delta S_s - h(\Delta S_s)\right) \quad \text{and} \quad S(h)_t := S_t - \check{S}(h)_t\,, \qquad t\geq 0\,,
$$

where the process $\{\Delta S_t\}_{t\geq 0}$ is defined by $\Delta S_t := S_t - S_{t-}$ and $\Delta S_0 := S_0$. Note that in a case of continuous process $\{S_t\}_{t\geq 0}$ we have $\check{S}(h)_t \equiv 0$ and $S(h)_t = S_t$. If $\{S_t\}_{t\geq 0}$ is not continuous, let's say it jumps at time t, we consider two cases. If the jump at time t is small, that is $h(\Delta S_t) = \Delta S_t$ then $\Delta S(h)_t = \Delta S_t$. If the jump at time t is big, that is $h(\Delta S_t) \leq c < \Delta S_t$, where $c > 0$ is such that $\|h\|_\infty < c$, then $\Delta S(h)_t = h(\Delta S_t) \leq c < \Delta S_t$. From this we conclude that process $\{S(h)_t\}_{t\geq 0}$ has bounded jumpes and therefore from [50, Lemma I.4.24] it follows that it is a special semimartingale. From Proposition 2.6.9 we know it admits a unique decomposition

$$S(h)_t = S_0 + M(h)_t + B(h)_t\,, \tag{2.20}$$

where $\{M(h)_t\}_{t\geq 0}$ is a local martingale, and $\{B(h)_t\}_{t\geq 0}$ is a predictable process of finite variation.

Definition 2.6.13. Let $\{S_t\}_{t\geq 0}$ be a semimartingale, and let $h : \mathbb{R}^{\mathrm{d}} \longrightarrow \mathbb{R}^{\mathrm{d}}$ be a truncation function. Furthermore, let $\{B(h)_t\}_{t\geq 0}$ be the predictable process of finite variation appearing in (2.20), let $N(\omega, \mathrm{d}y, \mathrm{d}s)$ be the compensator of the jump measure

$$\mu(\omega, \mathrm{d}y, \mathrm{d}s) := \sum_{s:\,\Delta S_s(\omega)\neq 0} \delta_{(\Delta S_s(\omega),s)}(\mathrm{d}y, \mathrm{d}s)$$

of the process $\{S_t\}_{t\geq 0}$, and let $\{C_t\}_{t\geq 0} = \{\big(C_t^{ij}\big)_{1\leq i,j\leq \mathrm{d}}\}_{t\geq 0}$ be the quadratic co-variation process for $\{S_t^c\}_{t\geq 0}$, that is, $C_t^{ij} = \langle S_t^{i,c}, S_t^{j,c}\rangle$. Then (B, C, N) is called the *characteristics* of the semimartingale $\{S_t\}_{t\geq 0}$ (relative to $h(x)$). In addition, by defining $\tilde{C}(h)_t^{ij} := \langle M(h)_t^i, M(h)_t^j\rangle$, $i, j = 1, \ldots, \mathrm{d}$, where $\{M(h)_t\}_{t\geq 0}$ is the local martingale appearing in (2.20), $(B, \tilde{C}, N)$ is called the *modified characteristics* of the semimartingale $\{S_t\}_{t\geq 0}$ (relative to $h(x)$).

Note that in the case of a continuous semimartingale $\{S_t\}_{t\geq 0}$ we have that $\{S_t\}_{t\geq 0}$ is a special semimartingale with decomposition $S_t = S_0 + M_t + B_t$. Therefore characteristics, which coincide with modified characteristics, are $(B, C, 0)$ where $C = \langle M, M\rangle$.

Example 2.6.14. Let $\{X_t\}_{t\geq 0}$ be a LTP with a Lévy triplet $(b(x), c(x), \nu(x, \mathrm{d}y))$. Then, according to [50, Proposition II.2.17], [91, Lemma 3.2 and Theorem 3.5], the (modified) characteristics of a $\{X_t\}_{t\geq 0}$ (with respect to a truncation function $h(x)$) are given by

$$
\begin{aligned}
B(h)_t^i &= \int_0^t b_i(X_s)\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \left(h_i(y) - y_i \mathbb{1}_{B_1(0)}(y)\right) \nu(X_s,\mathrm{d}y)\,\mathrm{d}s\,,\\
C_t^{ij} &= \int_0^t c_{ij}(X_s)\,\mathrm{d}s\,,\\
N(\mathrm{d}y,\mathrm{d}s) &= \nu(X_s,\mathrm{d}y)\,\mathrm{d}s\,,\\
\tilde{C}(h)_t^{ij} &= \int_0^t c_{ij}(X_s)\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} h_i(y)\,h_j(y)\,\nu(X_s,\mathrm{d}y)\,\mathrm{d}s\\
&= \int_0^t c_{ij}(X_s)\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} h_i(y)\,h_j(y)\,N(\mathrm{d}y,\mathrm{d}s)\,,
\end{aligned}
$$

for $t \geq 0$ and $i,j = 1,\ldots,d$.

□

This characterization of a LTP as a semimartingale will prove useful in the following manner. Next theorem is a direct consequence of [50, Theorem VIII.2.17]

Theorem 2.6.15. *Let $\{S_t^\varepsilon\}_{t\geq 0}$ be a* d*-dimensional semimartingale with modified characteristics $\left(B^\varepsilon, \widetilde{C}^\varepsilon, N^\varepsilon\right)$ and let $\{W_t\}_{t\geq 0}$ be a d-dimensional zero-drift Brownian motion determined by covariance matrix Σ. Then if the following conditions hold*

$$
\sup_{0\leq s\leq t} B_s^{\varepsilon,i} \xrightarrow[\varepsilon\to 0]{\mathbb{P}_x} 0\,, \tag{2.21}
$$

for all $t \geq 0$ and $i = 1,\ldots,\mathrm{d}$,

$$
\widetilde{C}_t^{\varepsilon,ij} \xrightarrow[\varepsilon\to 0]{\mathbb{P}_x} \Sigma^{ij}\,t\,, \tag{2.22}
$$

for all $t \geq 0$ and $i,j = 1,\ldots,\mathrm{d}$, and

$$
\int_0^t \int_{\mathbb{R}^d} g(y) N^\varepsilon(\mathrm{d}y,\mathrm{d}s) \xrightarrow[\varepsilon\to 0]{\mathbb{P}_x} 0\,, \tag{2.23}
$$

for all $t \geq 0$ and $g \in C_b(\mathbb{R}^\mathrm{d})$ vanishing in a neighbourhood of the origin, then

$$
\{S_t^\varepsilon\}_{t\geq 0} \overset{\varepsilon\to 0}{\Longrightarrow} \{W_t\}_{t\geq 0}\,.
$$

Here $\xrightarrow{\mathbb{P}_x}$ stands for the convergence in probability and $\Rightarrow$ denotes the convergence in the space of càdlàg functions endowed with the Skorohod J_1*-topology.*

Theorem 2.6.16. *Let $\{S_t^\varepsilon\}_{t\geq 0}$ be a* d*-dimensional semimartingale with characteristics $(B^\varepsilon, C^\varepsilon, 0)$ such that $|B_t^{\varepsilon,i}| \leq b_i t$ and $|C_t^{\varepsilon,ij}| \leq c_{i,j} t$ for all $t \geq 0$ and $i,j = 1,\ldots,\mathrm{d}$ and* *let* $\{S_t\}_{t\geq 0}$ *be a* d*-dimensional semimartingale with characteristics* $(B, C, 0)$*. Then if the following conditions hold*

$$B_t^{\varepsilon,i} \xrightarrow[\varepsilon\to 0]{\mathbb{P}} B_i\,, \tag{2.24}$$

for all $t \geq 0$ *and* $i = 1, \dots, \mathrm{d}$*, and*

$$C_t^{\varepsilon,ij} \xrightarrow[\varepsilon\to 0]{\mathbb{P}} C_t^{ij} \tag{2.25}$$

for all $t \geq 0$ *and* $i, j = 1, \dots, \mathrm{d}$*, then*

$$\{S_t^\varepsilon\}_{t\geq 0} \overset{\varepsilon\to 0}{\Longrightarrow} \{S_t\}_{t\geq 0}\,. \tag{2.26}$$

Here $\xrightarrow{\mathbb{P}_x}$ *stands for the convergence in probability and* $\Rightarrow$ *denotes the convergence in the space of continuous functions endowed with the locally uniform topology.*

Proof. From [50, Theorem VI.3.21] we see that processes $\{B_t^\varepsilon\}_{t\geq 0}$ and $\{C_t^\varepsilon\}_{t\geq 0}$ are tight. Consequently, [50, Theorem VI.4.18] implies tightness of $\{S_t^\varepsilon\}_{t\geq 0}$. In order to prove (2.26) it remains to prove finite-dimensional convergence in law of process $\{S_t^\varepsilon\}_{t\geq 0}$ to $\{S_t\}_{t\geq 0}$. According to [50, Theorem VIII.2.4] this will hold if conditions (2.24) and (2.25) are met. ∎

In an even more specific case, as expected with even fewer assumptions we get the same result. The next theorem is a direct consequence of [50, Theorem VIII.2.17]

Theorem 2.6.17. *Let* $\{S_t^\varepsilon\}_{t\geq 0}$ *be a* d*-dimensional semimartingale with characteristics* $(0, C^\varepsilon, 0)$ *and* $i, j = 1, \dots, \mathrm{d}$ *and let* $\{W_t\}_{t\geq 0}$ *be a d-dimensional zero-drift Brownian motion determined by covariance matrix* Σ*. Then if the following condition holds*

$$C_t^{\varepsilon,ij} \xrightarrow[\varepsilon\to 0]{\mathbb{P}} \Sigma^{ij}\, t \tag{2.27}$$

for all $t \geq 0$ *and* $i, j = 1, \dots, \mathrm{d}$*, then*

$$\{S_t^\varepsilon\}_{t\geq 0} \overset{\varepsilon\to 0}{\Longrightarrow} \{W_t\}_{t\geq 0}\,. \tag{2.28}$$

Here $\xrightarrow{\mathbb{P}_x}$ *stands for the convergence in probability and* $\Rightarrow$ *denotes the convergence in the space of continuous functions endowed with the locally uniform topology.*

2.7. DIFFUSION PROCESSES

In this section we take a closer look at a LTP $\{X_t\}_{t\geq 0}$ such that $\nu(x, \mathrm{d}y) \equiv 0$, that is a continuous LTP. We call these processes diffusions and they will be the central object of Chapter 4 in this thesis. Since a significant part of the conclusions made for general LTPs are trivial in this case we will acknowledge the similarities but construct the theory independently. That is, in this section we will define a diffusion process $\{X_t\}_{t\geq 0}$ by not relying on the theory of LTPs.

Let $\mathscr{L}^\varepsilon$ be a second-order elliptic differential operator of the form

$$\mathscr{L}^\varepsilon f(x) = \langle a(x/\varepsilon) + \varepsilon^{-1} b(x/\varepsilon), \nabla f(x)\rangle + \frac{1}{2}\mathrm{Tr}\left(c(x/\varepsilon)\nabla\nabla^{\mathrm{T}} f(x)\right), \tag{2.29}$$

with coefficients $a(x) = (a_i(x))_{i=1,\dots,\mathrm{d}}$, $b(x) = (b_i(x))_{i=1,\dots,\mathrm{d}}$ and $c(x) = (c_{ij}(x))_{i,j=1,\dots,\mathrm{d}}$ satisfying

(D) (i) there is $\sigma(x) = (\sigma_{ij}(x))_{i=1,\dots,\mathrm{d},\, j=1,\dots,\mathrm{n}}$ such that $c(x) = \sigma(x)\sigma(x)^{\mathrm{T}}$ for all $x \in \mathbb{R}^{\mathrm{d}}$;

(ii) a_i, b_i and σ_{ij}, $i = 1,\dots,\mathrm{d}$, $j = 1,\dots,\mathrm{n}$, are continuous and τ-periodic;

(iii) there is $\Theta > 0$ and a non-decreasing concave function $\theta : (0,\infty) \to (0,\infty)$ satisfying

$$\int_{0+} \frac{\mathrm{d}u}{\theta(u)} = \infty,$$

such that for all $x, y \in [0,\tau]$,

$$\begin{aligned}&\max\left\{\|\sigma(x) - \sigma(y)\|_{\mathrm{HS}}^2, \langle x - y, a(x) - a(y)\rangle, \langle x - y, b(x) - b(y)\rangle\right\}\\ &\leq \Theta\,|x-y|\,\theta(|x-y|).\end{aligned} \tag{2.30}$$

As in the case of LTPs $\varepsilon > 0$ is a small parameter defined as a microstructure period intended to tend to zero. If we compare (2.29) to (2.15) we see that, as already mentioned $\nu(x, \mathrm{d}y) \equiv 0$ but also that there is an additional drift term a. We will see in the fourth chapter that this term significantly complicates the situation and additional assumptions will be needed when $a \not\equiv 0$.

According to [108, Theorems 2.2 and 2.4], for any $\varepsilon > 0$, $x \in \mathbb{R}^{\mathrm{d}}$ and a given standard n-dimensional Brownian motion $\{W_t\}_{t\geq 0}$ (defined on a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$

satisfying the usual conditions), the following stochastic differential equation (SDE):

$$\begin{aligned} \mathrm{d}X^{\varepsilon}(x,t) &= a\left(X^{\varepsilon}(x,t)/\varepsilon\right)\,\mathrm{d}t + \frac{1}{\varepsilon}\,b\left(X^{\varepsilon}(x,t)/\varepsilon\right)\,\mathrm{d}t + \sigma\left(X^{\varepsilon}(x,t)/\varepsilon\right)\,\mathrm{d}W_t \\ X^{\varepsilon}(x,0) &= x \in \mathbb{R}^{\mathrm{d}}, \end{aligned} \tag{2.31}$$

admits a unique strong solution $\{X^{\varepsilon}(x,t)\}_{t\geq 0}$ which is a conservative (non-explosive) strong Markov process with continuous sample paths, and transition kernel $p^{\varepsilon}(t,x,\mathrm{d}y) = \mathbb{P}\left(X^{\varepsilon}(x,t) \in \mathrm{d}y\right)$, $t \geq 0$, $x \in \mathbb{R}^{\mathrm{d}}$.

Definition 2.7.1. If $\mathscr{L}^{\varepsilon}$ is a second-order elliptic differential operator given in (2.29) with coefficients a,b,c satisfying condition **(D)** then we say that the unique strong solution $\{X^{\varepsilon}(x,t)\}_{t\geq 0}$ to stochastic differential equation (2.31) is a d-dimensional diffusion process associated to operator $\mathscr{L}^{\varepsilon}$.

Notice that here we use a different notation (as opposite to the rest of this chapter), where we indicate the starting point of the process $\{X(x,t)\}_{t\geq 0}$ as an argument. This will prove more convenient when discussing diffusion processes because we will consider a single probability space for different processes (each $\varepsilon > 0$ defines a different diffusion process). We will use this notation throughout this section as well as in Chapter 4 and in parts of Chapter 5.

Remark 2.7.2. A more standard assumption instead of **(D)** (ii) and (iii) is to assume

($\tilde{\mathbf{D}}$) (ii) a_i, b_i and σ_{ij}, $i = 1,\ldots,\mathrm{d}$, $j = 1,\ldots,\mathrm{n}$, are Lipschitz continuous and τ-periodic.

If coefficients are not τ-periodic then a standard assumption is one of linear growth, but clearly this is trivially satisfied in the periodic setting. Notice that if **($\tilde{\mathbf{D}}$)** (ii) holds then (2.30) is satisfied with $\theta(u) = u$ and therefore the assumptions in **(D)** (ii) and (iii) are more general.

□

Process $\{X^{\varepsilon}(x,t)\}_{t\geq 0}$ is a special semimartingale and (using notation from Definition 2.6.8) its unique special semimartingale decomposition is

$$\begin{aligned} S_0 &= x \\ B_t &= \int_0^t a\left(X^{\varepsilon}(x,s)/\varepsilon\right) + \frac{1}{\varepsilon}\,b\left(X^{\varepsilon}(x,s)/\varepsilon\right)\,\mathrm{d}s \\ M_t &= \int_0^t \sigma\left(X^{\varepsilon}(x,s)/\varepsilon\right)\,\mathrm{d}W_s\,. \end{aligned}$$

Its characteristics are given by

$$\begin{aligned} B_t^i &= \int_0^t a\,(X^\varepsilon(x,s)/\varepsilon) + \frac{1}{\varepsilon}\,b\,(X^\varepsilon(x,s)/\varepsilon)\,\mathrm{d}s\,, \\ C_t^{ij} = \tilde{C}_t^{ij} &= \int_0^t c\,(X^\varepsilon(x,s)/\varepsilon)\,\mathrm{d}s\,, \\ N(\mathrm{d}y,\mathrm{d}s) &\equiv 0\,. \end{aligned}$$

Denote by

$$P_t^\varepsilon f(x) := \mathbb{E}\,[f(X(x,t))] = \int_{\mathbb{R}^\mathrm{d}} f(y)p^\varepsilon(t,x,\mathrm{d}y)\,, \qquad t \geq 0\,, f \in B_b(\mathbb{R}^\mathrm{d})\,,$$

the corresponding operator semigroup and by $(\mathscr{A}^\varepsilon, \mathscr{D}_{\mathscr{A}^\varepsilon})$ the B_b-generator of process $\{X^\varepsilon(x,t)\}_{t\geq 0}$. Notice that equation (2.31) is a special case of equation (2.10) from Example 2.3.7, where we take $Z_t = 0$, $\Phi_1(x) = a\,(x/\varepsilon) + \varepsilon^{-1}b\,(x/\varepsilon)$ and $\Phi_2(x) = \sigma(x)$. From [58, 1.1 Theorem] it follows that

Proposition 2.7.3. *Process $\{X^\varepsilon(x,t)\}_{t\geq 0}$ defined as above is a Feller process and $C_c^2\left(\mathbb{R}^\mathrm{d},\mathbb{R}\right)$ is contained in the domain of its generator.*

Let us also note that conservativeness of process $\{X^\varepsilon(x,t)\}_{t\geq 0}$ implies that it is also a C_b-Feller process and that if instead of condition **(D)** (ii) and (iii) we assume $(\tilde{\mathbf{D}})$(ii) the Feller property follows from [92, Corollary 3.3.] and [92, Theorem 3.5.] implies that $C_c^\infty\left(\mathbb{R}^\mathrm{d},\mathbb{R}\right) \subseteq \mathscr{D}_{\mathscr{A}^\varepsilon}$.

Corollary 2.7.4. *Let $\{X^\varepsilon(x,t)\}_{t\geq 0}$ be a* d*-dimensional diffusion process associated to a second-order elliptic differential operator $\mathscr{L}^\varepsilon$ given in (2.29) with coefficients a,b,c satisfying **(D)**. Then $\{X^\varepsilon(x,t)\}_{t\geq 0}$ is a LTP and its Lévy triplet is $(a(x/\varepsilon)+\varepsilon^{-1}b(x/\varepsilon), c(x/\varepsilon), 0)$.*

Clearly condition **(D)**(ii) implies **(P)**.

By employing Itô's formula, that is Theorem 2.6.12 for $f \in C_b^2(\mathbb{R}^\mathrm{d},\mathbb{R})$ we have

$$\begin{aligned} &(P_t^\varepsilon f(x) - f(x))\,/t = \mathbb{E}\Bigg[\frac{1}{t}\int\limits_0^t \langle a(X^\varepsilon(x,s)/\varepsilon) + \varepsilon^{-1}b(X^\varepsilon(x,s)/\varepsilon), \nabla f\,(X^\varepsilon(x,s))\rangle\,\mathrm{d}s + \\ &\frac{1}{t}\int\limits_0^t \langle\sigma(X^\varepsilon(x,s)/\varepsilon), \nabla f\,(X^\varepsilon(x,s))\rangle \mathrm{d}W_s + \frac{1}{2}\int\limits_0^t \mathrm{Tr}\left(c(X_s^\varepsilon/\varepsilon)\nabla\nabla^\mathrm{T} f\,(X_s^\varepsilon)\right)\mathrm{d}s\Bigg] = \\ &\mathbb{E}\left[\frac{1}{t}\int\limits_0^t \mathscr{L}^\varepsilon f\,(X^\varepsilon(x,s))\,\mathrm{d}s\right] = \frac{1}{t}\int\limits_0^t \mathbb{E}\,[\mathscr{L}^\varepsilon f\,(X^\varepsilon(x,s))]\,\mathrm{d}s = \frac{1}{t}\int\limits_0^t P_s^\varepsilon \mathscr{L}^\varepsilon f\,(x)\,\mathrm{d}s\,. \end{aligned}$$

and therefore, using a fact that Proposition 2.7.3 implies that $\{P_t^\varepsilon\}_{t\geq 0}$ is strongly continuous, we conclude

$$\lim_{t\to 0}\|(P_t^\varepsilon f - f)/t - \mathscr{L}^\varepsilon f\|_\infty = \lim_{t\to 0}\left\|\frac{1}{t}\int_0^t P_s^\varepsilon \mathscr{L}^\varepsilon f - \mathscr{L}^\varepsilon f\,\mathrm{d}s\right\|_\infty \leq$$

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \|P_s^\varepsilon \mathscr{L}^\varepsilon f - \mathscr{L}^\varepsilon f\|_\infty\,\mathrm{d}s = \|P_0^\varepsilon \mathscr{L}^\varepsilon f - \mathscr{L}^\varepsilon f\|_\infty = 0 \qquad \forall f\in C_\infty^2(\mathbb{R}^{\mathrm{d}},\mathbb{R}),$$

that is $\mathscr{A}^\varepsilon|_{C_\infty^2(\mathbb{R}^{\mathrm{d}},\mathbb{R})} = \mathscr{L}^\varepsilon$ and [15, Theorem 2.37] (again using Proposition 2.7.3) states that also $\mathscr{A}^\varepsilon|_{C_b^2(\mathbb{R}^{\mathrm{d}},\mathbb{R})} = \mathscr{L}^\varepsilon$.

Following [38] (see also [12, Lemma 3.4.1]), for $\varepsilon > 0$ let $\tilde{X}^\varepsilon(x,t) := \varepsilon^{-1}X^\varepsilon(\varepsilon x, \varepsilon^2 t)$, $t\geq 0$. From (2.31) we have

$$\varepsilon\tilde{X}^\varepsilon(x/\varepsilon, t/\varepsilon^2) - \varepsilon\cdot x/\varepsilon = X^\varepsilon(x,t) - x =$$

$$\int_0^t a\left(\tilde{X}^\varepsilon(x/\varepsilon, s/\varepsilon^2)\right)\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t b\left(\tilde{X}^\varepsilon(x/\varepsilon, s/\varepsilon^2)\right)\mathrm{d}s + \int_0^t \sigma\left(\tilde{X}^\varepsilon(x/\varepsilon, s/\varepsilon^2)\right)\mathrm{d}W_s =$$

$$\varepsilon^2\int_0^{t/\varepsilon^2} a\left(\tilde{X}^\varepsilon(x/\varepsilon, s)\right)\mathrm{d}s + \varepsilon\int_0^{t/\varepsilon^2} b\left(\tilde{X}^\varepsilon(x/\varepsilon, s)\right)\mathrm{d}s + \varepsilon\int_0^{t/\varepsilon^2} \sigma\left(\tilde{X}^\varepsilon(x/\varepsilon, s)\right)\mathrm{d}W_s^\varepsilon,$$

where $W_t^\varepsilon := \varepsilon^{-1}W_{\varepsilon^2 t}$, $t\geq 0$. Clearly, $\{\tilde{X}^\varepsilon(x,t)\}_{t\geq 0}$ satisfies

$$\begin{aligned} \mathrm{d}\tilde{X}^\varepsilon(x,t) &= \left(\varepsilon a\left(\tilde{X}^\varepsilon(x,t)\right) + b\left(\tilde{X}^\varepsilon(x,t)\right)\right)\mathrm{d}t + \sigma\left(\tilde{X}^\varepsilon(x,t)\right)\mathrm{d}W_t^\varepsilon \\ \tilde{X}^\varepsilon(x,0) &= x\in\mathbb{R}^{\mathrm{d}}. \end{aligned} \tag{2.32}$$

Observe that $\{W_t^\varepsilon\}_{t\geq 0} \overset{(\mathrm{d})}{=} \{W_t\}_{t\geq 0}$, although it is not a martingale with respect to $\{\mathscr{F}_t\}_{t\geq 0}$, where $\overset{(\mathrm{d})}{=}$ denotes the equality in distribution. Clearly, for every $\varepsilon > 0$ the processes $\{\tilde{X}^\varepsilon(x,t)\}_{t\geq 0}$, share the same structural properties as $\{X^\varepsilon(x,t)\}_{t\geq 0}$, mentioned above. Let also $\{\tilde{X}^0(x,t)\}_{t\geq 0}$ be a solution to

$$\begin{aligned} \mathrm{d}\tilde{X}^0(x,t) &= b\left(\tilde{X}^0(x,t)\right)\mathrm{d}t + \sigma\left(\tilde{X}^0(x,t)\right)\mathrm{d}W_t \\ \tilde{X}^0(x,0) &= x\in\mathbb{R}^{\mathrm{d}}. \end{aligned} \tag{2.33}$$

For $\varepsilon\geq 0$ denote by $\tilde{p}^\varepsilon(t,x,\mathrm{d}y) = \mathbb{P}\left(\tilde{X}^\varepsilon(x,t)\in\mathrm{d}y\right)$, $t\geq 0$, $x\in\mathbb{R}^{\mathrm{d}}$, $\{\tilde{P}_t^\varepsilon\}_{t\geq 0}$ and $(\tilde{\mathscr{A}}^\varepsilon, \mathscr{D}_{\tilde{\mathscr{A}}^\varepsilon})$ the corresponding transition kernel, operator semigroup and B_b-generator, respectively. Using the same arguments as in Proposition 2.7.3 we conclude

Proposition 2.7.5. *Process $\{\tilde{X}^0(x,t)\}_{t\geq 0}$ is a Feller process with Feller generator $(\tilde{\mathscr{A}}^0, \mathscr{D}_{\tilde{\mathscr{A}}^0})$ and $C_c^2\left(\mathbb{R}^{\mathrm{d}}\right) \subseteq \mathscr{D}_{\tilde{\mathscr{A}}^0}$.*

Since equation (2.33) coincides with equation (2.32) for $\varepsilon = 0$ it is reasonable to assume that the solution of (2.32) converges to solution of (2.33) in some sense, when $\varepsilon \to 0$. Next proposition, which is a direct consequence of Theorem 2.6.16, formalizes this idea.

Proposition 2.7.6. *Let $\{\tilde{X}^\varepsilon(x,t)\}_{t\geq 0}$ and $\{\tilde{X}^0(x,t)\}_{t\geq 0}$ be as above. Then*

$$\{\tilde{X}^\varepsilon(x,t)\}_{t\geq 0} \xrightarrow[\varepsilon\to 0]{(\mathrm{d})} \{\tilde{X}^0(x,t)\}_{t\geq 0}, \tag{2.34}$$

where, $\xrightarrow{(\mathrm{d})}$ denotes the convergence in the space of continuous functions endowed with the locally uniform topology.

In particular, for any $t \geq 0$ and $x \in \mathbb{R}^{\mathrm{d}}$,

$$\lim_{\varepsilon\to 0} \tilde{P}_t^\varepsilon f(x) = \lim_{\varepsilon\to 0} \mathbb{E}\left[f(\tilde{X}^\varepsilon(x,t))\right] = \mathbb{E}\left[f(\tilde{X}^0(x,t))\right] = \tilde{P}_t^0 f(x), \quad \text{for any } f \in C_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R}).$$

We can get better than this if we take into account the periodic structure of processes $\{\tilde{X}^\varepsilon(x,t)\}_{t\geq 0}$ and $\{\tilde{X}^0(x,t)\}_{t\geq 0}$. Due to τ-periodicity of the coefficients, $\{\tilde{X}^\varepsilon(x+k_\tau,t)\}_{t\geq 0}$ and $\{\tilde{X}^\varepsilon(x,t)+k_\tau\}_{t\geq 0}$, $\varepsilon \geq 0$, $x \in \mathbb{R}^{\mathrm{d}}$, $k_\tau \in \mathbb{Z}_\tau^{\mathrm{d}}$, are indistinguishable. In particular,

$$\tilde{p}^\varepsilon(t, x+k_\tau, B) = \tilde{p}^\varepsilon(t, x, B-k_\tau)$$

for all $\varepsilon \geq 0$, $t \geq 0$, $x \in \mathbb{R}^{\mathrm{d}}$, $k_\tau \in \mathbb{Z}_\tau^{\mathrm{d}}$ and $B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}})$, which implies that $\{\tilde{P}_t^\varepsilon\}_{t\geq 0}$ preserves the class of τ-periodic functions in $B_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$. As in Section 2.4 this and a straightforward adaptation of [57, Proposition 3.8.3] entails that $\{\Pi_\tau(\tilde{X}^\varepsilon)(x_\tau,t)\}_{t\geq 0}$ is a Markov process on $(\mathbb{T}_\tau^{\mathrm{d}}, \mathscr{B}(\mathbb{T}_\tau^{\mathrm{d}}))$ which we will denote by $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$, $x \in \mathbb{T}_\tau^{\mathrm{d}}$. Denote by $\tilde{p}^{\varepsilon,\tau}(t,x,\mathrm{d}y)$, $\{\tilde{P}_t^{\varepsilon,\tau}\}_{t\geq 0}$, $(\tilde{\mathscr{A}}^{\varepsilon,\tau}, \mathscr{D}_{\tilde{\mathscr{A}}^{\varepsilon,\tau}})$ its transition kernel, semigroup on the space $(B_b(\mathbb{T}_\tau^{\mathrm{d}}), \|\cdot\|_\infty)$ and infinitesimal generator of process $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$ respectively.

Proposition 2.7.7. *Under **(D)**, for any $t \geq 0$ and τ-periodic $f \in C_b(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ it holds that*

$$\lim_{\varepsilon\to 0} \|\tilde{P}_t^\varepsilon f - \tilde{P}_t^0 f\|_\infty = 0.$$

Proof. From equations Proposition 2.7.6 and Lemma 2.4.2 we see that

$$\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0} \xrightarrow[\varepsilon\to 0]{(\mathrm{d})} \{\tilde{X}^{0,\tau}(x,t)\}_{t\geq 0}.$$

Now, since $\mathbb{T}_\tau^{\mathrm{d}}$ is compact, the assertion follows from [53, Theorem 17.25]. ■

3. CLT for LTPs with Small Jumps

Throughout this chapter we assume that $\{X_t\}_{t\geq 0}$ is a LTP satisfying conditions **(C)**, **(SJ)** and **(P)**. The main theorem of this chapter is the Central limit theorem in the last section. In each section proceeding it we add additional assumptions for process $\{X_t\}_{t\geq 0}$, give examples of processes satisfying these conditions and explain why they are important.

3.1. Geometric Ergodicity

For the rest of this chapter we assume that

(FI) $\{X_t\}_{t\geq 0}$ is strong Feller and open-set irreducible.

Note that from Remark 2.4.3 condition **(FI)** implies that $\{X_t^\tau\}_{t\geq 0}$ is strong Feller and open-set irreducible, too.

Example 3.1.1. Here we give several examples of LTPs satisfying strong Feller property. We will not prove this but give references.

(i) Let $\{X_t\}_{t\geq 0}$ be a d-dimensional diffusion process associated to operator

$$\mathscr{L}f(x) = \langle b(x), \nabla f(x)\rangle + \frac{1}{2}\operatorname{Tr}\Big(c(x)\nabla\nabla^{\mathrm{T}} f(x)\Big),$$

with measurable drift coefficient b, such that diffusion coefficient c is continuous and positive definite, then according to [86, Theorem V.24.1], process $\{X_t\}_{t\geq 0}$ is strong Feller (where we used the fact that in periodic setting there always is a constant $\Lambda > 0$ such that

$$|c_{ij}(x)| + |b_i(x)|^2 \leq \Lambda(1+|x|^2), \qquad x\in\mathbb{R}^{\mathrm{d}},\ i,j = 1,\dots,\mathrm{d}). \tag{3.1}$$

Let us also remark that when $\{X_t\}_{t\geq 0}$ is a d-diffusion process associated to a second-order elliptic operator in divergence form

$$\mathscr{L}f(x) = \nabla\left(c(x)\cdot\nabla f(x)\right) \tag{3.2}$$

with c bounded, measurable and uniformly elliptic, strong Feller property of $\{X_t\}_{t\geq 0}$ has been discussed in [6], [71] and [100].

(ii) Suppose that $b, c \in C_b\left(\mathbb{R}^{\mathrm{d}}\right)$ and $c(x)$ positive definite. Further suppose that function $x \mapsto \int_B (1\wedge|y|^2)\,\nu(x,\mathrm{d}y)$ is continuous and bounded for any $B \in \mathscr{B}(\mathbb{R}^d)$. Then, according to [15, Theorems 3.23, 3.24 and 3.25] and [100, Theorem 4.3 and its remark], $\{X_t\}_{t\geq 0}$ is strong Feller.

(iii) Recently, there are lots of developments on heat kernel (that is, the transition density function) estimates of Feller processes. The reader is referred to [18, 19, 20, 21, 42, 55, 56] and the references therein for more details. In particular, let

$$\mathscr{L}f(x) = \int_{\mathbb{R}^{\mathrm{d}}}\left(f(x+y)-f(x)-\langle\nabla f(x),y\rangle\mathbb{1}_{B_1(0)}(y)\right)\frac{\kappa(x,y)}{|y|^{\mathrm{d}+\alpha(x)}}\,\mathrm{d}y\,,$$

where $\alpha : \mathbb{R}^{\mathrm{d}} \to (0,2)$ is a Hölder continuous function such that

$$\begin{aligned} &0 < \alpha_1 \leq \alpha(x) \leq \alpha_2 < 2\,, \qquad x\in\mathbb{R}^{\mathrm{d}}\,,\\ &|\alpha(x)-\alpha(y)| \leq c_1(|x-y|^{\beta_1}\wedge 1)\,, \qquad x,y\in\mathbb{R}^{\mathrm{d}}\,, \end{aligned}$$

for some constants $c_1 > 0$ and $\beta_1 \in (0,1]$, and $\kappa : \mathbb{R}^{\mathrm{d}}\times\mathbb{R}^{\mathrm{d}} \to (0,\infty)$ is a measurable function satisfying

$$\begin{aligned} &\kappa(x,y) = \kappa(x,-y)\,, \qquad x,y\in\mathbb{R}^{\mathrm{d}}\,,\\ &0 < \kappa_1 \leq \kappa(x,y) \leq \kappa_2 < \infty\,, \qquad x,y\in\mathbb{R}^{\mathrm{d}}\,,\\ &|\kappa(x,y)-\kappa(\bar{x},y)| \leq c_2(|x-\bar{x}|^{\beta_2}\wedge 1)\,, \qquad x,\bar{x},y\in\mathbb{R}^{\mathrm{d}}\,, \end{aligned}$$

for some constants $c_2 > 0$ and $\beta_2 \in (0,1]$. If $(\alpha_2/\alpha_1)-1 < \bar{\beta}_0/\alpha_2$, with $\bar{\beta}_0 \in (0,\beta_0]\cap(0,\alpha_2/2)$ and $\beta_0 = \min\{\beta_1,\beta_2\}$, then, by [18, Thereoms 1.1 and 1.3], $(\mathscr{L}, C_c^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R}))$ generates a LTP. Furthermore, by upper bounds as well as Hölder regularity and gradient estimates of the heat kernel (see [18, Thereoms 1.1 and 1.3, and Remark 1.4]), this associated process is strong Feller.

When $\alpha(x) \equiv \alpha \in (0,2)$ and $\kappa(x,y) \equiv \alpha 2^{\alpha-1}\frac{\Gamma((\alpha+\mathrm{d})/2)}{\pi^{\mathrm{d}/2}\Gamma(1-(\alpha/2))}$, operator $\mathscr{L}$ is a fractional Laplacian operator $-(-\Delta)^{\alpha/2}$, which is the infinitesimal generator of the rotationally symmetric α-stable Lévy process in $\mathbb{R}^{\mathrm{d}}$.

(iv) Let $\{X_t\}_{t\geq 0}$ and $\{\tilde{X}_t\}_{t\geq 0}$ be LTPs with semigroups $\{P_t\}_{t\geq 0}$ and $\{\tilde{P}_t\}_{t\geq 0}$, and Feller generators $(\mathscr{A}^\infty, \mathscr{D}_{\mathscr{A}^\infty})$ and $(\tilde{\mathscr{A}}^\infty, \mathscr{D}_{\tilde{\mathscr{A}}^\infty})$, respectively. Suppose that $\{X_t\}_{t\geq 0}$ is strong Feller. Lemma 2.3.3 gives us the formula

$$P_t f = \tilde{P}_t f + \int_0^t P_s(\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty)\tilde{P}_{t-s} f\,\mathrm{d}s\,, \qquad f \in \mathscr{D}_{\mathscr{A}^\infty} \cap \mathscr{D}_{\tilde{\mathscr{A}}^\infty}\,,$$

and, since both processes are LTPs, Corollary 2.3.4, if $\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty$ is a bounded operator on $(B_b(\mathbb{R}^{\mathrm{d}},\mathbb{R}), \|\cdot\|_\infty)$, implies that $\{\tilde{X}_t\}_{t\geq 0}$ is also strong Feller. The assertion above roughly asserts that a bounded perturbation preserves the strong Feller property. Below is a concrete example.

We will use the fact that $\mathscr{L}$ given in (iii) is a strong Feller. Let

$$\begin{aligned}\tilde{\mathscr{L}} f(x) &= \int_{B_1(0)} (f(x+y) - f(x) - \langle \nabla f(x), y\rangle)\,\frac{\kappa(x,y)}{|y|^{\mathrm{d}+\alpha(x)}}\,\mathrm{d}y\\ &\quad + \int_{B_1^c(0)} (f(x+y) - f(x))\,\frac{\gamma(x,y)}{|y|^{\mathrm{d}+\delta}}\,\mathrm{d}y\,,\end{aligned}$$

where α and κ satisfy all the assumptions in (iii), $0 < \delta \leq \alpha_1$ and $\kappa_2 \leq \gamma(x,y)$ bounded and such that $x \mapsto \gamma(x,y)$ is continuous for almost every $y \in \mathbb{R}^{\mathrm{d}}$. To see that $\tilde{\mathscr{L}}$ is strong Feller from the previous discussion it remains to see that $\tilde{\mathscr{L}}$ is a LTP and that

$$(\tilde{\mathscr{L}} - \mathscr{L}) f(x) = \int_{B_1^c(0)} (f(x+y) - f(x)) \left(\frac{\gamma(x,y)}{|y|^{\mathrm{d}+\delta}} - \frac{\kappa(x,y)}{|y|^{\mathrm{d}+\alpha(x)}}\right)\mathrm{d}y$$

is bounded. To see the latter

$$\begin{aligned}\|(\tilde{\mathscr{L}} - \mathscr{L})f\|_\infty &\leq 2\|f\|_\infty \left(\|\gamma\|_\infty \int_1^\infty r^{-\mathrm{d}-\delta} r^{\mathrm{d}-1}\,\mathrm{d}r + \kappa_2 \int_1^\infty r^{-\mathrm{d}-\alpha(x)} r^{\mathrm{d}-1}\,\mathrm{d}r\right)\\ &\leq 2\|f\|_\infty \left(\frac{\|\gamma\|_\infty}{\delta} + \frac{\kappa_2}{\alpha_1}\right)\end{aligned}$$

is bounded on $(B_b(\mathbb{R}^{\mathrm{d}},\mathbb{R}), \|\cdot\|_\infty)$. Now, according to [15, Lemma 1.28] and [97, Proposition 2.1], $\tilde{\mathscr{L}} = \mathscr{L} + (\tilde{\mathscr{L}} - \mathscr{L})$ generates a LTP if $\tilde{\mathscr{L}} - \mathscr{L}$ satisfies a positive maximum principle (2.7). To see this note that $\frac{\gamma(x,y)}{|y|^{\mathrm{d}+\delta}} \geq \frac{\kappa(x,y)}{|y|^{\mathrm{d}+\alpha(x)}}$ for all $x, y \in \mathbb{R}^{\mathrm{d}}$ and that if $f(x_0) = \sup_{x\in\mathbb{R}^{\mathrm{d}}} f(x)$ then $f(x_0+y) - f(x_0) \leq 0$. This yields the strong Feller property of the process associated with $\tilde{\mathscr{L}}$.

□

We remark also that the strong Feller property of LTPs has been discussed in [94]. In the special case when $\{X_t\}_{t\geq 0}$ is given through SDE (2.10), the strong Feller property (and the open-set irreducibility) has been discussed in [60] under the assumption that $\nu_Z(\mathbb{R}^m) < \infty$, and in [66, 67] for an arbitrary $\nu_Z(\mathrm{d}y)$, that is, an arbitrary pure-jump Lévy process $\{Z_t\}_{t\geq 0}$. Observe that in both situations non-degeneracy of $\Phi_2(x)\Phi_2^{\mathrm{T}}(x)$ has been assumed. In the case when $\Phi_3(x) \equiv \Phi_3 \in \mathbb{R}^{\mathrm{d}\times \mathrm{n}}$ the problem has been considered in [4, 62, 65], and for non-constant (and non-degenerate) $\Phi_3(x)$ in [64].

Example 3.1.2. Now we turn to several examples of LTPs which are open-set irreducible and again we only give references.

(i) According to [86, Theorems V.20.1 and V.24.1] and [30, Theorem 7.3.8], a diffusion process will be open-set irreducible (and strong Feller) if b and c are locally Hölder continuous, $c(x)$ is positive definite, and (3.1) holds true. As mentioned before (3.1) trivially holds true in the periodic case.

Also, when $b \in C_b^1(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$, $c \in C_b^2(\mathbb{R}^{\mathrm{d}}, \mathbb{S}^{\mathrm{d}})$, $\partial_{ij}c_{kl}$ is uniformly continuous for all $i, j, k, l = 1, \ldots, \mathrm{d}$, and $c(x)$ is positive definite, the open-set irreducibility (and the strong Feller property) of the process follows from the support theorem for diffusions, see [39, Lemma 6.1.1] and [47, p. 517]. For support theorem of jump processes one can refer to [98].

(ii) If $\{X_t\}_{t\geq 0}$ is a diffusion process generated by a second-order elliptic operator in divergence form (3.2) with uniformly elliptic, bounded and measurable diffusion coefficient, the open-set irreducibility (and the strong Feller property) follows from the corresponding heat kernel estimates (see [6, 71, 101]).

The diffusion processes with jumps or pure jump process considered in [18, 19, 20, 21, 42, 55, 56] are also open-set irreducible, which is a direct consequence of lower bounds of heat kernel obtained in these references.

(iii) Let $\mathscr{L}$ and $\tilde{\mathscr{L}}$ be the operators from Example 3.1.1 (iv). According to [18, Thereom 1.3], the LTP corresponding to $\mathscr{L}$ is open-set irreducible. Further, observe that

$$\sup_{x\in\mathbb{R}^{\mathrm{d}}} \int_{B_1^c(0)} \frac{\kappa(x,y)}{|y|^{\mathrm{d}+\alpha(x)}}\, \mathrm{d}y < \infty .$$

Thus, by [11, Lemma 3.1] and [10, Lemma 3.6], the process associated with the operator $\tilde{\mathscr{L}}$ is also open-set irreducible.

$\square$

For open-set irreducibility of LTPs of the form (2.8) we refer the reader to [4], [60] and [66, 67].

In the following proposition, which slightly generalizes [44, Lemma 2], we show that a LTP will be open-set irreducible if the corresponding Lévy measure shows enough jump activity.

Proposition 3.1.3. *The process $\{X_t\}_{t\geq 0}$ will be open-set irreducible if there are constants $R > r \geq 0$ such that*

(i) $\inf_{x\in K} \nu(x,O) > 0$ *for every non-empty open set $O \subseteq B_R(0)\setminus B_r(0)$, and every non-empty compact set $K \subset \mathbb{R}^{\mathrm{d}}$;*

(ii) the function $x \mapsto \int_{\mathbb{R}^{\mathrm{d}}} f(y+x)\nu(x,\mathrm{d}y)$ is lower semi-continuous for every non-negative lower semi-continuous function $f : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$.

Proof. Fix $\varepsilon,\rho > 0$ such that $0 < \varepsilon < \rho$ and $0 < \rho < \frac{R-r}{4}$. We wish to prove that for any $x,y \in \mathbb{R}^{\mathrm{d}}$ with $r+2\rho < |x-y| < R-2\rho$, there is $t_* = t_*(x,y,\rho,\varepsilon) > 0$ such that

$$p\left(t,z,B_\rho(y)\right) > 0\,, \qquad \text{for all } z \in B_\rho(x)\,,\ t \in (0,t_*]\,.$$

because the assertion then follows by employing the Chapman-Kolmogorov equation.

To see this take $f_\varepsilon \in C_c^\infty(\mathbb{R}^{\mathrm{d}})$ be such that $0 \leq f_\varepsilon \leq 1$, $\operatorname{supp} f_\varepsilon \subset B_\rho(y)$ and

$$f_\varepsilon(y) = \begin{cases} 1, & y \in B_{\rho-\varepsilon}(y) \\ 0, & y \in B_\rho^c(y)\,. \end{cases}$$

This implies that $0 \leq f_\varepsilon \leq \mathbb{1}_{B_\rho(y)}$ and $f_\varepsilon(z) = 0$, $\nabla f_\varepsilon(z) = 0$, $\nabla^2 f_\varepsilon(z) = 0$ for any $z \in B_\rho^c(y)$. By assumption (**LTP**) we know that $f_\varepsilon \in \mathscr{D}_{\mathscr{A}^\infty}$ and therefore

$$\lim_{t\to 0}\left\|\frac{P_t f_\varepsilon - f_\varepsilon}{t} - \mathscr{A}^\infty f_\varepsilon\right\|_\infty = 0\,.$$

From this we get

$$\liminf_{t\to 0}\inf_{z\in B_\rho(x)}\frac{p\left(t,z,B_\rho(y)\right)}{t} = \liminf_{t\to 0}\inf_{z\in B_\rho(x)}\frac{\mathbb{E}_z\left[\mathbb{1}_{B_\rho(y)}(X_t)\right]}{t} \geq \liminf_{t\to 0}\inf_{z\in B_\rho(x)}\frac{P_t f_\varepsilon(z)}{t} =$$

$$\liminf_{t\to 0}\inf_{z\in B_\rho(x)}\left|\frac{P_t f_\varepsilon(z)}{t}\right| = \liminf_{t\to 0}\inf_{z\in B_\rho(x)}\left|\frac{P_t f_\varepsilon(z)-f_\varepsilon(z)}{t}-\mathscr{A}^\infty f_\varepsilon(z)+\mathscr{A}^\infty f_\varepsilon(z)\right| \geq$$
$$\liminf_{t\to 0}\left(\inf_{z\in B_\rho(x)}|\mathscr{A}^\infty f_\varepsilon(z)|-\sup_{z\in\mathbb{R}^\mathrm{d}}\left|\frac{P_t f_\varepsilon(z)-f_\varepsilon(z)}{t}-\mathscr{A}^\infty f_\varepsilon(z)\right|\right) =$$
$$\inf_{z\in B_\rho(x)}|\mathscr{A}^\infty f_\varepsilon(z)|-\lim_{t\to 0}\left\|\frac{P_t f_\varepsilon - f_\varepsilon}{t}-\mathscr{A}^\infty f_\varepsilon\right\|_\infty =$$
$$\inf_{z\in B_\rho(x)}\Bigg|\langle b(z),\nabla f_\varepsilon(z)\rangle+\frac{1}{2}\mathrm{Tr}\,c(z)\nabla^2 f_\varepsilon(z)+$$
$$\int_{\mathbb{R}^d}\Big(f_\varepsilon(z+v)-f_\varepsilon(z)-\langle v,\nabla f_\varepsilon(z)\rangle 1_{B_1(0)}(v)\Big)\,\nu(z,\mathrm{d}v)\Bigg| =$$
$$\inf_{z\in B_\rho(x)}\int_{\mathbb{R}^d} f_\varepsilon(z+v)\nu(z,\mathrm{d}v) \geq \inf_{z\in B_\rho(x)}\int_{\mathbb{R}^\mathrm{d}}\mathbb{1}_{B_{\rho-\varepsilon}(y)}(z+v)\,\nu(z,\mathrm{d}v) =$$
$$\inf_{z\in B_\rho(x)}\nu(z,B_{\rho-\varepsilon}(y-z)).$$

Assume now that $\inf_{z\in B_\rho(x)}\nu\big(z,B_{\rho-\varepsilon}(y-z)\big)=0$. Then there is a sequence $\{z_n\}_{n\in\mathbb{N}}\subset B_\rho(x)$ converging to $z_0\in\bar{B}_\rho(x)$, such that

$$0 = \liminf_{n\to\infty}\nu\left(z_n,B_{\rho-\varepsilon}(y-z_n)\right) = \liminf_{n\to\infty}\int_{\mathbb{R}^d}\mathbb{1}_{B_{\rho-\varepsilon}(y)}(u+z_n)\,\nu(z_n,\mathrm{d}u).$$

However, since $z\mapsto \mathbb{1}_{B_{\rho-\varepsilon}(y)}(z)$ is a lower semi-continuous function, from the second condition of this propositionwe have that

$$\begin{aligned} &0 \geq \liminf_{n\to\infty}\int_{\mathbb{R}^d}\mathbb{1}_{B_{\rho-\varepsilon}(y)}(u+z_n)\,\nu(z_n,\mathrm{d}u) \geq \\ &\int_{\mathbb{R}^d}\mathbb{1}_{B_{\rho-\varepsilon}(y)}(u+z_0)\,\nu(z_0,\mathrm{d}u) = \nu\left(z_0,B_{\rho-\varepsilon}(y-z_0)\right). \end{aligned} \tag{3.3}$$

This will lead to a contradiction if we prove that $B_{\rho-\varepsilon}(y-z_0)\subseteq B_R(0)\setminus B_r(0)$, because the first condition of this proposition will then imply that

$$\nu\left(z_0,,B_{\rho-\varepsilon}(y-z_0)\right)=\nu\left(z_0,(B_R(0)\setminus B_r(0))\cap B_{\rho-\varepsilon}(y-z_0)\right) > 0.$$

To this end take $u\in B_{\rho-\varepsilon}(y-z_0)$, we have

$$r < r+\varepsilon = (r+2\rho)-\rho-(\rho-\varepsilon) < |x-y|-|z_0-x|-|u-(y-z_0)| \leq |u| =$$
$$|u-(y-z_0)+(y-x)+(x-z_0)| \leq |u-(y-z_0)|+|y-x|+|x-z_0| <$$
$$\rho-\varepsilon+R-2\rho+\rho = R-\varepsilon < R.$$

Hence, $\liminf_{t\to 0}\inf_{z\in B_\rho(x)}\frac{p(t,z,B_\rho(y))}{t} > 0$, that is there exists $t_*=t_*(x,y,\rho,\varepsilon)>0$ such that

$$p\left(t,z,B_\rho(y)\right) > 0, \qquad \text{for all } z\in B_\rho(x)\,,\ t\in(0,t_*]\,,$$

which concludes the proof. ■

Now that we know some examples of processes satisfying condition (**FI**) lets see how this condition can be useful.

Proposition 3.1.4. *The process $\{X_t^\tau\}_{t\geq 0}$ admits a unique invariant probability measure $\pi(\mathrm{d}x)$ such that*

$$\sup_{x\in\mathbb{T}_\tau^{\mathrm{d}}} \|p^\tau(t,x,\mathrm{d}y)-\pi(\mathrm{d}y)\|_{\mathrm{TV}} \leq \Gamma \mathrm{e}^{-\gamma t}, \qquad t\geq 0 \tag{3.4}$$

for some $\gamma,\Gamma>0$, that is process $\{X_t^\tau\}_{t\geq 0}$ is geometrically ergodic.

Proof. Since $\{X_t^\tau\}_{t\geq 0}$ is strong Feller and open-set irreducible from Proposition 2.2.18 we see that it is also ψ-irreducible. Since it is also C_b-Feller and $\mathbb{T}_\tau^{\mathrm{d}}$ is compact according to Proposition 2.2.4 and Proposition 2.2.8 process $\{X_t^\tau\}_{t\geq 0}$ admits one, and only one, invariant probability measure $\pi(\mathrm{d}x)$. According to Theorem 2.2.16 it is geometrically ergodic. ■

Remark 3.1.5. Alternatively, Proposition 3.1.4 is a consequence of [62, Remark 3.2] and [107, Theorem 1.1] or [70, Theorem 6.1] and [105, Theorem 5.1] (by setting $V(x)\equiv 1$ and $c=d=1$). Also, if instead of (**FI**) we assume

($\widetilde{\mathbf{FI}}$) $\{X_t\}_{t\geq 0}$ admits a density function $p_t(x,y)$ with respect to Lebesgue measure, that is $p(t,x,\mathrm{d}y)=p_t(x,y)\mathrm{d}y$, such that

(i) for any $t>0$, the function $(x,y)\mapsto p_t(x,y)$ is continuous on $\mathbb{R}^d\times\mathbb{R}^d$;

(ii) there is a non-empty open set $O\subseteq\mathbb{R}^{\mathrm{d}}$ such that $p_t(x,y)>0$ for all $t>0$, $x\in\mathbb{R}^{\mathrm{d}}$ and $y\in O$,

which guarantees that Döblin's irreducibility condition holds true (see [28, page 256]), then Proposition 3.1.4 follows from [12, Theorem 3.1].

□

3.2. Poisson Equation

In this section we investigate the regularity properties of a solution β, which will prove essential to the proof of the main theorem in this chapter, to the Poisson equation

$$\mathscr{A}^b\beta = b^* - \pi(b^*), \tag{3.5}$$

where

$$b^*(x) := b(x) + \int_{B_1^c(0)} y\,\nu(x,\mathrm{d}y) \text{ and, as always, } \pi(b^*) = \int_{\mathbb{T}_\tau^d} b^*(x)\,\pi(\mathrm{d}x)\,. \tag{3.6}$$

Observe that in Lemma 2.5.2 we only used the fact that $b_\tau^* \in C(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$. In order to use Itô's formula for β additional smoothness is required. For that purpose in the rest of this chapter we assume

(PE) b^* is of class $C_b^{\psi}(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$ for some Hölder exponent $\psi(r)$, and

(i) for some $t_0 > 0$, any $t \in (0,t_0]$ and any τ-periodic $f \in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$,

$$\|P_t f\|_\psi \le C(t)\|f\|_\infty\,,$$

where $\int_0^{t_0} C(t)\,\mathrm{d}t < \infty$;

(ii) for some $\lambda > 0$ and any τ-periodic $f \in C_b^{\psi}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ with $\int_{\mathbb{T}_\tau^{\mathrm{d}}} f_\tau(x)\,\pi(\mathrm{d}x) = 0$, the Poisson equation

$$\lambda u - \mathscr{A}^b u = f \tag{3.7}$$

admits a unique τ-periodic solution $u_{\lambda,f} \in C_b^{\varphi\psi}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ for some Hölder exponent $\varphi(r)$.

Remark 3.2.1. From Example 2.1.1 we see that condition **(PE)** is satisfied if for some $k_1, k_2 \in \mathbb{N}_0$ and $\alpha_1, \alpha_2 \in (0,1)$ instead of $C_b^{\psi}(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$ we have $C_b^{k_1,\alpha_1}(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$ and instead of $C_b^{\varphi\psi}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ we have $C_b^{k_2,\alpha_2}(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$.

□

Theorem 3.2.2. *The Poisson equation* (3.5) *admits a τ-periodic solution $\beta \in C_b^{\varphi\psi}(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$. Moreover, β is the unique solution in the class of continuous and periodic solutions to* (3.5) *satisfying $\int_{\mathbb{T}_\tau^{\mathrm{d}}} \beta_\tau(x)\,\pi(\mathrm{d}x) = 0$.*

Proof. From Lemma 2.5.2 we know that the solution of (3.5) is a τ-periodic extension of $\beta_\tau(x) = -R^\tau(b^*_\tau - \pi(b^*_\tau))(x)$. Analogously the solution $u_{\lambda,f}$ to (3.7) is exactly the τ-periodic extension of $R^\tau_\lambda f_\tau(x)$ for τ-periodic $f \in C^\psi_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$. From the resolvent identity

$$R^\tau(b^*_\tau - \pi(b^*_\tau)) = R^\tau_\lambda((b^*_\tau - \pi(b^*_\tau)) + \lambda R^\tau(b^*_\tau - \pi(b^*_\tau)))\,, \tag{3.8}$$

the fact that $b^*_\tau - \pi(b^*_\tau)$ is of class $C^\psi_b(\mathbb{T}^{\mathrm{d}}_\tau,\mathbb{R}^{\mathrm{d}})$ and **(PE)**(ii) we see that in order to prove that $\beta \in C^{\varphi\psi}_b(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$ it remain to show that $R^\tau f_\tau \in C^\psi(\mathbb{T}^{\mathrm{d}}_\tau,\mathbb{R})$ for any τ-periodic $f \in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ such that $\int_{\mathbb{T}^{\mathrm{d}}_\tau} f_\tau(x)\,\pi(\mathrm{d}x) = 0$. To see this, take t_0 from **(PE)**(i). we have

$$\int_0^\infty \|P^\tau_t f_\tau\|_\psi\,\mathrm{d}t = \int_0^{t_0} \|P^\tau_t f_\tau\|_\psi\,\mathrm{d}t + \int_{t_0}^\infty \|P^\tau_t, f_\tau\|_\psi\,\mathrm{d}t$$

For the first integral by **(PE)**(i), we have

$$\int_0^{t_0} \|P^\tau_t f_\tau\|_\psi\,\mathrm{d}t \le \|f_\tau\|_\infty \int_0^{t_0} C(t)\,\mathrm{d}t < \infty.$$

Also, since for any $t > 0$ we know that $P^\tau_t f_\tau \in C(\mathbb{T}^{\mathrm{d}}_\tau,\mathbb{R})$ and

$$\int_{\mathbb{T}^{\mathrm{d}}_\tau} P^\tau_t f_\tau(x)\,\pi(\mathrm{d}x) = \int_{\mathbb{T}^{\mathrm{d}}_\tau}\int_{\mathbb{T}^d_\tau} f_\tau(y)p^\tau(t,x,\mathrm{d}y)\,\pi(\mathrm{d}x) = \int_{\mathbb{T}^d_\tau} f_\tau(x)\,\pi(\mathrm{d}x) = 0$$

(PE)(i) and Corollary 2.2.10 imply

$$\int_{t_0}^\infty \|P^\tau_t f_\tau\|_\psi\,\mathrm{d}t = \int_{t_0}^\infty \|P^\tau_{t_0}\left(P^\tau_{t-t_0} f_\tau\right)\|_\psi\,\mathrm{d}t \le C(t_0)\int_{t_0}^\infty \|P^\tau_{t-t_0} f_\tau\|_\infty\,\mathrm{d}t \le$$
$$\Gamma C(t_0)\|f_\tau\|_\infty \int_{t_0}^\infty e^{-\gamma(t-t_0)}\,\mathrm{d}t < \infty.$$

To show that $R^\tau f_\tau \in C^\psi_b(\mathbb{T}^{\mathrm{d}}_\tau,\mathbb{R})$ it remains to see $\|\int_0^\infty P^\tau_t f_\tau\,\mathrm{d}t\|_\psi \le \int_0^\infty \|P^\tau_t f_\tau\|_\psi\,\mathrm{d}t$. Suppose that $m_\psi \in (0,1)$, then

$$\begin{aligned}
\left\|\int_0^\infty P^\tau_t f_\tau\,\mathrm{d}t\right\|_\psi &= \left[\int_0^\infty P^\tau_t f_\tau\,\mathrm{d}t\right]_{0,\psi} \\
&= \sup_{x\in\mathbb{R}^{\mathrm{d}}}\sup_{h\in\bar{B}_1(0)\setminus\{0\}} \frac{|\int_0^\infty P^\tau_t f_\tau((x+h)_\tau)\,\mathrm{d}t - \int_0^\infty P^\tau_t f_\tau(x_\tau)\,\mathrm{d}t|}{\psi(|h|)} \\
&\le \sup_{x\in\mathbb{R}^{\mathrm{d}}}\sup_{h\in\bar{B}_1(0)\setminus\{0\}} \int_0^\infty \frac{|P^\tau_t f_\tau((x+h)_\tau) - P^\tau_t f_\tau(x_\tau)|}{\psi(|h|)}\,\mathrm{d}t \le \int_0^\infty [P^\tau_t f_\tau]_{0,\psi}\,\mathrm{d}t\,.
\end{aligned}$$

For $m_\psi \ge 1$ we proceed as follows

$$\begin{aligned}
\left\|\frac{\partial}{\partial x_1}\int_0^\infty P^\tau_t f_\tau\,\mathrm{d}t\right\|_\infty &= \sup_{x\in\mathbb{R}^{\mathrm{d}}}\lim_{h\to 0}\left|\frac{\int_0^\infty P^\tau_t f_\tau((x+h)_\tau)\,\mathrm{d}t - \int_0^\infty P^\tau_t f_\tau(x_\tau)\,\mathrm{d}t}{h}\right| \\
&\le \sup_{x\in\mathbb{R}^{\mathrm{d}}}\lim_{h\to 0}\int_0^\infty \left|\frac{P^\tau_t f_\tau((x+h)_\tau) - P^\tau_t f_\tau(x_\tau)}{h}\right|\mathrm{d}t \\
&\le \int_0^\infty \left\|\frac{\partial}{\partial x_1}P^\tau_t f_\tau\right\|_\infty \mathrm{d}t\,,
\end{aligned}$$

where the last inequality follows from previously proven by the dominated convergence theorem. For the derivations of higher order we proceed analogously. Since the uniqueness is also guaranteed by Lemma 2.5.2 this completes the proof. ■

Example 3.2.3. Here we give several examples of LTPs satisfying condition **(PE)**. All of the examples will satisfy this condition in the sense of Remark 3.2.1, that is with Hölder spaces and not general Hölder spaces and we will only state what $k_1, k_2, \alpha_1, \alpha_2$ are, without every time mentioning Remark 3.2.1. For examples of processes satisfying condition **(PE)** in general Hölder spaces see [89].

(i) (Diffusion processes) Let $\varepsilon \in (0,1)$ and let $\{X_t\}_{t\geq 0}$ be a d-dimensional diffusion process associated to operator

$$\mathscr{L}f(x) := \langle b(x), \nabla f(x)\rangle + \frac{1}{2}\operatorname{Tr}\left(\mathrm{D}\left(c\nabla f\right)(x)\right),$$

with coefficients $b \in C_b^{\varepsilon}(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$, $c \in C_b^{1,\varepsilon}(\mathbb{R}^{\mathrm{d}}, \mathbb{S}^{\mathrm{d}})$ and $c(x)$ being also positive definite for every $x \in \mathbb{R}^{\mathrm{d}}$. Then, $b^* = b \in C_b^{\varepsilon}(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$, **(PE)**(i) with arbitrary $t_0 > 0$, $k_1 = 0$ and $\alpha_1 = \varepsilon$ follows from [73, the proof of Lemma 2.3]. Also, a straightforward adaptation of [73, Theorem 2.1], together with [48, Chapter 4.8] and [74, Proposition 4.2], implies that **(PE)**(ii) is satisfied with $k_2 = 2, \alpha_2 = \varepsilon$.

(ii) (Diffusion processes with jumps) Let $\varepsilon \in (0,1)$. Assume that b and c are as in (i), and that

$$\nu(x,\mathrm{d}y) := \frac{\gamma(x,y)}{|y|^{\mathrm{d}+\alpha}}\mathbb{1}_{B_1^c(0)}(y)\mathrm{d}y,$$

where $\alpha > 1$ and $x \mapsto \gamma(x,y)$ is positive, ε-Hölder continuous and bounded. Since $b \in C_b^{\varepsilon}(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$ let us first see that $x \mapsto \int_{B_1^c(0)} y\,\nu(x,\mathrm{d}y) \in C_b^{\varepsilon}(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$. Let $C > 0$ be such that $|\gamma(x,y) - \gamma(z,y)| \leq C\|x-z\|^{\varepsilon}$, for all $x, z \in \mathbb{R}^{\mathrm{d}}$, then

$$\left|\int_{B_1^c(0)} y\,\nu(x,\mathrm{d}y) - \int_{B_1^c(0)} y\,\nu(z,\mathrm{d}y)\right| = \left|\int_{B_1^c(0)} y\,\frac{\gamma(x,y)}{|y|^{\mathrm{d}+\alpha}}\mathrm{d}y - \int_{B_1^c(0)} y\,\frac{\gamma(z,y)}{|y|^{\mathrm{d}+\alpha}}\mathrm{d}y\right| \leq$$
$$\int_{B_1^c(0)} \frac{|y|}{|y|^{\mathrm{d}+\alpha}}\,|\gamma(x,y) - \gamma(z,y)|\,\mathrm{d}y \leq \int_{B_1^c(0)} \frac{|y|}{|y|^{\mathrm{d}+\alpha}} C\|x-z\|^{\varepsilon}\mathrm{d}y =$$
$$C\|x-z\|^{\varepsilon}\int_1^{\infty} r^{\mathrm{d}-1}r^1 r^{-\mathrm{d}-\alpha}\,\mathrm{d}r = \frac{C}{\alpha-1}\|x-z\|^{\varepsilon} \qquad \text{for all } x, z \in \mathbb{R}^{\mathrm{d}}.$$

Let us see that $\{X_t\}_{t\geq 0}$ satisfies **(PE)**(i). Denote by $\{P_t\}_{t\geq 0}$ the semigroup of $\{X_t\}_{t\geq 0}$, and let $\{\tilde{P}_t\}_{t\geq 0}$ be the semigroup of the diffusion process with coefficients b and c. Also, denote by $(\mathscr{A}^\infty, \mathscr{D}_{\mathscr{A}^\infty})$ and $(\tilde{\mathscr{A}}^\infty, \mathscr{D}_{\tilde{\mathscr{A}}^\infty})$ the corresponding C_∞-generators, respectively. Then Lemma 2.3.3 gives us the formula

$$P_t f = \tilde{P}_t f + \int_0^t \tilde{P}_s(\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty)P_{t-s} f \,\mathrm{d}s, \qquad f \in \mathscr{D}_{\mathscr{A}^\infty} \cap \mathscr{D}_{\tilde{\mathscr{A}}^\infty}. \tag{3.9}$$

and, since both processes are LTPs, Corollary 2.3.4, if $\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty$ is a bounded operator on $(B_b(\mathbb{R}^\mathrm{d},\mathbb{R}), \|\cdot\|_\infty)$, implies that it also holds for any $f \in B_b(\mathbb{R}^\mathrm{d},\mathbb{R})$. To see that operation $\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty$ take $f \in B_b(\mathbb{R}^\mathrm{d},\mathbb{R})$

$$\begin{aligned} \left\|\left(\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty\right) f\right\|_\infty &= \left\|\int_{B_1^c(0)} (f(\cdot+y) - f(\cdot))\frac{\gamma(\cdot,y)}{|y|^{\mathrm{d}+\alpha}}\,\mathrm{d}y\right\|_\infty \\ &\leq 2\|f\|_\infty\|\gamma\|_\infty \int_1^\infty r^{\mathrm{d}-1} r^{-\mathrm{d}-\alpha}\,\mathrm{d}r = \frac{2\|\gamma\|_\infty}{\alpha}\|f\|_\infty. \end{aligned} \tag{3.10}$$

According to (i), there is a measurable function $\tilde{C}_\varepsilon : (0,\infty) \to (0,\infty)$ such that for all $t \geq 0$ we have $\int_0^t \tilde{C}_\varepsilon(s)\,\mathrm{d}s < \infty$ and $\left\|\tilde{P}_t f\right\|_\varepsilon \leq \tilde{C}_\varepsilon(t)\,\|f\|_\infty$ and all τ-periodic functions $f \in C_b(\mathbb{R}^\mathrm{d},\mathbb{R})$. For fixed τ-periodic $f \in C_b(\mathbb{R}^\mathrm{d},\mathbb{R})$, since $\{X_t\}_{t\geq 0}$ is a C_b-Feller process, for all $0 \leq s \leq t < \infty$ we have $P_{t-s} f \in C_b(\mathbb{R}^\mathrm{d},\mathbb{R})$ and from (3.10), using dominated convergence theorem, we see that $\left(\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty\right) P_{t-s} f \in C_b(\mathbb{R}^\mathrm{d},\mathbb{R})$. This, together with (3.9), implies that $P_t f \in C_b^\varepsilon(\mathbb{R}^\mathrm{d},\mathbb{R})$ for every $t \geq 0$ and we have

$$\|P_t f\|_\varepsilon \leq \left\|\tilde{P}_t f\right\|_\varepsilon + \int_0^t \left\|\tilde{P}_s(\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty)P_{t-s} f\right\|_\varepsilon \mathrm{d}s \leq C_\varepsilon(t)\,\|f\|_\infty\,,$$

where $C_\varepsilon(t) = \tilde{C}_\varepsilon(t) + \left\|\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty\right\| \int_0^t \tilde{C}_\varepsilon(s)\,\mathrm{d}s$. Also,

$$\int_0^t C_\varepsilon(s)\,\mathrm{d}s \leq \left(1 + t\left\|\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty\right\|\right) \int_0^t \tilde{C}_\varepsilon(s)\,\mathrm{d}s, \qquad t > 0,$$

where $\left\|\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty\right\|$ stands for the operator norm of $\mathscr{A}^\infty - \tilde{\mathscr{A}}^\infty$. Thus, $\{X_t\}_{t\geq 0}$ satisfies **(PE)**(i) with $k_1 = 0, \alpha_1 = \varepsilon$.

Condition **(PE)**(ii) with $k_2 = 2, \alpha_2 = \varepsilon$ follows again from [73, Theorem 2.1], since the following conditions are met

(a)

$$\begin{aligned} \sup_{x\in\mathbb{R}^\mathrm{d}} \int_{\mathbb{R}^\mathrm{d}} \frac{|y|^{1+\varepsilon}}{1+|y|^{1+\varepsilon}}\,\nu(x,\mathrm{d}y) &= \sup_{x\in\mathbb{R}^\mathrm{d}} \int_{B_1^c(0)} \frac{|y|^{1+\varepsilon}}{1+|y|^{1+\varepsilon}} \frac{\gamma(x,y)}{|y|^{\mathrm{d}+\alpha}}\,\mathrm{d}y \\ &\leq \|\gamma\|_\infty \int_1^\infty r^{\mathrm{d}-1} r^{-\mathrm{d}-\alpha}\,\mathrm{d}r = \frac{\|\gamma\|_\infty}{\alpha} < \infty, \end{aligned}$$

(b)

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^{\mathrm{d}}}\int_{B_r(0)}|y|^{1+\varepsilon}\,\nu(x,\mathrm{d}y)\,=\,0\,,$$

(c)

$$\begin{aligned}\lim_{R\to\infty}\sup_{x\in\mathbb{R}^{\mathrm{d}}}\int_{B_R^c(0)}\nu(x,\mathrm{d}y)\,&=\,\lim_{R\to\infty}\sup_{x\in\mathbb{R}^{\mathrm{d}}}\int_{B_R^c(0)}\frac{\gamma(x,y)}{|y|^{\mathrm{d}+\alpha}}\,\mathrm{d}y\\ &\le\,\|\gamma\|_\infty\lim_{R\to\infty}\int_R^\infty r^{\mathrm{d}-1}r^{-\mathrm{d}-\alpha}\,\mathrm{d}r\,=\,\|\gamma\|_\infty\lim_{R\to\infty}\frac{R^{-\alpha}}{\alpha}\,=\,0\,,\end{aligned}$$

(d)

$$\begin{aligned}&\sup_{x,z\in\mathbb{R}^{\mathrm{d}}}\|x-z\|^{-\varepsilon}\int_{\mathbb{R}^{\mathrm{d}}}\frac{|y|^{1+\varepsilon}}{1+|y|^{1+\varepsilon}}\,|\nu(x,\mathrm{d}y)-\nu(z,\mathrm{d}y)|\,=\\ &\sup_{x,z\in\mathbb{R}^{\mathrm{d}}}\|x-z\|^{-\varepsilon}\int_{B_1^c(0)}\frac{|y|^{1+\varepsilon}}{1+|y|^{1+\varepsilon}}\,\frac{1}{|y|^{\mathrm{d}+\alpha}}\,|\gamma(x,y)-\gamma(z,y)|\,\mathrm{d}y\,\le\\ &C\int_1^\infty r^{\mathrm{d}-1}r^{-\mathrm{d}-\alpha}\mathrm{d}y\,=\,\frac{C}{\alpha}\,<\,\infty\,.\end{aligned}$$

(iii) (Pure-jump LTPs) In the pure jump case, sufficient conditions for **(PE)**(i) are given in [63, Theorem 1.1]. Also, when the underlying process is given as a solution to an SDE of the form (2.10), we refer to [62, 64, 65] and the references therein.

To construct an example satisfying **(PE)**(ii), we can again employ a perturbation method. Let $\{X_t\}_{t\ge0}$ and $\{\tilde{X}_t\}_{t\ge0}$ be LTPs with semigroups $\{P_t\}_{t\ge0}$ and $\{\tilde{P}_t\}_{t\ge0}$, and B_b-generators $(\mathscr{A}^b,\mathscr{D}_{\mathscr{A}^b})$ and $(\tilde{\mathscr{A}}^b,\mathscr{D}_{\tilde{\mathscr{A}}^b})$, respectively. Assume that $\mathscr{A}^b$ satisfies **(PE)**(ii) for some $k_1,k_2\in\mathbb{N}_0$ and $\alpha_1,\alpha_2\in(0,1)$. Further, assume that $\mathscr{A}^b-\tilde{\mathscr{A}}^b$ is a bounded operator on $(B_b(\mathbb{R}^{\mathrm{d}},\mathbb{R}),\|\cdot\|_\infty)$, and that $(\mathscr{A}^b-\tilde{\mathscr{A}}^b)f\in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ for every $f\in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$. Then,

$$\tilde{P}_t f\,=\,P_t f+\int_0^t P_s(\mathscr{A}^b-\tilde{\mathscr{A}}^b)\tilde{P}_{t-s}f\,\mathrm{d}s\,,\qquad f\in\mathscr{D}_{\mathscr{A}^b}\cap\mathscr{D}_{\tilde{\mathscr{A}}^b}\,.$$

Similarly as before, the above relation holds for all $f\in B_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$. Thus, for any $\lambda>0$ and any τ-periodic $f\in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$,

$$\begin{aligned}\tilde{P}_t^\tau f_\tau\,&=\,P_t^\tau f_\tau+\int_0^t P_s^\tau(\mathscr{A}^b-\tilde{\mathscr{A}}^b)\tilde{P}_{t-s}^\tau f_\tau\,\mathrm{d}s\,,\\ \int_0^\infty e^{-\lambda t}\tilde{P}_t^\tau f_\tau\,\mathrm{d}t\,&=\,\int_0^\infty e^{-\lambda t}P_t^\tau f_\tau\,\mathrm{d}t+\int_0^\infty\int_0^t e^{-\lambda s}P_s^\tau(\mathscr{A}^b-\tilde{\mathscr{A}}^b)e^{-\lambda(t-s)}\tilde{P}_{t-s}^\tau f_\tau\,\mathrm{d}s\,\mathrm{d}t\,,\\ \tilde{R}_\lambda^\tau f_\tau\,&=\,R_\lambda^\tau f_\tau+\frac{1}{2}R_\lambda^\tau(\mathscr{A}^b-\tilde{\mathscr{A}}^b)\tilde{R}_\lambda^\tau f_\tau\,.\end{aligned}$$

Assume now that $\{\tilde{P}_t\}_{t\geq 0}$ satisfies **(PE)**(i) with k_1, α_1, and that $(\mathscr{A}^b - \tilde{\mathscr{A}}^b)f \in C_b^{k_1,\alpha_1}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ for every $f \in C_b^{k_1,\alpha_1}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$. Then, according to the proof of Theorem 3.2.2, for any τ-periodic $f \in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ with $\int_{\mathbb{T}_\tau^{\mathrm{d}}} f_\tau(x)\,\pi(\mathrm{d}x) = 0$, $\tilde{R}_\lambda^\tau f \in C^{k_1,\alpha_1}(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R})$ and so $(\mathscr{A}^b - \tilde{\mathscr{A}}^b)\tilde{R}_\lambda^\tau f_\tau \in C^{k_1,\alpha_1}(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R})$. Hence, for any τ-periodic $f \in C_b^{k_1,\alpha_1}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$, $\tilde{R}_\lambda^\tau f_\tau \in C^{k_2,\alpha_2}(\mathbb{T}_\tau^{\mathrm{d}},\mathbb{R})$, that is, the corresponding τ-periodic extension is a solution to (3.7). Finally, uniqueness follows from the fact that any solution $u(x)$ to (3.7) must have the representation $\int_0^\infty e^{-\lambda t}\tilde{P}_t f\,\mathrm{d}t$, since $u = (\lambda - \tilde{\mathscr{A}}^b)^{-1} f$.

Below we give concrete examples of LTPs $\{X_t\}_{t\geq 0}$ and $\{\tilde{X}_t\}_{t\geq 0}$ satisfying the above assumptions.

Let $\alpha \in (0,2)$ and $n : \mathbb{R}^d \setminus \{0\} \to [\underline{\Gamma}, \overline{\Gamma}]$, with $0 \leq \underline{\Gamma} \leq \overline{\Gamma} < \infty$, be measurable. Then

$$\nu_0(\mathrm{d}y) := \mathbb{1}_{B_1(0)}(y)\,\frac{n(y)}{|y|^{\mathrm{d}+\alpha}}\,\mathrm{d}y$$

is a Lévy measure, indeed, we have

$$\int_{\mathbb{R}^{\mathrm{d}}} \left(1 \wedge |y|^2\right) \frac{n(y)}{|y|^{\mathrm{d}+\alpha}}\,\mathrm{d}y = \int_{B_1(0)} |y|^2 \frac{n(y)}{|y|^{\mathrm{d}+\alpha}}\,\mathrm{d}y = \overline{\Gamma}\int_0^1 r^{\mathrm{d}-1+2-d-\alpha}\,\mathrm{d}r = \overline{\Gamma}\frac{1}{\alpha} < \infty.$$

Denote by $\{X_t\}_{t\geq 0}$ the Lévy process generated by the Lévy triplet $(0,0,\nu_0(\mathrm{d}y))$. Also, let $(\mathscr{A}^b, \mathscr{D}_{\mathscr{A}^b})$ be the corresponding B_b-generator. Then, according to [59, Example 4.2] $\{X_t\}_{t\geq 0}$ satisfies **(PE)**(ii) with $k_2 = 2, \alpha_2 = \alpha_1$.

Further, let $\{\tilde{X}_t\}_{t\geq 0}$ be a LTP generated by $(0,0,\nu(x,\mathrm{d}y))$ with

$$\nu(x,\mathrm{d}y) = \nu_0(\mathrm{d}y) + \frac{\gamma(x,y)}{|y|^{\mathrm{d}+\beta}}\mathbb{1}_{B_1^c(0)}(y)\,\mathrm{d}y,$$

where $\beta > 1$ and $x \mapsto \gamma(x,y)$ is positive, ε-Hölder continuous and bounded for almost every $y \in B_1^c(0)$ (see in Example 3.1.1 (iv) that this Lévy kernel generates a LTP). Denote by $(\tilde{\mathscr{A}}^b, \mathscr{D}_{\tilde{\mathscr{A}}^b})$ the corresponding B_b-generator. As in (ii) we see that $\mathscr{A}^b - \tilde{\mathscr{A}}^b$ is bounded on $(B_b(\mathbb{R}^{\mathrm{d}},\mathbb{R}), \|\cdot\|_\infty)$, and $(\mathscr{A}^b - \tilde{\mathscr{A}}^b)f \in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ for every $f \in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$. Furthermore, $(\mathscr{A}^b - \tilde{\mathscr{A}}^b)f \in C_b^\varepsilon(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ for every $f \in C_b^\varepsilon(\mathbb{R}^{\mathrm{d}},\mathbb{R})$. With these at hand, we can follow the argument in (ii) to check that **(PE)**(ii) is satisfied.

□

3.3. Generalization of Itô's Formula

As mentioned in a previous section we wish to use Itô's formula applied to a function β. In this section we slightly generalize [82, Lemma 4.2] (see also [31]), and prove Itô's formula for a pure-jump LTP with respect to a not necessarily twice continuously differentiable function. But first lets consider the case when function in question is twice continuously differentiable.

Lemma 3.3.1. *Let $f \in C_b^2(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ and b^* be given in (3.6). Then the following Itô's formula holds*

$$\begin{aligned} f(X_t) = & f(X_0) + \int_0^t \langle \nabla f(X_s), b^*(X_s)\rangle \,\mathrm{d}s + \int_0^t \left\langle \nabla f(X_s), \tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s \right\rangle \\ & + \int_0^t \int_{\mathbb{R}} \left(f(X_{s-} + k(X_{s-}, z)) - f(X_{s-})\right)\left(\tilde{\mu}(\cdot, \mathrm{d}z, \mathrm{d}s) - \tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right) \\ & + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t \partial_{ij} f(X_s) c_{ij}(X_s)\,\mathrm{d}s \qquad (3.11) \\ & + \int_0^t \int_{\mathbb{R}} \left(f(X_s + k(X_s, z)) - f(X_s) - \langle \nabla f(X_s), k(X_s, z)\rangle\right)\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s, \end{aligned}$$

for all $t \geq 0$.

Proof. Since $f \in C_b^2(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ we can use Itô's formula (2.6.12) for process $\{X_t\}_{t\geq 0}$ with representation given in (2.14). First we observe that $X_t^c = \int_0^t \tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s$ and

$$\begin{aligned} & \sum_{s\leq t}\left(f(X_s) - f(X_{s-}) - \sum_{i=1}^{\mathrm{d}} \frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i\right) = \\ & \int_0^t \int_{\mathbb{R}^{\mathrm{d}}} \left(f(X_{s-} + z) - f(X_{s-}) - \langle \nabla f(X_{s-}), z\rangle\right)\mu(\cdot, \mathrm{d}z, \mathrm{d}s), \quad \forall t \geq 0. \end{aligned}$$

Using (2.12) and the definition of b^* given in (3.6) we get

$$\begin{aligned} f(X_t) = & f(X_0) + \int_0^t \langle \nabla f(X_s), b(X_s)\rangle \,\mathrm{d}s + \int_0^t \left\langle \nabla f(X_s), \tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s \right\rangle \\ & + \int_0^t \int_{\mathbb{R}} \langle \nabla f(X_s), k(X_s, z)\rangle \mathbb{1}_{\{u:|k(X_s,u)|\geq 1\}}(z)\,\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s \\ & + \int_0^t \int_{\mathbb{R}} \langle \nabla f(X_{s-}), k(X_{s-}, z)\rangle \left(\tilde{\mu}(\cdot, \mathrm{d}z, \mathrm{d}s) - \tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right) \\ & + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t \partial_{ij} f(X_s) c_{ij}(X_s)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \int_{\mathbb{R}} \left(f(X_{s-}+k(X_{s-},z))-f(X_{s-})-\langle \nabla f(X_{s-}),k(X_{s-},z)\rangle\right)\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)\\
&=f(X_0)+\int_0^t \langle \nabla f(X_s),b(X_s)\rangle\,\mathrm{d}s+\int_0^t \left\langle \nabla f(X_s),\tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s\right\rangle\\
&+\int_0^t \left\langle \nabla f(X_s),\int_{\mathbb{R}} k(X_s,z)\mathbb{1}_{\{u:|k(X_s,u)|\geq 1\}}(z)\,\tilde{\nu}(\mathrm{d}z)\right\rangle \mathrm{d}s\\
&+\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}f(X_s)c_{ij}(X_s)\,\mathrm{d}s\\
&+\int_0^t \int_{\mathbb{R}} \left(f(X_{s-}+k(X_{s-},z))-f(X_{s-})\right)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right)\\
&+\int_0^t \int_{\mathbb{R}} \left(f(X_s+k(X_s,z))-f(X_s)-\langle \nabla f(X_s),k(X_s,z)\rangle\right)\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\\
&=f(X_0)+\int_0^t \langle \nabla f(X_s),b^*(X_s)\rangle\,\mathrm{d}s+\int_0^t \left\langle \nabla f(X_s),\tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s\right\rangle\\
&+\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}f(X_s)c_{ij}(X_s)\,\mathrm{d}s\\
&+\int_0^t \int_{\mathbb{R}} \left(f(X_{s-}+k(X_{s-},z))-f(X_{s-})\right)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right)\\
&+\int_0^t \int_{\mathbb{R}} \left(f(X_s+k(X_s,z))-f(X_s)-\langle \nabla f(X_s),k(X_s,z)\rangle\right)\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}
$$

■

Notice that if $\{X_t\}_{t\geq 0}$ is pure-jump, that is $c(x)\equiv 0$ in equation (3.13) only the first derivative of function appears. This leads one to suspect that in that case assumption of $f\in C_b^2(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ is not necessary. It is however not enough to assume $f\in C_b^1(\mathbb{R}^{\mathrm{d}},\mathbb{R})$. One possible sufficient assumption is given in the following proposition.

Proposition 3.3.2. *Assume that $\{X_t\}_{t\geq 0}$ is pure-jump, that is $c(x)\equiv 0$, and $f\in C_b^1(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ such that*

$$\sup_{x\in\mathbb{R}^{\mathrm{d}}}\int_{B_1(0)}|y|\,\nu(x,\mathrm{d}y)\;<\;\infty. \tag{3.12}$$

Then

$$
\begin{aligned}
f(X_t)\;&=f(X_0)+\int_0^t \langle \nabla f(X_s),b^*(X_s)\rangle\,\mathrm{d}s\\
&+\int_0^t \int_{\mathbb{R}} \left(f(X_{s-}+k(X_{s-},z))-f(X_{s-})\right)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right)\\
&+\int_0^t \int_{\mathbb{R}} \left(f(X_s+k(X_s,z))-f(X_s)-\langle \nabla f(X_s),k(X_s,z)\rangle\right)\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\,,
\end{aligned}
\tag{3.13}
$$

holds true for all $t\geq 0$ where b^* is given in (3.6).

Proof. Let $\chi \in C_c^\infty(\mathbb{R}^{\mathrm{d}})$, $0 \le \chi \le 1$, be such that $\int_{\mathbb{R}^{\mathrm{d}}} \chi(x)\,\mathrm{d}x = 1$. For $n \in \mathbb{N}$ define $\chi_n(x) := n^d \chi(nx)$, and $f_n(x) := (\chi_n * f)(x)$, where $*$ stands for the standard convolution operator. We have $\|f_n\|_1 \le \|f\|_1$ for $n \in \mathbb{N}$, to see this we first observe that $\|f_n\|_\infty \le \|f\|_\infty$ and

$$\left\|\partial^i f_n\right\|_\infty = \left\|\chi_n * \partial^i f\right\|_\infty = \left\|\int\limits_{\mathbb{R}^{\mathrm{d}}} \partial^i f(\cdot - y)\chi_n(y)\,\mathrm{d}y\right\|_\infty \le \int\limits_{\mathbb{R}^{\mathrm{d}}} \left\|\partial^i f\right\|_\infty \chi_n(y)\,\mathrm{d}y = \left\|\partial^i f\right\|_\infty .$$

Also for all $x \in \mathbb{R}^{\mathrm{d}}$ we have $\lim_{n\to\infty} f_n(x) = f(x)$ and $\lim_{n\to\infty} \nabla f_n(x) = \nabla f(x)$ because $\lim_{n\to\infty} \partial f_n(x) = \lim_{n\to\infty} (\chi_n * \partial f)(x) = \partial f(x)$. Since clearly $\{f_n\}_{n\in\mathbb{N}} \subset C_b^\infty(\mathbb{R}^{\mathrm{d}})$ we can employ the previous Lemma to conclude (3.13) for function f_n.

Now, by letting $n \to \infty$ we see that the left hand-side converges to $f(X_t)$ and the first terms on the right-hand side converges to $f(X_0)$. For the remaining terms on the right-hand side we will need to employ the dominated convergence theorem. In the case of the second term this is easy since ∇f_n is bounded and so is b^*. Therefore the second term converges to $\int_0^t \langle \nabla f(X_{s-}), b^*(X_{s-})\rangle\,\mathrm{d}s$.

For the third term by employing the isometry formula, we have

$$\widetilde{\mathbb{E}}_x\Big[\Big(\int_0^t \int_{\mathbb{R}} (f_n(X_{s-} + k(X_{s-},z)) - f_n(X_{s-}) - f(X_{s-} + k(X_{s-},z)) + f(X_{s-}))$$
$$\big(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s) - \tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\big)\Big)^2\Big] =$$
$$\widetilde{\mathbb{E}}_x\Big[\int_0^t \int_{\mathbb{R}} (f_n(X_s + k(X_s,z)) - f_n(X_s) - f(X_{s-} + k(X_s,z)) + f(X_s))^2\, \tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\Big].$$

For $g \in C^1(\mathbb{R}^{\mathrm{d}})$ we have

$$g(x+y) - g(x) = \int_0^1 \langle \nabla g(x+ry), y\rangle\,\mathrm{d}r. \tag{3.14}$$

If we apply this for $g = f_n - f$ and use Cauchy inequality we get

$$|f_n(x+y) - f_n(x) - f(x+y) + f(x)|^2 \le \left(\int_0^1 (|\nabla f_n(x+ry)| + |\nabla f(x+ry)|)\,|y|\,\mathrm{d}r\right)^2$$
$$\le 4\,\|f\|_1^2\,|y|^2 .$$

Condition (**C3**) then implies that

$$\sup_{x\in\mathbb{R}^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} |f_n(x+y) - f_n(x) - f(x+y) + f(x)|^2\, \nu(x,\mathrm{d}y) \le 4\,\|f\|_1^2 \int_{\mathbb{R}^{\mathrm{d}}} |y|^2 \nu(x,\mathrm{d}y) < \infty$$

The dominated convergence theorem therefore implies

$$\int_0^t \int_{\mathbb{R}} (f_n(X_{s-}+k(X_{s-},z)) - f_n(X_{s-})) \left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s) - \tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right) \xrightarrow{L^2}$$
$$\int_0^t \int_{\mathbb{R}} (f(X_{s-}+k(X_{s-},z)) - f(X_{s-})) \left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s) - \tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right).$$

From this we conclude that there is a subsequence such that the third term on the left-hand side in (3.13) converges (in this subsequence) to

$$\int_0^t \int_{\mathbb{R}^{\mathrm{d}}} (f(X_{s-}+k(X_{s-},z)) - f(X_{s-})) \left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s) - \tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right).$$

It remains to consider convergence of the last term in (3.13) and to that end we will use (3.14) with $g = f_n$. Cauchy inequality then implies

$$|f_n(x+y) - f_n(x) - \langle \nabla f_n(x), y\rangle| = \int_0^1 \langle \nabla f_n(x+ry) - \nabla f_n(x), y\rangle \,\mathrm{d}r \leq$$
$$\int_0^1 |\nabla f_n(x+ry) - \nabla f_n(x)||y|\,\mathrm{d}r \leq 2\,\|f\|_1\,|y|$$

Therefore using condition (3.12) we conclude

$$\sup_{x\in\mathbb{R}^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} |f_n(x+y) - f_n(x) - \langle \nabla f_n(x), y\rangle|\,\nu(x,\mathrm{d}y) \leq 2\,\|f\|_1 \int_{\mathbb{R}^{\mathrm{d}}} |y|\nu(x,\mathrm{d}y) < \infty$$

Thus, the dominated convergence theorem implies that the last term converges to

$$\int_0^t \int_{\mathbb{R}^d} \Big(f(X_s+k(X_s,z)) - f(X_s) - \langle \nabla f(X_s), k(X_s,z)\rangle\Big)\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\,.$$

which proves the desired result. ■

Alternate sufficient condition is given in the next proposition.

Proposition 3.3.3. *Assume that $\{X_t\}_{t\geq 0}$ is pure-jump, that is $c(x)\equiv 0$, and $f \in C_b^{\phi}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ with Hölder exponent $\phi(r)$ such that $m_\phi > 1$ and*

$$\sup_{x\in\mathbb{R}^d} \int_{B_1(0)} \phi(|y|)\,\nu(x,\mathrm{d}y) < \infty. \tag{3.15}$$

Then (3.13) holds true for all $t \geq 0$, where b^* is given in (3.6).

Proof. Without loss of generality, assume that $m_\phi \in (1,2]$, in this case for $f \in C_b^{\phi}(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ we have

$$\|f\|_\phi = \|f\|_\infty + \sum_{i=1}^{\mathrm{d}} \left\|\partial^i f\right\|_\infty + \sum_{i=1}^{\mathrm{d}} [\partial^i f]_{-1,\phi}$$

We proceed the same as in the proof of the previous proposition with few miner alternations. This time we have $\|f_n\|_\phi \le \|f\|_\phi$ for $n \in \mathbb{N}$. To see this, atop of the observations already made, notice that

$$
\begin{aligned}
[\partial^i f_n]_{-1,\phi} &= \sup_{x\in\mathbb{R}^{\mathrm{d}}} \sup_{h\in\bar{B}_1(0)\setminus\{0\}} \frac{|\chi_n * \partial^i f(x+h) - \chi_n * \partial^i f(x)|}{\phi(|h|)|h|^{-1}} \\
&\le \sup_{x\in\mathbb{R}^{\mathrm{d}}} \sup_{h\in\bar{B}_1(0)\setminus\{0\}} \int_{\mathbb{R}^{\mathrm{d}}} \frac{|\partial^i f(x+h-y) - \partial^i f(x-y)|}{\phi(|h|)|h|^{-1}} \chi_n(y)\,\mathrm{d}y \\
&\le \int_{\mathbb{R}^{\mathrm{d}}} \sup_{x\in\mathbb{R}^{\mathrm{d}}} \sup_{h\in\bar{B}_1(0)\setminus\{0\}} \frac{|\partial^i f(x+h-y) - \partial^i f(x-y)|}{\phi(|h|)|h|^{-1}} \chi_n(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^{\mathrm{d}}} \sup_{x\in\mathbb{R}^{\mathrm{d}}} \sup_{h\in\bar{B}_1(0)\setminus\{0\}} \frac{|\partial^i f(x+h) - \partial^i f(x)|}{\phi(|h|)|h|^{-1}} \chi_n(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^{\mathrm{d}}} [\partial^i f]_{-1,\phi} \chi_n(y)\,\mathrm{d}s = [\partial^i f]_{-1,\phi}\,.
\end{aligned}
$$

Another alternation needs to be make in the arguments made for estimates of the last term on the right-hand side of (3.13). We again start with the following inequality

$$
\begin{aligned}
&|f_n(x+y) - f_n(x) - \langle \nabla f_n(x), y\rangle| \le \int_0^1 |\nabla f_n(x+ry) - \nabla f_n(x)||y|\,\mathrm{d}r = \\
&\int_0^1 |\nabla f_n(x+ry) - \nabla f_n(x)||y\mathbb{1}_{B_1(0)}(y)|\,\mathrm{d}r + \int_0^1 |\nabla f_n(x+ry) - \nabla f_n(x)||y\mathbb{1}_{B_1^c(0)}(y)|\,\mathrm{d}r\,.
\end{aligned}
$$

To see that the second integral is finite we can, similarly to before, use the fact that $|\nabla f_n(x+ry) - \nabla f_n(x)| \le 2\|f_n\|_1 \le 2\|f\|_\phi$ because $|y\mathbb{1}_{B_1^c(0)}(y)| \le y^2\mathbb{1}_{B_1^c(0)}(y)$ and therefore we will be able to use (**C3**). For the first integral we observe that for $y \in B_1(0)\setminus\{0\}$ and $r \in (0,1]$ and some constant $\kappa \in (0,1]$ we have

$$
\begin{aligned}
&\|f\|_\phi \ge \|f_n\|_\phi \ge \sum_{i=1}^{\mathrm{d}} \sup_{x\in\mathbb{R}^{\mathrm{d}}} \sup_{h\in\bar{B}_1(0)\setminus\{0\}} \frac{|\partial^i f_n(x+ry) - \partial^i f_n(x)|}{\phi(|h|)|h|^{-1}} \ge \\
&\sum_{i=1}^{\mathrm{d}} \frac{|\partial^i f_n(x+h) - \partial^i f_n(x)|}{\phi(|yr|)|yr|^{-1}} \ge \frac{\kappa|\nabla f_n(x+ry) - \nabla f_n(x)|}{\phi(|y|)|y|^{-1}}\,,
\end{aligned}
$$

where last inequality holds because function $y \mapsto \phi(|y|)|y|^{-1}$ is almost increasing in $(0,1]$. Therefore using condition (3.15) and (**C3**) we conclude

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} |f_n(x+y) - f_n(x) - \langle \nabla f_n(x), y\rangle|\,\nu(x,\mathrm{d}y) \le \\
&\sup_{x\in\mathbb{R}^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} \|f\|_\phi \frac{\phi(|y|)}{\kappa} \mathbb{1}_{B_1(0)}(y)\,\nu(x,\mathrm{d}y) + \sup_{x\in\mathbb{R}^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} 2|y|^2 \mathbb{1}_{B_1^c(0)}(y)\,\nu(x,\mathrm{d}y) < \infty\,.
\end{aligned}
$$

Thus, the dominated convergence theorem implies that the last term converges to

$$\int_0^t \int_{\mathbb{R}^d} \Big(f(X_s + k(X_s,z)) - f(X_s) - \langle \nabla f(X_s), k(X_s,z) \rangle \Big) \tilde{\nu}(\mathrm{d}z)\, \mathrm{d}s\,.$$

which proves the desired result. ■

This leads us to assume the following condition for the rest of this chapter

(GI) One of the following holds

(1) $\beta \in C_b^2(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$ if $c(x) \not\equiv 0$;

(2) $\beta \in C_b^1(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$ if $c(x) \equiv 0$ and

$$\sup_{x \in \mathbb{R}^{\mathrm{d}}} \int_{B_1(0)} |y|\, \nu(x, \mathrm{d}y) < \infty\,. \tag{3.16}$$

(3) $m_{\varphi\psi} > 1$ if $c(x) \equiv 0$ and

$$\sup_{x \in \mathbb{R}^{\mathrm{d}}} \int_{B_1(0)} \varphi(|y|)\psi(|y|)\, \nu(x, \mathrm{d}y) < \infty; \tag{3.17}$$

Remark 3.3.4. If the conditions of Remark 3.2.1 are satisfied instead of **(PE)** then in Theorem 3.2.2 we conclude that $\beta \in C_b^{k_2,\alpha_2}(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}})$. If this is so then condition **(GI)** (iii) becomes $k_2 > 0$ if $c(x) \equiv 0$ and

$$\sup_{x \in \mathbb{R}^{\mathrm{d}}} \int_{B_1(0)} |y|^{k_2+\alpha_2} \nu(x, \mathrm{d}y) < \infty\,.$$

□

3.4. Central Limit Theorem

We are now in position to state the main result of this chapter. In the proof we follow the approach from [35].

Theorem 3.4.1. *Let $\{X_t\}_{t\geq 0}$ be a* d*-dimensional LTP with semigroup $\{P_t\}_{t\geq 0}$ and Lévy triplet $(b(x), c(x), \nu(x,\mathrm{d}y))$, satisfying **(SJ)**, **(P)**, **(FI)**, **(PE)** and **(GI)**. Then for any initial distribution of $\{X_t\}_{t\geq 0}$,*

$$\left\{\varepsilon X_{\varepsilon^{-2}t} - \varepsilon^{-1}\pi(b^*)t\right\}_{t\geq 0} \xRightarrow{\varepsilon\to 0} \{W_t^{\Sigma}\}_{t\geq 0}\,. \tag{3.18}$$

Here, b^* is given in (3.6), $\pi(\mathrm{d}x)$ is a measure whose existence and uniqueness is given by Proposition 3.1.4 and $\Rightarrow$ denotes the convergence in the space of càdlàg functions endowed with the Skorohod J_1*-topology, and $\{W_t^{\Sigma}\}_{t\geq 0}$ is a* d*-dimensional zero-drift Brownian motion determined by covariance matrix Σ given by*

$$\begin{aligned}\Sigma = \pi\Bigg(&(\mathbb{I}_{\mathrm{d}} - \mathrm{D}\beta)\,c\,(\mathbb{I}_{\mathrm{d}} - \mathrm{D}\beta)^{\mathrm{T}} + \\ &\int_{\mathbb{R}^{\mathrm{d}}} (y - \beta(\cdot + y) + \beta(\cdot))\,(y - \beta(\cdot + y) + \beta(\cdot))^{\mathrm{T}}\,\nu(\cdot,\mathrm{d}y)\Bigg),\end{aligned} \tag{3.19}$$

where β is a solution to (3.5) whose existence and uniqueness is given by Theorem 3.2.2.

Proof. Let $\beta \in C_b^{\varphi\psi}(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$ be a τ-periodic solution to (3.5). Because of condition **(GI)** according to Lema 3.3.1 (if **(GI)** (1) holds) or Proposition 3.3.2 (if **(GI)** (2) holds) or Proposition 3.3.3 (if **(GI)** (3) holds) we can apply Itô's formula to the process $\{\beta(X_t)\}_{t\geq 0}$ and therefore

$$\begin{aligned}\beta(X_t) - \beta(X_0) &= \int_0^t \langle\nabla\beta(X_s), b^*(X_s)\rangle\,\mathrm{d}s + \int_0^t \left\langle\nabla\beta(X_s), \tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s\right\rangle \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t c_{ij}(X_s)\partial_{ij}\beta(X_s)\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}} (\beta(X_{s-} + k(X_{s-},z)) - \beta(X_{s-}))\,(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s) - \tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s) \\ &\quad + \int_0^t\int_{\mathbb{R}} (\beta(X_s + k(X_s,z)) - \beta(X_s) - \langle\nabla\beta(X_s), k(X_s,z)\rangle)\,\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\,,\end{aligned} \tag{3.20}$$

where we used the fact that if **(GI)** (2) or (3) is satisfied then $\sigma \equiv 0$ and $c_{ij} \equiv 0$ for all $i,j = 1,\ldots,\mathrm{d}$. From (3.5) we know that $b^*(X_t) - \pi(b^*) = \mathscr{A}^b\beta(X_t)$ for any $t \geq 0$ and therefore using (2.13) we get

$$\begin{aligned}
b^*(X_s)-\pi(b^*) &= \langle \nabla\beta(X_s), b(X_s)\rangle + \frac{1}{2}\sum_{i,j=1}^{d} c_{ij}(X_s)\partial_{ij}\beta(X_s) \\
&\quad + \int_{\mathbb{R}^d} \Big(\beta(X_s+z)-\beta(X_s)-\langle\nabla\beta(X_s),z\rangle\,\mathbb{1}_{B_1(0)}(z)\Big)\,\nu(X_s,\mathrm{d}z) \\
&= \langle \nabla\beta(X_s), b(X_s)\rangle + \frac{1}{2}\sum_{i,j=1}^{d} c_{ij}(X_s)\partial_{ij}\beta(X_s) \\
&\quad + \int_{\mathbb{R}} \left(\beta(X_s+k(X_s,z))-\beta(X_s)-\langle\nabla\beta(X_s),k(X_s,z)\rangle\right)\,\tilde{\nu}(\mathrm{d}z) \\
&\quad + \left\langle \nabla\beta(X_s), \int_{\mathbb{R}^d} z\mathbb{1}_{B_1^c(0)}(z)\,\nu(X_s,\mathrm{d}z)\right\rangle .
\end{aligned}$$

Combining this with equation (3.20) and the fact that $b^*(x) := b(x)+\int_{B_1^c(0)} z\,\nu(x,\mathrm{d}z)$ we conclude

$$\begin{aligned}
\beta(X_t)-\beta(X_0) &= \int_0^t b^*(X_s)-\pi(b^*)\,\mathrm{d}s + \int_0^t \left\langle \nabla\beta(X_s), \tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s\right\rangle \\
&\quad + \int_0^t\int_{\mathbb{R}} \left(\beta(X_{s-}+k(X_{s-},z))-\beta(X_{s-})\right)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right) .
\end{aligned} \tag{3.21}$$

Observe that due to boundedness of $\beta(x)$, $\{\varepsilon X_{\varepsilon^{-2}t}-\pi(b^*)\varepsilon^{-1}t-\varepsilon\beta(X_{\varepsilon^{-2}t})+\varepsilon\beta(X_0)\}_{t\geq 0}$ converges in the Skorohod space as $\varepsilon\to 0$ if, and only if, $\{\varepsilon X_{\varepsilon^{-2}t}-\pi(b^*)\varepsilon^{-1}t\}_{t\geq 0}$ converges, and if this is the case the limit is the same. We will use Theorem 2.6.15 to show this convergence. To this end denote by $S_t := X_t-\pi(b^*)t-\beta(X_t)+\beta(X_0)$ and let's prove that the process $\{S_t\}_{t\geq 0}$ is a semimartingale and determine its characteristics. By combining (2.14) and (3.21) we have that

$$\begin{aligned}
S_t &= x+\int_0^t b(X_s)-\pi(b^*)\mathrm{d}s+\int_0^t \tilde{\sigma}(X_s)\mathrm{d}\tilde{W}_s+\int_0^t\int_{\mathbb{R}} k(X_s,z)\mathbb{1}_{\{u:|k(X_s,u)|\geq 1\}}(z)\,\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s \\
&\quad + \int_0^t\int_{\mathbb{R}} k(X_{s-},z)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right) \\
&\quad - \int_0^t b^*(X_t)-\pi(b^*)\,\mathrm{d}s-\int_0^t \left\langle\nabla\beta(X_s),\tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s\right\rangle \\
&\quad - \int_0^t\int_{\mathbb{R}} \left(\beta(X_{s-}+k(X_{s-},z))-\beta(X_{s-})\right)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right) \\
&= x+\int_0^t \tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s-\int_0^t \left\langle\nabla\beta(X_s),\tilde{\sigma}(X_s)\,\mathrm{d}\tilde{W}_s\right\rangle \\
&\quad + \int_0^t\int_{\mathbb{R}} \left(k(X_{s-},z)-\beta(X_{s-}+k(X_{s-},z))+\beta(X_{s-})\right)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right) ,
\end{aligned}$$

where we used the fact that $\int_{\mathbb{R}} k(X_s,z)\mathbb{1}_{\{u:|k(X_s,u)|\geq 1\}}(z)\,\tilde{\nu}(\mathrm{d}z) = \int_{B_1^c(0)} z\,\nu(x,\mathrm{d}z)$.

We proceed as in Example 2.6.11 and Example 2.6.14. First notice that in a semimartingale decomposition of $\{S_t\}_{t\geq 0}$ the adapted process of finite variation $\{B_t\}_{t\geq 0}$ is a nul-process which is trivially predictable. Therefore $\{S_t\}_{t\geq 0}$ is a special semimartingale and again for the truncation function we can take $h(x)=x$. The continuous martingale part of $\{S_t\}_{t\geq 0}$ is a d-dimensional process such that for $i=1,\dots,\mathrm{d}$

$$\begin{aligned} S_t^{i,c} &= \sum_{l=1}^{\mathrm{d}}\int_0^t \tilde{\sigma}_{il}(X_s)\,\mathrm{d}\tilde{W}_s^l - \sum_{k,l=1}^{\mathrm{d}}\int_0^t \partial_k\beta_i(X_s)\tilde{\sigma}_{kl}(X_s)\,\mathrm{d}\tilde{W}_s^l \\ &= \sum_{l=1}^{\mathrm{d}}\int_0^t \sum_{k=1}^{\mathrm{d}} \left(\delta_{ki}-\partial_k\beta_i(X_s)\right)\tilde{\sigma}_{kl}(X_s)\,\mathrm{d}\tilde{W}_s^l\,. \end{aligned}$$

Since from equation (2.12) and (2.13) we know that

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}} \left(k(X_{s-},z)-\beta(X_{s-}+k(X_{s-},z))+\beta(X_{s-})\right)\left(\tilde{\mu}(\cdot,\mathrm{d}z,\mathrm{d}s)-\tilde{\nu}(\mathrm{d}z)\,\mathrm{d}s\right) = \\ &\int_0^t\int_{\mathbb{R}} \left(y-\beta(X_{s-}+y)+\beta(X_{s-})\right)\left(\mu(\cdot,\mathrm{d}y,\mathrm{d}s)-\nu(X_s,\mathrm{d}y)\,\mathrm{d}s\right)\,, \end{aligned}$$

the jump measure $\mu^S(\cdot,\mathrm{d}y,\mathrm{d}s)$ of process $\{S_t\}_{t\geq 0}$ is

$$\mu^S(\cdot,B,\mathrm{d}s) = \int_{\mathbb{R}^{\mathrm{d}}} \mathbb{1}_B\left(y-(\beta(X_s+y)-\beta(X_s))\right)\mu(\cdot,\mathrm{d}y,\mathrm{d}s)\,,$$

for $B\in\mathscr{B}(\mathbb{R}^{\mathrm{d}})$. We conclude that (modified) characteristic of $\{S_t\}_{t\geq 0}$ are as follows

$$\begin{aligned} B_t^i &\equiv 0\,, \\ C_t^{ij} &= \int_0^t \sum_{k,l=1}^{\mathrm{d}} \left(\delta_{ki}-\partial_k\beta_i(X_s)\right)c_{kl}(X_s)\left(\delta_{lj}-\partial_l\beta_j(X_s)\right)\mathrm{d}s\,, \\ N(B,\mathrm{d}s) &= \int_{\mathbb{R}^{\mathrm{d}}} \mathbb{1}_B\left(y-(\beta(X_s+y)-\beta(X_s))\right)\nu(X_s,\mathrm{d}y)\,\mathrm{d}s\,, \\ \widetilde{C}_t^{ij} &= C_t^{ij} + \int_0^t\int_{\mathbb{R}^{\mathrm{d}}} y_iy_j\,N(\mathrm{d}y,\mathrm{d}s) \\ &= \int_0^t \sum_{k,l=1}^{\mathrm{d}} \left(\delta_{ki}-\partial_k\beta_i(X_s)\right)c_{kl}(X_s)\left(\delta_{lj}-\partial_l\beta_j(X_s)\right)\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}^{\mathrm{d}}} \left(y_i-\beta_i(X_s+y)+\beta_i(X_s)\right)\left(y_j-\beta_j(X_s+y)+\beta_j(X_s)\right)\nu(X_s,\mathrm{d}y)\,\mathrm{d}s\,, \end{aligned}$$

for $t\geq 0$ and $i,j=1,\dots,\mathrm{d}$ and $B\in\mathscr{B}(\mathbb{R}^{\mathrm{d}})$. Consequently, for any $x\in\mathbb{R}^{\mathrm{d}}$,

$$\{\varepsilon X_{\varepsilon^{-2}t}-\pi(b^*)\varepsilon^{-1}t-\varepsilon\beta(X_{\varepsilon^{-2}t})+\varepsilon\beta(X_0)\}_{t\geq 0}\,,\qquad \varepsilon>0\,,$$

is a $\mathbb{P}_x$- semimartingale whose (modified) characteristics (relative to $h(x)=x$) are given by

$$
\begin{aligned}
B_t^{\varepsilon,i} &= 0\,,\\
C_t^{\varepsilon,ij} &= \varepsilon^2\int_0^{\varepsilon^{-2}t}\sum_{k,l=1}^{\mathrm{d}}\left(\delta_{ki}-\partial_k\beta_i(X_s)\right)c_{kl}(X_s)\left(\delta_{lj}-\partial_l\beta_j(X_s)\right)\mathrm{d}s\,,\\
N^{\varepsilon}(B,\mathrm{d}s) &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^{\mathrm{d}}}\mathbb{1}_B\left(\varepsilon y-\varepsilon\big(\beta(X_{\varepsilon^{-2}s}+y)-\beta(X_{\varepsilon^{-2}s})\big)\right)\nu(X_{\varepsilon^{-2}s},\mathrm{d}y)\,\mathrm{d}s\,,\\
\widetilde{C}_t^{\varepsilon,ij} &= C_t^{\varepsilon,ij}+\varepsilon^2\int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^{\mathrm{d}}}\prod_{k=i,j}\left(y_k-\beta_k(X_s+y)+\beta_k(X_s)\right)\nu(X_s,\mathrm{d}y)\,\mathrm{d}s\,,
\end{aligned}
$$

for $t\geq 0$ and $i,j=1,\dots,\mathrm{d}$ and $B\in\mathscr{B}(\mathbb{R}^{\mathrm{d}})$.

Now we are ready to check the conditions in Theorem 2.6.15. First note that condition (2.21) is trivially satisfied. To prove the convergence in (2.22), first observe that due to τ-periodicity of all components we can replace $\{X_t\}_{t\geq 0}$ by $\{X_t^{\tau}\}_{t\geq 0}$, which is, due to Proposition 3.1.4, an ergodic Markov process. The assertion now follows as a direct consequence of Theorem 2.2.19 (Birkhoff ergodic theorem).

It remains to prove the relation in (2.23). To this end, fix $g\in C_b(\mathbb{R}^d)$ that vanishes on $B_\delta(0)$ for some $\delta>0$. We wish to prove that

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d} g(y)N^{\varepsilon}(\mathrm{d}y,\mathrm{d}s) =\\
&\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{R}^d} g\left(\varepsilon y-\varepsilon\left(\beta(X_{\varepsilon^{-2}s}+y)-\beta(X_{\varepsilon^{-2}s})\right)\right)\nu(X_{\varepsilon^{-2}s},\mathrm{d}y)\,\mathrm{d}s \xrightarrow[\varepsilon\to 0]{\mathbb{P}_x} 0\,.
\end{aligned}
$$

We will do this by proving the convergnece in L^2, to this end define

$$
\begin{aligned}
F^{\varepsilon}(x) :=& \frac{1}{\varepsilon^2}\int_{\mathbb{R}^{\mathrm{d}}} g\left(\varepsilon y-\varepsilon\big(\beta(x+y)-\beta(x)\big)\right)\nu(x,\mathrm{d}y)\\
&-\frac{1}{\varepsilon^2}\int_{\mathbb{T}_\tau^d}\int_{\mathbb{R}^{\mathrm{d}}} g\left(\varepsilon y-\varepsilon\big(\beta(z+y)-\beta(z)\big)\right)\nu(z,\mathrm{d}y)\,\pi(\mathrm{d}z)\,.
\end{aligned}
$$

Clearly, for any $\varepsilon>0$ F^{ε} satisfies $F^{\varepsilon}(X_t)=F^{\varepsilon}(X_t^{\tau})$ for $t\geq 0$, and

$$
\int_{\mathbb{T}_\tau^d} F^{\varepsilon}(x)\,\pi(\mathrm{d}x) = 0\,.
$$

It is also τ-periodic and bounded, we have

$$
\|F^{\varepsilon}\|_\infty \leq \frac{2\,\|g\|_\infty}{\varepsilon^2}\sup_{x_\tau\in\mathbb{T}_\tau^{\mathrm{d}}}\left\|\int_{\mathbb{R}^d}\mathbb{1}_{B_\delta^c(0)}\left(\varepsilon y-\varepsilon\left(\beta(x+y)-\beta(x)\right)\right)\nu(x,\mathrm{d}y)\right\|\,.
$$

Now, by the Markov property and geometric ergodicity of $\{X_t^\tau\}_{t\geq 0}$, we have

$$\mathbb{E}_x\left[\left(\int_0^t F^\varepsilon(X_{\varepsilon^{-2}s})\,\mathrm{d}s\right)^2\right] = \mathbb{E}_{x_\tau}^\tau\left[\left(\int_0^t F^\varepsilon\left(X_{\varepsilon^{-2}s}^\tau\right)\,\mathrm{d}s\right)^2\right] =$$

$$\mathbb{E}_{x_\tau}^\tau\left[\int_0^t F^\varepsilon\left(X_{\varepsilon^{-2}s}^\tau\right)\,\mathrm{d}s\int_0^t F^\varepsilon\left(X_{\varepsilon^{-2}u}^\tau\right)\,\mathrm{d}u\right] = 2\int_0^t\int_0^s \mathbb{E}_{x_\tau}^\tau\left[F^\varepsilon\left(X_{\varepsilon^{-2}s}^\tau\right)F^\varepsilon\left(X_{\varepsilon^{-2}u}^\tau\right)\right]\mathrm{d}u\,\mathrm{d}s =$$

$$2\int_0^t\int_0^s \mathbb{E}_{x_\tau}^\tau\left[F^\varepsilon\left(X_{\varepsilon^{-2}u}^\tau\right)\mathbb{E}_{x_\tau}^\tau\left[F^\varepsilon\left(X_{\varepsilon^{-2}s}^\tau\right)\mid X_{\varepsilon^{-2}u}^\tau\right]\right]\mathrm{d}u\,\mathrm{d}s =$$

$$2\int_0^t\int_0^s \mathbb{E}_{x_\tau}^\tau\left[F^\varepsilon\left(X_{\varepsilon^{-2}u}^\tau\right)P_{\varepsilon^{-2}(s-u)}^\tau F^\varepsilon\left(X_{\varepsilon^{-2}u}^\tau\right)\right]\mathrm{d}u\,\mathrm{d}s \leq 2\int_0^t\int_0^s \|F^\varepsilon\|_\infty\left\|P_{\varepsilon^{-2}(s-u)}^\tau F^\varepsilon\right\|_\infty \mathrm{d}u\,\mathrm{d}s \leq$$

$$2\Gamma\|F^\varepsilon\|_\infty^2\int_0^t\int_0^s e^{-\gamma\varepsilon^{-2}(s-u)}\,\mathrm{d}u\,\mathrm{d}s = 2\Gamma\|F^\varepsilon\|_\infty^2\,\frac{\gamma\varepsilon^{-2}t+e^{-\gamma\varepsilon^{-2}t}-1}{\gamma^2\varepsilon^{-4}} \leq \frac{2\Gamma\varepsilon^2 t}{\gamma}\|F^\varepsilon\|_\infty^2 \leq$$

$$\frac{8\Gamma\|g\|_\infty^2 t}{\varepsilon^2\gamma}\sup_{x_\tau\in\mathbb{T}_\tau^{\mathrm{d}}}\left(\int_{\mathbb{R}^d}\mathbb{1}_{B_\delta^c}\left(\varepsilon y-\varepsilon\left(\beta(x+y)-\beta(x)\right)\right)\nu(x,\mathrm{d}y)\right)^2.$$

Since β is bounded we can take $\varepsilon>0$ be such that $2\varepsilon\|\beta\|_\infty<\delta/2$. Then,

$$\delta < \|\varepsilon y-\varepsilon(\beta(x+y)-\beta(x))\| \leq \|\varepsilon y\|+2\varepsilon\|\beta\|_\infty \leq \|\varepsilon y\|+\frac{\delta}{2}$$

implies that $\mathbb{1}_{B_\delta^c}\left(\varepsilon y-\varepsilon\left(\beta(x+y)-\beta(x)\right)\right) \leq \mathbb{1}_{B_{\delta/2}^c}(\varepsilon y)$ and therefore

$$\mathbb{E}_x\left[\left(\int_0^t F^\varepsilon(X_{\varepsilon^{-2}s})\,\mathrm{d}s\right)^2\right] \leq \frac{8\Gamma\|g\|_\infty^2 t}{\varepsilon^2\gamma}\sup_{x_\tau\in\mathbb{T}_\tau^{\mathrm{d}}}\left(\int_{\mathbb{R}^{\mathrm{d}}}\mathbb{1}_{B_{\delta/2}^c}(\varepsilon y)\,\nu(x,\mathrm{d}y)\right)^2 =$$

$$\frac{8\Gamma\|g\|_\infty^2 t}{\varepsilon^2\gamma}\left(\frac{2\varepsilon}{\delta}\right)^4\sup_{x_\tau\in\mathbb{T}_\tau^{\mathrm{d}}}\left(\int_{B_{\delta/2\varepsilon}^c}\left(\frac{\delta}{2\varepsilon}\right)^2\nu(x,\mathrm{d}y)\right)^2 \leq$$

$$\left(\frac{8\sqrt{2}\Gamma^{1/2}\|g\|_\infty\varepsilon t^{1/2}}{\gamma^{1/2}\delta^2}\sup_{x_\tau\in\mathbb{T}_\tau^{\mathrm{d}}}\int_{B_{\delta/2\varepsilon}^c}|y|^2\,\nu(x,\mathrm{d}y)\right)^2.$$

Similarly we conclude

$$\mathbb{E}_x\left[\left(\int_0^t\frac{1}{\varepsilon^2}\int_{\mathbb{T}_\tau^{\mathrm{d}}}\int_{\mathbb{R}^{\mathrm{d}}}g\left(\varepsilon y-\varepsilon\left(\beta(z+y)-\beta(z)\right)\right)\,\nu(z,\mathrm{d}y)\,\pi(\mathrm{d}z)\,\mathrm{d}s\right)^2\right] =$$

$$\frac{t^2}{\varepsilon^4}\left(\int_{\mathbb{T}_\tau^{\mathrm{d}}}\int_{\mathbb{R}^{\mathrm{d}}}g\left(\varepsilon y-\varepsilon\left(\beta(z+y)-\beta(z)\right)\right)\,\nu(z,\mathrm{d}y)\,\pi(\mathrm{d}z)\right)^2 \leq$$

$$\frac{t^2}{\varepsilon^4}\|g\|_\infty^2\left(\frac{2\varepsilon}{\delta}\right)^4\left(\int_{\mathbb{T}_\tau^{\mathrm{d}}}\int_{B_{\delta/2\varepsilon}^c}\left(\frac{\delta}{2\varepsilon}\right)^2\nu(z,\mathrm{d}y)\,\pi(\mathrm{d}z)\right)^2 \leq \left(\frac{4\|g\|_\infty t}{\delta^2}\sup_{x_\tau\in\mathbb{T}_\tau^{\mathrm{d}}}\int_{B_{\delta/2\varepsilon}^c}|y|^2\,\nu(x,\mathrm{d}y)\right)^2.$$

Consequently,

$$\left(\mathbb{E}_x\left[\left(\int_0^t\int_{\mathbb{R}^d} g(y)\,N^{\varepsilon}(\mathrm{d}y,\mathrm{d}s)\right)^2\right]\right)^{\frac{1}{2}} \leq \left(\mathbb{E}_x\left[\left(\int_0^t F^{\varepsilon}(X_{\varepsilon^{-2}s})\,\mathrm{d}s\right)^2\right]\right)^{\frac{1}{2}} +$$
$$\left(\mathbb{E}_x\left[\left(\frac{t}{\varepsilon^2}\int_{\mathbb{T}^d_\tau}\int_{\mathbb{R}^d} g\left(\varepsilon y-\varepsilon\left(\beta(z+y)-\beta(z)\right)\right)\nu(z,\mathrm{d}y)\,\pi(\mathrm{d}z)\right)^2\right]\right)^{\frac{1}{2}} \leq$$
$$\left(\frac{8\sqrt{2}\Gamma^{1/2}\,\|g\|_\infty\,\varepsilon t^{1/2}}{\gamma^{1/2}\delta^2}+\frac{4\,\|g\|_\infty\,t}{\delta^2}\right)\sup_{x_\tau\in\mathbb{T}^d_\tau}\int_{B^c_{\delta/2\varepsilon}}|y|^2\,\nu(x,\mathrm{d}y)\,,$$

and since from (**SJ**) we know that $\lim\limits_{\varepsilon\to 0}\sup\limits_{x_\tau\in\mathbb{T}^d_\tau}\int_{B_{\delta/2\varepsilon}}|y|^2\,\nu(x,\mathrm{d}y)=\sup\limits_{x_\tau\in\mathbb{T}^d_\tau}\int_{\mathbb{R}^d}|y|^2\,\nu(x,\mathrm{d}y)<\infty$ this concludes the proof.

■

4. CLT for Degenerate Diffusions

In this chapter we prove a Central limit theorem for diffusion processes associated to the second-order differential operator given in (2.29) satisfying condition (**D**) (see Definition 2.7.1). As mentioned in the Section 2.7 we will use notation $\{X(x,t)\}_{t\geq 0}$ to denote such diffusion process with starting point $x\in\mathbb{R}^{\mathrm{d}}$, that is $\mathbb{P}(X(x,0)=x)=1$.

4.1. Diffusions with "Degenerate" Diffusion Term

In this chapter we assume that

(DD) (i) there is an open connected set $\mathscr{O}\subset[0,\tau]$ such that the matrix $c(x)$ is positive definite on $\overline{\mathscr{O}}$, that is,

$$\langle c(x)\xi,\xi\rangle > 0 \qquad \forall (x,\xi)\in\overline{\mathscr{O}}\times\mathbb{R}^{\mathrm{d}}\setminus\{0\};$$

(ii) a, b and c are γ-Hölder continuous for some $0<\gamma\leq 1$, that is, there is $\Gamma>0$ such that for all $x,y\in\mathbb{R}^{\mathrm{d}}$

$$|a(x)-a(y)|+|b(x)-b(y)|+\|c(x)-c(y)\|_{\mathrm{HS}}\leq\Gamma|x-y|^{\gamma}. \tag{4.1}$$

Remark 4.1.1. (ii) For a given symmetric, non-negative definite and Borel measurable $\mathrm{d}\times\mathrm{d}$-matrix-valued function c there is a unique non-negative definite and Borel measurable $\mathrm{d}\times\mathrm{d}$-matrix-valued function $\mathring{\sigma}$ such that $c(x)=\mathring{\sigma}(x)\mathring{\sigma}(x)^{\mathrm{T}}$ for all $x\in\mathbb{R}^{\mathrm{d}}$. In general, it is not clear that smoothness (Hölder continuity or differentiability) of c implies smoothness of $\mathring{\sigma}$. However, if $c(x)$ is additionally positive definite or twice continuously differentiable, this will be the case (see [39, Lemma 6.1.1 and Theorem 6.1.2]). In particular, under **(DD)**, $\mathring{\sigma}$ will be γ-Hölder continuous on $\mathscr{O}$.

(iii) Condition **(D)** holds true if for all $x, y \in \mathbb{R}^{\mathrm{d}}$,

$$|x-y|\,(|a(x)-a(y)|+|b(x)-b(y)|)+\|\sigma(x)-\sigma(y)\|_{\mathrm{HS}}^2 \leq \Theta|x-y|\,\theta(|x-y|)\,. \tag{4.2}$$

This together with periodicity of σ, automatically implies $1/2$-Hölder continuity of σ. Moreover, since

$$\begin{aligned}\|c(x)-c(y)\|_{\mathrm{HS}} &= \|\sigma(x)\sigma(x)^{\mathrm{T}}-\sigma(y)\sigma(y)^{\mathrm{T}}\|_{\mathrm{HS}}\\ &\leq \|(\sigma(x)-\sigma(y))\,\sigma(x)^{\mathrm{T}}\|_{\mathrm{HS}}+\|\sigma(y)\,(\sigma(x)^{\mathrm{T}}-\sigma(y)^{\mathrm{T}})\|_{\mathrm{HS}}\\ &\leq 2\sup_{z\in\mathbb{R}^{\mathrm{d}}}\|\sigma(z)\|_{\mathrm{HS}}\,\|\sigma(x)-\sigma(y)\|_{\mathrm{HS}}\end{aligned}$$

it also implies $1/2$-Hölder continuity of c. In addition, if $\limsup_{u\to 0}\theta(u)/u^{\gamma}<\infty$ for some $\gamma\in(0,1]$, it is easy to see that (4.2) implies γ-Hölder continuity of a and b, and $(1+\gamma)/2$-Hölder continuity of σ and c.

(iii) Assumptions **(D)** and **(DD)** imply that $c(x)$ is uniformly elliptic on $\overline{\mathscr{O}}$, that is, there is $\alpha>0$ such that

$$\langle c(x)\xi,\xi\rangle \geq \alpha|\xi|^2 \qquad \forall\,(x,\xi)\in\overline{\mathscr{O}}\times\mathbb{R}^{\mathrm{d}}\,.$$

Indeed, since for every $x\in\overline{\mathscr{O}}$ the matrix $c(x)$ is symmetric and positive definite, the corresponding eigenvalues $\lambda_1(x),\dots,\lambda_{\mathrm{d}}(x)$ are real and positive. Also, since $\lambda_1(x),\dots,\lambda_{\mathrm{d}}(x)$ are roots of the polynomial $\lambda\mapsto\det(c(x)-\lambda\mathbb{I}_{\mathrm{d}})$ we see that each $\lambda_i\colon\overline{\mathscr{O}}\to(0,\infty)$ is continuous. Hence, due to compactness of $\overline{\mathscr{O}}$, we conclude that there is $\alpha>0$ such that $c(x)-\alpha\mathbb{I}_{\mathrm{d}}$ is positive definite on $\overline{\mathscr{O}}$, which proves the assertion.

$\square$

Under the assumption that $c(x)$ is uniformly elliptic and twice continuously differentiable, and that $a,b\in C^2(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$ (in particular conditions **(D)** and **(DD)** are automatically satisfied with $\theta(v)=v$, $\gamma=1$ and $\mathscr{O}=(0,\tau_1)\times\cdots\times(0,\tau_{\mathrm{d}})$), in [12, Theorem 3.4.4] the CLT for process $\{X^{\varepsilon}(x,t)\}_{t\geq 0}$ has been shown. A crucial step in this proof is an application of Itô's formula to the process $\{\beta(X^{\varepsilon}(x,t))\}_{t\geq 0}$, where β is a solution of the Poisson equation which, in the uniformly elliptic case, is in $C^2(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$. On the other hand, in the case when the coefficient $c(x)$ can be degenerate it is not clear how to conclude necessary smoothness of $\beta(x)$. From Lemma 2.5.2 we know that a first step in that direction is showing geometrical ergodicity of C_b-Feller process $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$.

4.2. Geometric Ergodicity

In Section 4.1 we have allowed a diffusion coefficient of process $\{\tilde{X}^\varepsilon(x,t)\}_{t\geq 0}$ to be degenerate outside some open connected set $\mathscr{O} \subset [0,\tau]$. In order to control the behaviour of process outside $\mathscr{O}$ we will therefore need some additional assumption. What we need to make sure is that with positive probability our process doesn't get stuck in this area outside of $\mathscr{O}$. More formally we proceed as follows.

For $\varepsilon \geq 0$ and $B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}})$, let $\tilde{\tau}_B^{\varepsilon,x} := \inf\{t \geq 0\colon \tilde{X}^\varepsilon(x,t) \in B\}$ be the first entry time of B by $\{\tilde{X}^\varepsilon(x,t)\}_{t\geq 0}$. For the rest of this chapter we assume

(RT) for all $\varepsilon \geq 0$ and $x \in \mathbb{R}^{\mathrm{d}}$,

$$\mathbb{P}\left(\tilde{\tau}^{\varepsilon,x}_{\mathscr{O}+\tau} < \infty\right) > 0\,,$$

where $\mathscr{O}+\tau := \{x + k_\tau\colon x \in \mathscr{O},\ k_\tau \in \mathbb{Z}^{\mathrm{d}}_\tau\}$.

Example 4.2.1. Let $\{X(x,t)\}_{t\geq 0}$ be a 1-dimensional diffusion process associated to second-order elliptic differential operator $\mathscr{L}$ of the form

$$\mathscr{L} f(x) = b(x)\cdot f'(x) + \frac{1}{2}c(x)\cdot f''(x)\,,$$

where coefficients $b(x) \geq 0$ and $c(x) \geq 0$ are τ-periodic and such that $\operatorname{supp} c = \overline{\mathscr{O}}$ and

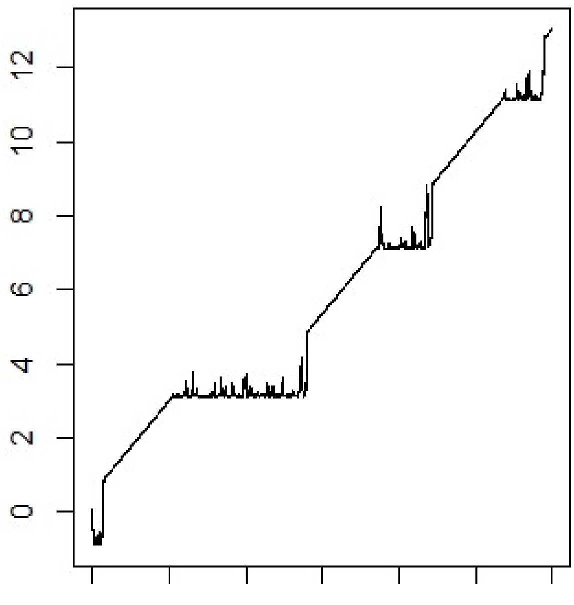

Figure 4.1: Simulation of process $\{X(x,t)\}_{t\geq 0}$

$\inf_{x\in\mathscr{O}^c} b(x) > 0$ for some open $\mathscr{O} \in \mathbb{R}$. Then conditions **(D)**, **(DD)** and **(RT)** are satisfied. In Figure 4.1 simulation of such process with $\tau = 4$ and $\mathscr{O} = B_1(0)$, $b(x) \equiv 0.5$ and

$$\sigma(x) = \begin{cases} 0, & x \in [1,3] \\ 8e^{\frac{-1}{1-x^2}} & x \in [-1,1], \end{cases}$$

is shown.

□

Under **(D)** comment after Proposition 2.7.5 tougether with Remark 2.4.3 (i) states that $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$ is a C_b-Feller process and this together with Proposition 2.2.4 show that $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$ admits at least one invariant probability measure. Assuming additionally **(DD)** and **(RT)**, in what follows we show that $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$ admits one, and only one, invariant measure, and the corresponding marginals converge as $t \to \infty$ to the invariant measure in the total variation norm. We do this by showing that $\{\tilde{X}^{\varepsilon}(x,t)\}_{t\geq 0}$ is ψ-irreducible, which follows from next proposition.

Proposition 4.2.2. *Under **(D)**, **(DD)** and **(RT)**, there exists a measure $\psi(\mathrm{d}x)$ on $\mathbb{T}^{\mathrm{d}}_{\tau}$ such that*

(i) $\operatorname{supp}(\psi)$ *has nonempty interior;*

(ii) for every $x \in \mathbb{T}^{\mathrm{d}}_{\tau}$ and $\varepsilon \in [0,\varepsilon_0]$ there is $t_{x,\varepsilon} \geq 0$ such that

$$\psi(B) > 0 \implies \tilde{p}^{\varepsilon,\tau}(t,x,B) > 0 \qquad \forall t \geq t_{x,\varepsilon}\,. \tag{4.3}$$

Proof. Let's first take $x \in \overline{\mathscr{O}} + \tau$ and prove that there exists a measure $\psi(\mathrm{d}x)$ such that (4.3) holds for $t_{x,\varepsilon} = 0$. According to [30, Theorems 7.3.6 and 7.3.7] there is a strictly positive function $q^{\varepsilon}(t,x,y)$ on $(0,\infty)\times\overline{\mathscr{O}}+\tau\times\overline{\mathscr{O}}+\tau$, jointly continuous in t, x and y, satisfying

$$\mathbb{E}\left[f(\tilde{X}^{\varepsilon}(x,t))\,\mathbb{1}_{(t,\infty]}\left(\tilde{\tau}^{\varepsilon,x}_{(\overline{\mathscr{O}}+\tau)^c}\right)\right] = \int_{\overline{\mathscr{O}}+\tau} f(y)\,q^{\varepsilon}(t,x,y)\,\mathrm{d}y$$

for all $t > 0$, $x \in \overline{\mathscr{O}} + \tau$ and $f \in C_b(\mathbb{R}^{\mathrm{d}})$. Clearly, by employing dominated convergence theorem, the above relation holds also for $f = \mathbb{1}_{\mathscr{U}+\tau}$ for any open set $\mathscr{U} \subseteq \mathscr{O}$. Denote by $\mathscr{D}$ the class of all $B \in \mathscr{B}(\mathscr{O})$ such that

$$\mathbb{P}\left(\tilde{X}^{\varepsilon}(x,t) \in B+\tau\,,\, \tilde{\tau}^{\varepsilon,x}_{(\overline{\mathscr{O}}+\tau)^c} > t\right) = \int_{B+\tau} q^{\varepsilon}(t,x,y)\,\mathrm{d}y\,.$$

Clearly, $\mathscr{D}$ contains the π-system of open rectangles in $\mathscr{B}(\mathscr{O})$, and forms a λ-system. Hence, by employing Dynkin's π-λ theorem we conclude that $\mathscr{D}=\mathscr{B}(\mathscr{O})$. Consequently, for all $t>0$, $x\in\overline{\mathscr{O}}+\tau$ and $B\in\mathscr{B}([0,\tau])$ we have

$$\tilde{p}^{\varepsilon}(t,x,B+\tau)\geq\mathbb{P}\left(\tilde{X}^{\varepsilon}(x,t)\in(B\cap\mathscr{O})+\tau\,,\,\tilde{\tau}^{\varepsilon,x}_{(\overline{\mathscr{O}}+\tau)^c}>t\right)=\int_{(B\cap\mathscr{O})+\tau}q^{\varepsilon}(t,x,y)\,\mathrm{d}y\,.$$

Set now $\psi(B+\tau):=\lambda((B\cap\mathscr{O})+\tau)$, $B\in\mathscr{B}([0,\tau])$, where $\lambda(\mathrm{d}x)$ stands for the Lebesgue measure on $\mathbb{R}^{\mathrm{d}}$. Clearly, by construction, $\psi(\mathrm{d}x)$ is a measure on σ-algebra $\mathscr{B}([0,\tau])+\tau:=\{B+\tau\colon B\in\mathscr{B}([0,\tau])\}$, $\mathrm{supp}(\psi)$ has non-empty interior, and for $B\in\mathscr{B}([0,\tau])$ it holds that

$$\psi(B+\tau)>0\Longrightarrow\tilde{p}^{\varepsilon}(t,x,B+\tau)>0\qquad\forall\,(t,x)\in(0,\infty)\times\overline{\mathscr{O}}+\tau\,.\tag{4.4}$$

It remains to show that for each $x\in([0,\tau]\setminus\overline{\mathscr{O}})+\tau$ there is $t_{x,\varepsilon}\geq0$ such that the implication in (4.4) holds for all $t\geq t_{x,\varepsilon}$. Since $\{\tilde{X}^{\varepsilon}(x,t)\}_{t\geq0}$ has continuous sample paths and $\mathscr{O}$ is an open set, we have that

$$\begin{aligned}\sum_{t\in\mathbb{Q}_+}\tilde{p}^{\varepsilon}(t,x,\mathscr{O}+\tau)&\geq\mathbb{P}\left(\exists t\in\mathbb{Q}_+\text{ such that }\tilde{X}^{\varepsilon}(x,t)\in\mathscr{O}+\tau\right)\\&=\mathbb{P}\left(\exists t\geq0\text{ such that }\tilde{X}^{\varepsilon}(x,t)\in\mathscr{O}+\tau\right)\\&=\mathbb{P}\left(\tilde{\tau}^{\varepsilon,x}_{\mathscr{O}+\tau}<\infty\right).\end{aligned}$$

From (**RT**) we see that there is $t_{x,\varepsilon}\in\mathbb{Q}_+$ such that $\tilde{p}^{\varepsilon}(t_{x,\varepsilon},x,\mathscr{O}+\tau)>0$. Let $B\in\mathscr{B}([0,\tau])$ be such that $\psi(B+\tau)>0$. For any $t>t_{x,\varepsilon}$ we then have

$$\begin{aligned}\tilde{p}^{\varepsilon}(t,x,B+\tau)&\geq\mathbb{P}\left(\tilde{X}^{\varepsilon}(x,t)\in B+\tau\,,\,\tilde{X}^{\varepsilon}(x,t_{x,\varepsilon})\in\mathscr{O}+\tau\right)\\&=\int_{\mathscr{O}+\tau}\tilde{p}^{\varepsilon}(t-t_{x,\varepsilon},y,B+\tau)\,\tilde{p}^{\varepsilon}(t_{x,\varepsilon},x,\mathrm{d}y)\,,\end{aligned}$$

which is strictly positive due to (4.4). The result now follows from (2.16). ■

Proposition 4.2.3. *For every $\varepsilon\geq0$ process $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq0}$ admits a unique invariant probability measure $\pi^{\varepsilon}(\mathrm{d}x)$ such that*

$$\sup_{x\in\mathbb{T}^{\mathrm{d}}_{\tau}}\|\tilde{p}^{\varepsilon,\tau}(t,x,\mathrm{d}y)-\pi^{\varepsilon}(\mathrm{d}y)\|_{\mathrm{TV}}\leq\Gamma\mathrm{e}^{-\gamma t}\qquad\forall\,\varepsilon,t\geq0\,,$$

for some $\gamma>0$ and $\Gamma>0$, that is process $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq0}$ is geometrically ergodic.

Proof. In remarks proceeding Proposition 4.2.2 we have already seen that there exists as invariant probability measure for process $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$. From Proposition 4.2.2 it immediately follows that $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$ is ψ-irreducible. According to Proposition 2.2.8 this entails that $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$ admits a unique invariant probability measure $\pi^{\varepsilon}(\mathrm{d}x)$. Next, the C_b-Feller property of $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$ and Proposition 4.2.2 together with Lemma 2.2.15 imply that $\mathbb{T}^{\mathrm{d}}_{\tau}$ is a petite set for process $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$. From Proposition 4.2.2 (which implies that $\sum_{i=1}^{\infty}\tilde{p}^{\varepsilon,\tau}(i,x,B)>0$ for all $x\in\mathbb{T}^{\mathrm{d}}_{\tau}$ whenever $\psi(B)>0$), and Proposition 2.2.12 we see that it is also a small set, which implies that $\{\tilde{X}^{\varepsilon,\tau}(x,t)\}_{t\geq 0}$ is aperiodic. The desired result now follows from Theorem 2.2.14. ■

Proposition 4.2.4.

$$\pi^{\varepsilon}(\mathrm{d}x)\xrightarrow[\varepsilon\to 0]{(\mathrm{w})}\pi^{0}(\mathrm{d}x)\,.$$

Proof. Since $\mathbb{T}^{\mathrm{d}}_{\tau}$ is compact the family of probability measures $\{\pi^{\varepsilon}(\mathrm{d}x)\}_{\varepsilon\geq 0}$ is tight. Hence, for any sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}\subset[0,\varepsilon_0]$ converging to 0 there is a further subsequence $\{\varepsilon_{i_j}\}_{j\in\mathbb{N}}$ such that $\{\pi^{\varepsilon_{i_j}}(\mathrm{d}x)\}_{j\in\mathbb{N}}$ converges weakly to some probability measure $\bar{\pi}^{0}(\mathrm{d}x)$. Take $f\in C(\mathbb{T}^{\mathrm{d}}_{\tau},\mathbb{R})$, and fix $t\geq 0$ and $\varepsilon>0$. From Proposition 2.7.7 we have that there is $0<\varepsilon_1\leq\varepsilon_0$ such that

$$\|\tilde{P}^{\varepsilon,\tau}_t f-\tilde{P}^{0,\tau}_t f\|_{\infty}\leq\varepsilon\qquad\forall\,\varepsilon\in[0,\varepsilon_1]\,.$$

We now have that

$$\begin{aligned}
&|\bar{\pi}^{0}(f)-\bar{\pi}^{0}\left(\tilde{P}^{0,\tau}_t f\right)|=\lim_{j\to\infty}|\pi^{\varepsilon_{i_j}}(f)-\bar{\pi}^{0}\left(\tilde{P}^{0,\tau}_t f\right)|=\\
&\lim_{j\to\infty}|\bar{\pi}^{\varepsilon_{i_j}}\left(\tilde{P}^{\varepsilon_{i_j},\tau}_t f\right)-\bar{\pi}^{0}\left(\tilde{P}^{0,\tau}_t f\right)|\leq\\
&\limsup_{j\to\infty}|\pi^{\varepsilon_{i_j}}\left(\tilde{P}^{\varepsilon_{i_j},\tau}_t f\right)-\bar{\pi}^{\varepsilon_{i_j}}\left(\tilde{P}^{0,\tau}_t f\right)|+\lim_{j\to\infty}|\pi^{\varepsilon_{i_j}}\left(\tilde{P}^{0,\tau}_t f\right)-\bar{\pi}^{0}\left(\tilde{P}^{0,\tau}_t f\right)|\leq\varepsilon\,,
\end{aligned}$$

which implies that $\bar{\pi}^{0}(\mathrm{d}x)$ is an invariant probability measure for process $\{X^{0,\tau}(x,t)\}_{t\geq 0}$. Thus, $\bar{\pi}^{0}(\mathrm{d}x)=\pi^{0}(\mathrm{d}x)$, which proves the assertion. ■

Now we are in a position to consider the solution β to the Poisson equation

$$\mathscr{A}^{0}\beta=b-\pi^{0}(b)\,. \tag{4.5}$$

Corollary 4.2.5. *Poisson equation (4.5) admits a τ-periodic solution $\beta\in C(\mathbb{R}^{\mathrm{d}})$. Moreover, β is the unique solution in the class of continuous and τ-periodic solutions to (4.5) satisfying $\int_{\mathbb{T}^{\mathrm{d}}_{\tau}}\beta_{\tau}(x)\,\pi^{0}(\mathrm{d}x)=0$.*

Proof. From Proposition 2.7.5 it follows that $\{\tilde{X}^0(x,t)\}_{t\geq 0}$ is C_b-Feller which implies that $\{\tilde{X}^{0,\tau}(x,t)\}_{t\geq 0}$ is C_b-Feller (see Remark 2.4.3). From Proposition 4.2.3, Proposition 2.5.1 and Lemma 2.5.2 we know that the solution to equation 4.5 is a well defined zero-resolvent

$$\beta(x) = -\int_0^\infty \tilde{P}_t^0 \left(b - \pi^0(b)\right)(x)\,\mathrm{d}t\,, \qquad x \in \mathbb{R}^{\mathrm{d}}\,.$$

Since the uniqueness in the class of continuous and τ-periodic solutions to (3.7) satisfying $\int_{\mathbb{T}_\tau^{\mathrm{d}}} \beta_\tau(x)\,\pi^0(\mathrm{d}x) = 0$ is also guaranteed by Lemma 2.5.2 this completes the proof. ■

4.3. Special Case $a \equiv 0$

In this section we will consider a special case when process $\{X^{\varepsilon}(x,t)\}_{t\geq 0}$ is a solution to the following SDE

$$\begin{aligned} \mathrm{d}X^{\varepsilon}(x,t) &= \frac{1}{\varepsilon} b\left(X^{\varepsilon}(x,t)/\varepsilon\right) \mathrm{d}t + \sigma\left(X^{\varepsilon}(x,t)/\varepsilon\right) \mathrm{d}W_t \\ X^{\varepsilon}(x,0) &= x \in \mathbb{R}^{\mathrm{d}}. \end{aligned}$$

That is a d-dimensional diffusion process associated to a second-order elliptic differential operator $\mathscr{L}^{\varepsilon}$ of the form

$$\mathscr{L}^{\varepsilon} f(x) = \langle \varepsilon^{-1} b(x/\varepsilon), \nabla f(x)\rangle + \frac{1}{2}\mathrm{Tr}\left(c(x/\varepsilon)\nabla\nabla^{\mathrm{T}} f(x)\right), \tag{4.6}$$

Recall that $X^{\varepsilon}(x,t) = \varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon, t/\varepsilon^2)$, $t \geq 0$. Clearly, $\{\tilde{X}^{\varepsilon}_t\}_{t\geq 0}$ satisfies

$$\begin{aligned} \mathrm{d}\tilde{X}^{\varepsilon}(x,t) &= b\left(\tilde{X}^{\varepsilon}(x,t)\right) + \sigma\left(\tilde{X}^{\varepsilon}(x,t)\right) \mathrm{d}W^{\varepsilon}_t \\ \tilde{X}^{\varepsilon}(x,0) &= x \in \mathbb{R}^{\mathrm{d}} \end{aligned} \tag{4.7}$$

and $\{\tilde{X}^{\varepsilon}(x,t)\}_{t\geq 0} \overset{(\mathrm{d})}{=} \{\tilde{X}^{0}(x,t)\}_{t\geq 0}$, where $\overset{(\mathrm{d})}{=}$ denotes the equality in distribution. In this case it will be easier to prove Central limit theorem and no regularity additional assumptions on process $\{X(x,t)\}_{t\geq 0}$ will be necessary.

Theorem 4.3.1. *Let $\{X^{\varepsilon}(x,t)\}_{t\geq 0}$ be a* d*-dimensional diffusion process associated to a second-order elliptic differential operator $\mathscr{L}^{\varepsilon}$ given in* (4.6) *with coefficients b and c satisfying* ***(D)****,* ***(DD)*** *and* ***(RT)****. Then*

$$\{X^{\varepsilon}(x,t) - \varepsilon^{-1}\pi^0(b)t\}_{t\geq 0} \underset{\varepsilon\to 0}{\overset{(\mathrm{d})}{\Longrightarrow}} \{W^{\Sigma}(x,t)\}_{t\geq 0}. \tag{4.8}$$

Here, $\Rightarrow$ denotes the convergence in the space of continuous functions endowed with the locally uniform topology, $\pi^0(\mathrm{d}x)$ is a measure whose existence and uniqueness is given by Proposition 4.2.3 and $\{W^{\Sigma}(x,t)\}_{t\geq 0}$ is a d*-dimensional zero-drift Brownian motion determined by covariance matrix Σ given by*

$$\Sigma = \pi^0\left(c - \bar{c} - \bar{c}^{\mathrm{T}} - \beta\left(\mathscr{A}^{0,\tau}\beta\right)^{\mathrm{T}} - \left(\mathscr{A}^{0,\tau}\beta\right)\beta^{\mathrm{T}}\right), \tag{4.9}$$

where $\bar{c} \in B(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}\times\mathrm{d}})$ is τ-periodic and such that $\pi^0(\|\bar{c}\|_{\mathrm{HS}}) < \infty$ and β is a solution to (4.5) *whose existence and uniqueness is given by Corollary 4.2.5.*

Proof. From the comments at the beginning of this chapter the statement of this theorem is equivalent to

$$\{\varepsilon\tilde{X}^0(x/\varepsilon,\varepsilon^{-2}t)-\varepsilon^{-1}\pi^0(b)t\}_{t\geq 0}\overset{\varepsilon\to 0}{\Longrightarrow}\{W^{\Sigma}(x,t)\}_{t\geq 0} \tag{4.10}$$

and this is what we will prove. Due to boundedness of $\beta(x)$, process $\{\varepsilon\tilde{X}^0(x/\varepsilon,t/\varepsilon^2)-x-\varepsilon^{-1}\pi^0(b)t\}_{t\geq 0}$ converges in law if, and only if process $\{\varepsilon\tilde{X}^0(x/\varepsilon,t/\varepsilon^2)-x-\varepsilon^{-1}\pi^0(b)t-\varepsilon\beta(\tilde{X}^0(x/\varepsilon,t/\varepsilon^2))+\varepsilon\beta(x/\varepsilon)\}_{t\geq 0}$ converges, and if this is the case the limit is the same. We have

$$\begin{aligned}
&\varepsilon\tilde{X}^0(x/\varepsilon,t/\varepsilon^2)-x-\varepsilon^{-1}\pi^0(b)t-\varepsilon\beta\left(\tilde{X}^0(x/\varepsilon,t/\varepsilon^2)\right)+\varepsilon\beta(x/\varepsilon)=\\
&-\varepsilon\beta\left(\tilde{X}^0(x/\varepsilon,t/\varepsilon^2)\right)+\varepsilon\beta(x/\varepsilon)+\varepsilon\int_0^{\varepsilon^{-2}t}\left(b\left(\tilde{X}^0(x/\varepsilon,s)\right)-\pi^0(b)\right)\mathrm{d}s\\
&\quad+\varepsilon\int_0^{\varepsilon^{-2}t}\sigma\left(\tilde{X}^0(x/\varepsilon,s)\right)\mathrm{d}W_s\qquad\forall t\geq 0.
\end{aligned} \tag{4.11}$$

Denote

$$\begin{aligned}
M_1(x,t)&:=\beta\left(\tilde{X}^0(x,t)\right)-\beta(x)-\int_0^t\left(b\left(\tilde{X}^0(x,s)\right)-\pi^0(b)\right)\mathrm{d}s,\qquad t\geq 0,\\
M_2(x,t)&:=\int_0^t\sigma\left(\tilde{X}^0(x,s)\right)\mathrm{d}W^0_s,\qquad t\geq 0.
\end{aligned}$$

Process $\{M_2(x,t)\}_{t\geq 0}$ is clearly an $\{\mathscr{F}_t\}_{t\geq 0}$-martingale and according to Proposition 2.3.2 so is the $\{M_1(x,t)\}_{t\geq 0}$. Hence, $\{\varepsilon\tilde{X}^0(x/\varepsilon,t/\varepsilon^2)-x-\varepsilon^{-1}\pi^0(b)t-\varepsilon\beta(\tilde{X}^0(x/\varepsilon,t/\varepsilon^2))+\varepsilon\beta(x/\varepsilon)\}_{t\geq 0}$ is also an $\{\mathscr{F}_t\}_{t\geq 0}$-martingale. According to Theorem 2.6.17, in order to conclude (4.8) it suffices to show that

$$\varepsilon^2\langle -M_1(x/\varepsilon,\cdot)+M_2(x/\varepsilon,\cdot)\rangle_{t/\varepsilon^2}\xrightarrow[\varepsilon\to 0]{\mathbb{P}}\Sigma t,\qquad t\geq 0. \tag{4.12}$$

For $i,j=1,\ldots,\mathrm{d}$ and $k,l=1,2$, we have that

$$\langle M_k^i(x,\cdot),M_l^j(x,\cdot)\rangle_t=4^{-1}\left(\langle M_k^i(x,\cdot)+M_l^j(x,\cdot)\rangle_t-\langle M_k^i(x,\cdot)-M_l^j(x,\cdot)\rangle_t\right),\qquad t\geq 0.$$

Let $\{\theta_t\}_{t\geq 0}$ be the family of shift operators on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0})$ satisfying $\theta_s\circ\tilde{X}^0(x,t)=\tilde{X}^0(x,s+t)$ for all $s,t\geq 0$ (see [72, p. 119]). Therefore for $s,t\geq 0$ we have that

$$\begin{aligned}
&\theta_t\circ M_1(x,s)=\beta\left(\tilde{X}^0(x,t+s)\right)-\beta\left(\tilde{X}^0(x,t)\right)-\int_0^s\left(b\left(\tilde{X}^0(x,t+u)\right)-\pi^0(b)\right)\mathrm{d}u=\\
&\beta\left(\tilde{X}^0(x,t+s)\right)-\beta(x)+\beta(x)-\beta\left(\tilde{X}^0(x,t)\right)-\int_t^{s+t}\left(b\left(\tilde{X}^0(x,u)\right)-\pi^0(b)\right)\mathrm{d}u=\\
&M_1(x,t+s)-M_1(x,t).
\end{aligned}$$

Observe that

$$M_2(x,t) = \tilde{X}^0(x,t) - x - \int_0^t b\left(\tilde{X}^0(x,s)\right) \mathrm{d}s\,, \qquad t \geq 0\,.$$

Therefore similarly to before for $s,t \geq 0$ we have that

$$\theta_t \circ M_2(x,s) = \tilde{X}^0(x,t+s) - \tilde{X}^0(x,t) - \int_0^s b\left(\tilde{X}^0(x,t+u)\right) \mathrm{d}u = M_2(x,t+s) - M_2(x,t)\,.$$

In other words, the processes $\{M_i(x,t)\}_{t\geq 0}$, $i = 1,2$, are continuous additive martingales with respect to $\{\tilde{X}^0(x,t)\}_{t\geq 0}$, in the sense of [16]. Then according to [23, Theorem 3.18 (v)] $\{\langle M_k^i(x,\cdot) \pm M_l^j(x,\cdot)\rangle_t\}_{t\geq 0}$ is also additive with respect to $\{\tilde{X}^0(x,t)\}_{t\geq 0}$. Next, form the martingale representation theorem we see that for each $i,j = 1,\dots \mathrm{d}$ and $k,l = 1,2$ it holds that $\mathrm{d}\langle M_k^i(x,\cdot) \pm M_l^j(x,\cdot)\rangle_t \ll \mathrm{d}t$. Using these two properties [23, Proposition 3.56] implies that for each $i,j = 1,\dots,\mathrm{d}$ and $k,l = 1,2$ there is a non-negative $\tilde{c}_{ij}^{k,l\pm} \in B(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ such that

$$\langle M_k^i(x,\cdot) \pm M_l^j(x,\cdot)\rangle_t = \int_0^t \tilde{c}_{ij}^{kl\pm}\left(\tilde{X}^0(x,s)\right) \mathrm{d}s\,, \qquad t \geq 0\,.$$

From τ-periodicity of the coefficients and the fact that $\{\tilde{X}^0(x+k_\tau,t)\}_{t\geq 0}$ and $\{\tilde{X}^0(x,t) + k_\tau\}_{t\geq 0}$ are indistinguishable for all $x \in \mathbb{R}^{\mathrm{d}}$ and $k_\tau \in \mathbb{Z}_\tau^{\mathrm{d}}$ we conclude that $\tilde{c}_{ij}^{k,l\pm}$ is τ-periodic. Due to boundedness of b,β and σ we have

$$\begin{aligned}
&\pi^0(\tilde{c}_{ij}^{kl\pm}) = \int_0^1 \pi^0(\tilde{c}_{ij}^{kl\pm})\,\mathrm{d}s = \int_0^1 \int_{\mathbb{T}_\tau^{\mathrm{d}}} \mathbb{E}\left[\tilde{c}_{ij}^{kl\pm}\left(\tilde{X}^{0,\tau}(x,s)\right)\right] \pi^0(\mathrm{d}x)\,\mathrm{d}s = \\
&\int_{\mathbb{T}_\tau^{\mathrm{d}}} \mathbb{E}\left[\int_0^1 \tilde{c}_{ij}^{kl\pm}\left(\tilde{X}^{0,\tau}(x,s)\right) \mathrm{d}s\right] \pi^0(\mathrm{d}x) = \int_{\mathbb{T}_\tau^{\mathrm{d}}} \mathbb{E}\left[\langle M_k^i(z_x,\cdot) \pm M_l^j(z_x,\cdot)\rangle_1\right] \pi^0(\mathrm{d}x) = \\
&\int_{\mathbb{T}_\tau^{\mathrm{d}}} \mathbb{E}\left[\left(M_k^i(z_x,1) \pm M_l^j(z_x,1)\right)^2\right] \pi^0(\mathrm{d}x) \leq \\
&2\int_{\mathbb{T}_\tau^{\mathrm{d}}} \mathbb{E}\left[M_k^i(z_x,1)^2\right] \pi^0(\mathrm{d}x) + 2\int_{\mathbb{T}_\tau^{\mathrm{d}}} \mathbb{E}\left[M_l^j(z_x,1)^2\right] \pi^0(\mathrm{d}x) < \infty\,,
\end{aligned}$$

where $z_x \in \Pi_\tau^{-1}(\{x\})$ is arbitrary. Set now $\bar{c}_{ij}^{kl}(x) := (\tilde{c}_{ij}^{kl+}(x) - \tilde{c}_{ij}^{kl-}(x))/4$, and $\bar{c}^{kl}(x) := (\bar{c}_{ij}^{kl}(x))_{i,j=1,\dots,\mathrm{d}}$. Clearly, for all $k,l = 1,2$ $\bar{c}^{kl} \in B(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}\times\mathrm{d}})$ is symmetric, non-negative definite and τ-periodic, and satisfies $\pi^0(\|\bar{c}^{kl}\|_{\mathrm{HS}}) < \infty$ and

$$\langle M_k(x,\cdot), M_l(x,\cdot)\rangle_t = \int_0^t \bar{c}^{kl}\left(\tilde{X}^0(x,s)\right) \mathrm{d}s\,, \qquad t \geq 0\,.$$

Directly from Proposition 4.2.3 and Theorem 2.2.19 (Birkhoff ergodic theorem) it follows that for all $t \geq 0$

$$\varepsilon^2 \langle M_k(x/\varepsilon,\cdot), M_l(x/\varepsilon,\cdot)\rangle_{t/\varepsilon^2} = \varepsilon^2 \int_0^{t/\varepsilon^2} \bar{c}^{kl}\left(\tilde{X}^{0,\tau}((x/\varepsilon)_\tau,s)\right) \mathrm{d}s \xrightarrow[\varepsilon\to 0]{\mathbb{P}\text{-a.s.}} \pi^0(\bar{c}^{kl})\,t\,. \tag{4.13}$$

It remains to determine $\pi^0(\bar{c}^{kl})$ for $k,l = 1,2$. For $k = l = 1$ we know that $\pi^0(\bar{c}^{11}) = \lim_{\varepsilon\to 0}\varepsilon^2\int_{\mathbb{T}^d_\tau}\mathbb{E}_{z_x}\left[\langle M_1(z_x,\cdot),M_1(z_x,\cdot)\rangle_{1/\varepsilon^2}\right]\pi^0(\mathrm{d}x)$. Dominated convergence theorem implies that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^2\int_{\mathbb{T}^d_\tau}\mathbb{E}_{z_x}\left[\langle M_1(z_x,\cdot),M_1(z_x,\cdot)\rangle_{1/\varepsilon^2}\right]\pi^0(\mathrm{d}x) = \\
&\lim_{\varepsilon\to 0}\varepsilon^2\int_{\mathbb{T}^d_\tau}\mathbb{E}\Bigg[\left(\int_0^{\varepsilon^{-2}}\tilde{\mathscr{A}}^0\beta\left(\tilde{X}^0(z_x,s)\right)\mathrm{d}s\right)\left(\int_0^{\varepsilon^{-2}}\tilde{\mathscr{A}}^0\beta\left(\tilde{X}^0(z_x,s)\right)\mathrm{d}s\right)^{\mathrm{T}} - \\
&\qquad\left(\int_0^{\varepsilon^{-2}}\tilde{\mathscr{A}}^0\beta\left(\tilde{X}^0(z_x,s)\right)\mathrm{d}s\right)\left(\beta\left(\tilde{X}^0(z_x,1/\varepsilon^2)\right)-\beta(z_x)\right)^{\mathrm{T}} - \\
&\qquad\left(\beta\left(\tilde{X}^0(z_x,1/\varepsilon^2)\right)-\beta(z_x)\right)\left(\int_0^{\varepsilon^{-2}}\tilde{\mathscr{A}}^0\beta\left(\tilde{X}^0(z_x,s)\right)\mathrm{d}s\right)^{\mathrm{T}}\Bigg]\pi^0(\mathrm{d}x) = \\
&\lim_{\varepsilon\to 0}\varepsilon^2\int_{\mathbb{T}^d_\tau}\mathbb{E}\Bigg[\left(\int_0^{\varepsilon^{-2}}\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\mathrm{d}s\right)\left(\int_0^{\varepsilon^{-2}}\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\mathrm{d}s\right)^{\mathrm{T}} - \\
&\qquad\left(\int_0^{\varepsilon^{-2}}\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\mathrm{d}s\right)\left(\beta\left(\tilde{X}^{0,\tau}(x,1/\varepsilon^2)\right)\right)^{\mathrm{T}} - \\
&\qquad\left(\beta\left(\tilde{X}^{0,\tau}(x,1/\varepsilon^2)\right)\right)\left(\int_0^{\varepsilon^{-2}}\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\mathrm{d}s\right)^{\mathrm{T}}\Bigg]\pi^0(\mathrm{d}x) + \\
&\quad\int_{\mathbb{T}^d_\tau}\left(\lim_{\varepsilon\to 0}\varepsilon^2\int_0^{\varepsilon^{-2}}\tilde{P}^{0,\tau}_s\tilde{\mathscr{A}}^{0,\tau}\beta(x)\,\mathrm{d}s\right)(\beta(x))^{\mathrm{T}}\pi^0(\mathrm{d}x) + \\
&\quad\int_{\mathbb{T}^d_\tau}\beta(x)\left(\lim_{\varepsilon\to 0}\varepsilon^2\int_0^{\varepsilon^{-2}}\tilde{P}^{0,\tau}_s\tilde{\mathscr{A}}^{0,\tau}\beta(x)\,\mathrm{d}s\right)^{\mathrm{T}}\pi^0(\mathrm{d}x) = \\
&\lim_{\varepsilon\to 0}\varepsilon^2\int_{\mathbb{T}^d_\tau}\mathbb{E}\Big[2\int_0^{\varepsilon^{-2}}\int_s^{\varepsilon^{-2}}\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,v)\right)\left(\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\right)^{\mathrm{T}}\mathrm{d}v\,\mathrm{d}s - \\
&\qquad\int_0^{\varepsilon^{-2}}\left(\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\right)\left(\beta\left(\tilde{X}^{0,\tau}(x,1/\varepsilon^2)\right)\right)^{\mathrm{T}}\mathrm{d}s - \\
&\qquad\int_0^{\varepsilon^{-2}}\left(\beta\left(\tilde{X}^{0,\tau}(x,1/\varepsilon^2)\right)\right)\left(\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\right)^{\mathrm{T}}\mathrm{d}s\Big]\pi^0(\mathrm{d}x) + \\
&\quad\pi^0(\tilde{\mathscr{A}}^{0,\tau}\beta)\pi^0(\beta^{\mathrm{T}}) + \pi^0\left((\tilde{\mathscr{A}}^{0,\tau}\beta)^{\mathrm{T}}\right)\pi^0(\beta) = \\
&\lim_{\varepsilon\to 0}\varepsilon^2\int_{\mathbb{T}^d_\tau}\int_0^{\varepsilon^{-2}}\mathbb{E}\Bigg[\left(\int_s^{\varepsilon^{-2}}\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,v)\right)\mathrm{d}v - \beta\left(\tilde{X}^{0,\tau}(x,1/\varepsilon^2)\right)\right)\left(\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\right)^{\mathrm{T}} + \\
&\qquad\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\left(\int_s^{\varepsilon^{-2}}\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,v)\right)\mathrm{d}v - \beta\left(\tilde{X}^{0,\tau}(x,1/\varepsilon^2)\right)\right)^{\mathrm{T}}\Bigg]\mathrm{d}s\,\pi^0(\mathrm{d}x).
\end{aligned}
$$

Set

$$
M_1^\tau(x,t) := \beta\left(\tilde{X}^{0,\tau}(x,t)\right) - \beta(x) - \int_0^t\tilde{\mathscr{A}}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\mathrm{d}s, \qquad t \geq 0.
$$

We have

$$\int_s^{\varepsilon^{-2}} \mathscr{A}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,v)\right)\mathrm{d}v - \beta\left(\tilde{X}^{0,\tau}(x,1/\varepsilon^2)\right) = M_1^\tau(x,s)M_1^\tau(x,1/\varepsilon^2) - \beta\left(\tilde{X}^{0,\tau}(x,s)\right).$$

Clearly, $\{M_1^\tau(x,t)\}_{t\geq 0}$ is a $\{\mathscr{F}_t\}_{t\geq 0}$-martingale and therefore

$$\begin{aligned}
&\mathbb{E}\left[\left(M_1^\tau(x,s) - M_1^\tau(x,1/\varepsilon^2)\right)\left(\mathscr{A}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\right)^{\mathrm{T}}\right] = \\
&\mathbb{E}\left[\mathbb{E}\left[\left(M_1^\tau(x,s) - M_1^\tau(x,1/\varepsilon^2)\right)\left(\mathscr{A}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\right)^{\mathrm{T}} \mid \mathscr{F}_s\right]\right] = 0\,.
\end{aligned}$$

We now have

$$\begin{aligned}
\pi^0(\bar{c}^{11}) &= -\lim_{\varepsilon\to 0}\varepsilon^2\int_{\mathbb{T}_\tau^d}\int_0^{\varepsilon^{-2}}\mathbb{E}\Big[\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\left(\mathscr{A}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\right)^{\mathrm{T}} \\
&\qquad + \mathscr{A}^{0,\tau}\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\left(\beta\left(\tilde{X}^{0,\tau}(x,s)\right)\right)^{\mathrm{T}}\Big]\,\mathrm{d}s\,\pi^0(\mathrm{d}x) \\
&= -\lim_{\varepsilon\to 0}\varepsilon^2\int_0^{\varepsilon^{-2}}\int_{\mathbb{T}_\tau^d}\tilde{P}_s^{0,\tau}\left(\beta\left(\mathscr{A}^{0,\tau}\beta\right)^{\mathrm{T}} + \mathscr{A}^{0,\tau}\beta\,(\beta)^{\mathrm{T}}\right)(x)\,\pi^0(\mathrm{d}x)\,\mathrm{d}s \\
&= -\pi^0\left(\beta\left(\mathscr{A}^{0,\tau}\beta\right)^{\mathrm{T}} + \left(\mathscr{A}^{0,\tau}\beta\right)\beta^{\mathrm{T}}\right).
\end{aligned}$$

For $k=l=2$ it follows from Example 2.6.6 that $\bar{c}^{22} = c$. For mixed terms we have $\langle M_1(x,\cdot), M_2(x,\cdot)\rangle_t = \langle M_2(x,\cdot), M_1(x,\cdot)\rangle_t^{\mathrm{T}}$, $t \geq 0$. Therefore $\bar{c} := \bar{c}^{12} = \left(\bar{c}^{21}\right)^{\mathrm{T}}$, which completes the proof. ■

4.4. Generalization of Itô's Formula

In the more general case when $a \not\equiv 0$ we will require additional smoothness of a solution β to the Poisson equation (4.5). For that we will follow the approach from [43]. Let $\sigma_j(x) := (\sigma_{ij}(x),\dots,\sigma_{dj}(x))^{\mathrm{T}}$, $j = 1,\dots,\mathrm{n}$, and let $\mathscr{U} \subseteq [0,\tau]$ where the parabolic Hörmander condition holds, that is, the set of $x \in [0,\tau]$ for which the Lie algebra generated by $(b(x),1) \cup \{(\sigma_1(x),0),\dots,(\sigma_{\mathrm{n}}(x),0)\}$ spans $\mathbb{R}^{\mathrm{d}+1}$. Observe that $\mathscr{O} \subseteq \mathscr{U}$. Let $\tilde{\tau}^{0,x}_B := \inf\{t \geq 0\colon \tilde{X}^0(x,t) \in B\}$ be the first entry time of B by $\{\tilde{X}^0(x,t)\}_{t\geq 0}$. In the rest of this chapter assume the following

(J) $\sigma \in C^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{n}})$, $a,b \in C^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R}^{\mathrm{d}})$, and

$$\inf_{t>0}\sup_{x\in\mathbb{R}^{\mathrm{d}}} \mathbb{E}\left[\|J^x_t\|_{\mathrm{HS}}\,\mathbb{1}_{[t,\infty]}(\tilde{\tau}^{0,x}_{\mathscr{U}+\tau})\right] < 1\,,$$

where $\{J^x_t\}_{t\geq 0}$ is the Jacobian of the stochastic flow associated to $\{\tilde{X}^0(x,t)\}_{t\geq 0}$, that is, a solution to

$$\begin{aligned} \mathrm{d}J^x_t &= \mathrm{D}b\left(\tilde{X}^0(x,t)\right) J^x_t\,\mathrm{d}t + \sum_{j=1}^{\mathrm{n}} \mathrm{D}\sigma_j\left(\tilde{X}^0(x,t)\right) J^x_t\,\mathrm{d}\left(W^0_j\right)_t \\ J^x_0 &= \mathbb{I}_{\mathrm{d}}\,. \end{aligned}$$

Under this condition we are able to conclude the following

Proposition 4.4.1. *Let $f \in C^k_b(\mathbb{R}^{\mathrm{d}})$, $k = 0,1,2$ then $\tilde{P}^0_t f \in C^k(\mathbb{R}^{\mathrm{d}})$ for any $t \geq 0$. More specifically if $f \in C^1(\mathbb{R}^{\mathrm{d}})$ is τ-periodic with $\pi^0(f) = 0$ then there are $\bar{\gamma} > 0$ and $\bar{\Gamma} > 0$, such that*

$$\|\nabla\tilde{P}^0_t f\|_\infty \leq \bar{\Gamma}\,(\|f\|_\infty + \|\nabla f\|_\infty)\,\mathrm{e}^{-\bar{\gamma}t} \tag{4.14}$$

for all $t \geq 0$.

Proof. According to [41, Lemma II.9.2 and Theorem II.9.5], smoothness of $\sigma(x)$ and $b(x)$ implies that $\tilde{P}^0_t f \in C^k(\mathbb{R}^{\mathrm{d}})$ for any $t \geq 0$ and $f \in C^k_b(\mathbb{R}^{\mathrm{d}})$, $k = 0,1,2$. Also, under **(D)**, **(DD)**, **(RT)** and **(J)**, in [43, Lemma 2.6] it has been shown that there are $\bar{\gamma} > 0$ and $\bar{\Gamma} > 0$, such that equation (4.14) holds for all $t \geq 0$ and τ-periodic $f \in C^1(\mathbb{R}^{\mathrm{d}})$ with $\pi^0(f) = 0$. ■

In particular, $\beta \in C^1(\mathbb{R}^{\mathrm{d}})$. This is still not enough to use a classical Itô's formula but it proves to be enough for us to derive the following Itô-type formula for the process $\{\beta(\tilde{X}_t^{\varepsilon})\}_{t\geq 0}$.

Let $f \in C(\mathbb{R}^{\mathrm{d}})$ be τ-periodic. Define

$$\zeta(x) := -\int_0^\infty \tilde{P}_t^0 \left(f - \pi^0(f)\right)(x)\,\mathrm{d}t\,, \qquad x \in \mathbb{R}^{\mathrm{d}}.$$

This function is again well defined, τ-periodic, continuous, and satisfies $\zeta \in \mathscr{D}_{\tilde{\mathscr{A}}^0}$ and $\tilde{\mathscr{A}}^0\zeta(x) = f(x) - \pi^0(f)$.

Lemma 4.4.2. *Assume $f \in C^2(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ is τ-periodic. Then it holds that*

$$\begin{aligned}\zeta\left(\tilde{X}^{\varepsilon}(x,t)\right) &= \zeta(x) + \int_0^t (f - \pi^0(f))\left(\tilde{X}^{\varepsilon}(x,s)\right)\mathrm{d}s + \varepsilon\int_0^t \left(\nabla\zeta^{\mathrm{T}}a\right)\left(\tilde{X}^{\varepsilon}(x,s)\right)\mathrm{d}s\\ &\quad + \int_0^t \left(\nabla\zeta^{\mathrm{T}}\sigma\right)\left(\tilde{X}^{\varepsilon}(x,s)\right)\mathrm{d}W_s^{\varepsilon} \qquad \forall t \geq 0.\end{aligned}$$

Proof. Since, as we have already commented, $x \mapsto \tilde{P}_s^0(f - \pi^0(f))(x)$ is twice continuously differentiable, for any $s \geq 0$, we can use Itô's formula Theorem 2.6.12 for process $\{\tilde{X}^{\varepsilon}(x,t)\}_{t\geq 0}$ with representation given in (2.32). Observe that $\tilde{X}^{\varepsilon,c}(x,t) = \int_0^t \tilde{\sigma}(\tilde{X}^{\varepsilon}(x,s))\,\mathrm{d}W_s^{\varepsilon}$. We have

$$\begin{aligned}\tilde{P}_s^0(f - \pi^0(f))\left(\tilde{X}^{\varepsilon}(x,t)\right) &= \tilde{P}_s^0(f - \pi^0(f))(x) + \int_0^t \left\langle \nabla\tilde{P}_s^0(f - \pi^0(f)), \varepsilon a + b\right\rangle\left(\tilde{X}^{\varepsilon}(x,u)\right)\mathrm{d}u\\ &\quad + \int_0^t \left\langle \nabla\tilde{P}_s^0(f - \pi^0(f))\left(\tilde{X}^{\varepsilon}(x,u)\right), \sigma\left(\tilde{X}^{\varepsilon}(x,u)\right)\mathrm{d}W_u^{\varepsilon}\right\rangle\\ &\quad + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t \partial_{ij}P_s^0(f - \pi^0(f))\left(\tilde{X}^{\varepsilon}(x,u)\right)c_{ij}(\tilde{X}^{\varepsilon}(x,u))\,\mathrm{d}u\\ &= \tilde{P}_s^0(f - \pi^0(f))(x) + \int_0^t \tilde{\mathscr{A}}^0\tilde{P}_s^0(f - \pi^0(f))\left(\tilde{X}^{\varepsilon}(x,u)\right)\mathrm{d}u\\ &\quad + \varepsilon\int_0^t \langle\nabla\tilde{P}_s^0(f - \pi^0(f)), a\rangle\left(\tilde{X}^{\varepsilon}(x,u)\right)\mathrm{d}u \qquad (4.15)\\ &\quad + \int_0^t \left\langle \nabla\tilde{P}_s^0(f - \pi^0(f))\left(\tilde{X}^{\varepsilon}(x,u)\right), \sigma\left(\tilde{X}^{\varepsilon}(x,u)\right)\mathrm{d}W_u^{\varepsilon}\right\rangle.\end{aligned}$$

By integrating the previous relation with respect to the time variable $s \in [0,\infty)$ (and recalling the definition of the function ζ), for all $t \geq 0$ we arrive at

$$\begin{aligned}\zeta\left(\tilde{X}^{\varepsilon}(x,t)\right) &= \zeta(x) - \int_0^\infty\int_0^t \tilde{\mathscr{A}}^0\tilde{P}_s^0(f - \pi^0(f))\left(\tilde{X}^{\varepsilon}(x,u)\right)\mathrm{d}u\,\mathrm{d}s\\ &\quad + \varepsilon\int_0^t \langle\nabla\zeta, a\rangle\left(\tilde{X}^{\varepsilon}(x,u)\right)\mathrm{d}u + \int_0^t (\nabla\zeta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x,u)\right)\mathrm{d}W_u^{\varepsilon}.\end{aligned} \qquad (4.16)$$

The last two integrals on the right-hand side in (4.16) are well defined, and follow from the last two terms in (4.15), because of (4.14). By observing that $\tilde{\mathscr{A}}^0\tilde{P}^0_s f(x) = \tilde{P}^0_s\tilde{\mathscr{A}}^0 f(x) = \tilde{P}^{0,\tau}_s\tilde{\mathscr{A}}^{0,\tau} f(x_\tau)$ (the last equality follows from Remark 2.4.4), and $\pi^0(\tilde{\mathscr{A}}^{0,\tau} f) = 0$, Corollary 2.2.10 together with Proposition 4.2.3 implies that the second integral on the right-hand side in (4.16) is well defined. It remains to prove that

$$-\int_0^\infty \tilde{\mathscr{A}}^0\tilde{P}^0_s(f-\pi^0(f))(x)\,\mathrm{d}s = (f-\pi^0(f))(x) \qquad \forall x\in\mathbb{R}^{\mathrm{d}}.$$

We have

$$-\int_0^\infty \tilde{\mathscr{A}}^0\tilde{P}^0_t(f-\pi^0(f))(x)\,\mathrm{d}t = -\int_0^\infty \lim_{s\to 0}\frac{\tilde{P}^0_{s+t}(f-\pi^0(f))(x)-\tilde{P}^0_t(f-\pi^0(f))(x)}{s}\,\mathrm{d}t\,.$$

By employing Proposition 2.3.1 and Corollary 2.2.10 we have

$$\begin{aligned}
&\frac{\|\tilde{P}^0_{s+t}(f-\pi^0(f))-\tilde{P}^0_t(f-\pi^0(f))\|_\infty}{s} = \frac{\|\int_t^{s+t}\tilde{P}^0_u\tilde{\mathscr{A}}^0(f-\pi^0(f))\,\mathrm{d}u\|_\infty}{s} \le \\
&\frac{\int_t^{s+t}\|\tilde{P}^0_u\tilde{\mathscr{A}}^0(f-\pi^0(f))\|_\infty\,\mathrm{d}u}{s} \le \frac{\Gamma\,\|\tilde{\mathscr{A}}^0(f-\pi^0(f))\|_\infty}{s}\int_t^{s+t}\mathrm{e}^{-\gamma u}\mathrm{d}u = \\
&\frac{\Gamma\,\|\tilde{\mathscr{A}}^0(f-\pi^0(f))\|_\infty}{s}\mathrm{e}^{-\gamma t}\left(\mathrm{e}^{-\gamma s}-1\right) \le \Gamma\,\|\tilde{\mathscr{A}}^0(f-\pi^0(f))\|_\infty\mathrm{e}^{-\gamma t}
\end{aligned}$$

for all $s,t\in(0,\infty)$. The result now follows from the dominated convergence theorem since

$$-\lim_{s\to 0}\frac{1}{s}\int_0^\infty \tilde{P}^0_{s+t}(f-\pi^0(f))(x)-\tilde{P}^0_t(f-\pi^0(f))(x)\,\mathrm{d}t = \tilde{\mathscr{A}}^0\zeta(x) = f(x)-\pi^0(f)\,.$$

■

4.5. CENTRAL LIMIT THEOREM

We are now ready to prove the main result of this chapter.

Theorem 4.5.1. *Let $\{X^\varepsilon(x,t)\}_{t\geq 0}$ be a* d*-dimensional diffusion process associated to a second-order elliptic differential operator $\mathscr{L}^\varepsilon$ given in* (2.29) *with coefficients a,b,c satisfying **(D)**, **(DD)**, **(RT)** and **(J)**. Then*

$$\{X^\varepsilon(x,t)-\varepsilon^{-1}\pi^0(b)t\}_{t\geq 0} \xrightarrow[\varepsilon\to 0]{\text{(d)}} \{W^{\mathsf{b},\mathsf{c}}(x,t)\}_{t\geq 0}\,. \tag{4.17}$$

Here, $\overset{\text{(d)}}{\Longrightarrow}$ denotes the convergence in the space of continuous functions endowed with the locally uniform topology, $\pi^0(\mathrm{d}x)$ is a measure whose existence and uniqueness is given by Proposition 4.2.3 *and $\{W^{\mathsf{b},\mathsf{c}}(x,t)\}_{t\geq 0}$ is a* d*-dimensional Brownian motion determined by covariance matrix and drift vector*

$$\mathsf{c} = \pi^0\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)^\mathrm{T}\right) \qquad \textit{and} \qquad \mathsf{b} = \pi^0\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,a\right), \tag{4.18}$$

respectively. Here β is a solution to (4.5) *whose existence and uniqueness is given by Corollary* 4.2.5.

Proof. Recall that $X^\varepsilon(x,t)=\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)$, $t\geq 0$. Hence, due to boundedness of β, $\{X^\varepsilon(x,t)-\varepsilon^{-1}\pi^0(b)t\}_{t\geq 0}$ converges in law if, and only if

$$\{\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)-\varepsilon^{-1}\pi^0(b)t-\varepsilon\beta(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2))+\varepsilon\beta(x/\varepsilon)\}_{t\geq 0} \tag{4.19}$$

converges in law, and if this is the case the limit is the same. By combining Lemma 4.4.2 (applied to b) with (2.32) for every $t\geq 0$ we have

$$\begin{aligned}
&\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)-\varepsilon^{-1}\pi^0(b)t-x-\varepsilon\beta\left(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)+\varepsilon\beta(x/\varepsilon)=\\
&\varepsilon^2\int_0^{\varepsilon^{-2}t} a\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s+\varepsilon\int_0^{\varepsilon^{-2}t} b\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s+\varepsilon\int_0^{\varepsilon^{-2}t}\sigma\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon-\\
&\varepsilon\int_0^{\varepsilon^{-2}t}\pi^0(b)\,\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(b-\pi^0(b))\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-\\
&\varepsilon^2\int_0^{\varepsilon^{-2}t}(\mathrm{D}\beta a)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\mathrm{D}\beta\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon=\\
&\varepsilon^2\int_0^{\varepsilon^{-2}t}((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,a)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s+\varepsilon\int_0^{\varepsilon^{-2}t}((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon\,.
\end{aligned}$$

Clearly, process in (4.19) is a semimartingale with bounded variation and predictable quadratic covariation parts

$$\left\{\varepsilon^2\int_0^{\varepsilon^{-2}t}\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right\}_{t\geq 0},$$

and

$$\left\{\varepsilon^2\int_0^{\varepsilon^{-2}t}\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)c\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)^\mathrm{T}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right\}_{t\geq 0},$$

respectively. From Theorem 2.6.16 we see that in order for the statement of the theorem to hold it remains to see

$$\varepsilon^2\int_0^{\varepsilon^{-2}t}\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\xrightarrow[\varepsilon\to 0]{\mathbb{P}}\mathsf{b}t,$$

and

$$\varepsilon^2\int_0^{\varepsilon^{-2}t}\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)c\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)^\mathrm{T}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\xrightarrow[\varepsilon\to 0]{\mathbb{P}}\mathsf{c}t$$

for all $t\geq 0$. We will do this by proving the convergence in L^2 for the bounded variation and an analogous relation holds for the predictable quadratic covariation part. First note that $\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a-\mathsf{b}\right)=\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a-\pi^0\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a\right)\right)$ and that instead of $\pi^0(\mathrm{d}x)$ we can take $\pi^\varepsilon(\mathrm{d}x)$. This can be done because

$$\begin{aligned}
&\varepsilon^2\left(\mathbb{E}\left[\left(\int_0^{\varepsilon^{-2}t}\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a-\mathsf{b}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right)^\mathrm{T}\right.\right.\\
&\qquad\left.\left.\left(\int_0^{\varepsilon^{-2}t}\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a-\mathsf{b}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right)\right]\right)^{1/2}\leq\\
&\varepsilon^2\left(\mathbb{E}\left[\left(\int_0^{\varepsilon^{-2}t}\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a-\pi^\varepsilon\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a\right)\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right)^\mathrm{T}\right.\right.\\
&\qquad\left.\left.\left(\int_0^{\varepsilon^{-2}t}\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a-\pi^\varepsilon\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a\right)\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right)\right]\right)^{1/2}\\
&\quad+\left|\pi^\varepsilon\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a\right)-\mathsf{b}\right|t
\end{aligned}$$

and Proposition 4.2.4. In order to make calculations easer to follow we define

$$F^\varepsilon(x):=\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a-\pi^\varepsilon\left(\left(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta\right)a\right)\right)(x).$$

Due to τ-periodicity, for any $\varepsilon>0$ F^ε satisfies $F^\varepsilon\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)=F^\varepsilon\left(\tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,s)\right)$ for $t\geq 0$, and

$$\int_{\mathbb{T}^d_\tau}F^\varepsilon(x)\,\pi^\varepsilon(\mathrm{d}x)=0.$$

It is also bounded, we have $\|F^\varepsilon\|_\infty \leq 2\|(\mathbb{I}_\mathrm{d} - \mathrm{D}\beta)\, a\|_\infty$. Now, by the Markov property and the fact that the geometric ergodicity of $\{\tilde{X}^{\varepsilon,\tau}(x_\tau,t)\}_{t\geq 0}$ allows us to employ Corollary 2.2.10, we have

$$
\begin{aligned}
&\varepsilon^4\mathbb{E}\left[\left(\int_0^{\varepsilon^{-2}t} F^\varepsilon\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s\right)^{\mathrm{T}}\left(\int_0^{\varepsilon^{-2}t} F^\varepsilon\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s\right)\right] = \\
&\varepsilon^4\mathbb{E}\left[\left(\int_0^{\varepsilon^{-2}t} F^\varepsilon\left(\tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,s)\right)\mathrm{d}s\right)^{\mathrm{T}}\left(\int_0^{\varepsilon^{-2}t} F^\varepsilon\left(\tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,s)\right)\mathrm{d}s\right)\right] = \\
&2\varepsilon^4\int_0^{\varepsilon^{-2}t}\int_0^s \mathbb{E}\left[F^\varepsilon\left(\tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,s)\right)^{\mathrm{T}} F^\varepsilon\left(\tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,u)\right)\right]\mathrm{d}u\,\mathrm{d}s = \\
&2\varepsilon^4\int_0^{\varepsilon^{-2}t}\int_0^s \mathbb{E}\Big[\mathbb{E}\left[F^\varepsilon\left(\tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,s)\right)^{\mathrm{T}} \mid \tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,u)\right] F^\varepsilon\left(\tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,u)\right)\Big]\mathrm{d}u\,\mathrm{d}s = \\
&2\varepsilon^4\int_0^{\varepsilon^{-2}t}\int_0^s \mathbb{E}\left[\left(\tilde{P}^{\varepsilon,\tau}_{s-u}F^\varepsilon\left(\tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,u)\right)\right)^{\mathrm{T}} F^\varepsilon\left(\tilde{X}^{\varepsilon,\tau}((x/\varepsilon)_\tau,u)\right)\right]\mathrm{d}u\,\mathrm{d}s \leq \\
&2\varepsilon^4\int_0^{\varepsilon^{-2}t}\int_0^s \left\|\tilde{P}^{\varepsilon,\tau}_{s-u}F^\varepsilon\right\|_\infty \|F^\varepsilon\|_\infty\,\mathrm{d}u\,\mathrm{d}s \leq 2\varepsilon^4\,\|F^\varepsilon\|_\infty^2\,\Gamma\int_0^{\varepsilon^{-2}t}\int_0^s \mathrm{e}^{-\gamma(s-u)}\mathrm{d}u\,\mathrm{d}s \leq \\
&\frac{8\varepsilon^4\Gamma\,\|(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,a\|_\infty^2}{\gamma^2}\left(\varepsilon^{-2}t + \mathrm{e}^{-\gamma\varepsilon^{-2}t} - 1\right).
\end{aligned}
$$

Thus the result follows by letting $\varepsilon \to 0$. As already mentioned analogous relation holds for the predictable quadratic covariation part where we take

$$F^\varepsilon(x) := \left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)^{\mathrm{T}} - \pi^\varepsilon\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)^{\mathrm{T}}\right)\right)(x)\,.$$

■

Example 4.5.2. We end this chapter by showing how one can construct an example of a diffusion process satisfying conditions **(D)**, **(DD)**, **(RT)** and **(J)**. Our example will additionally satisfy $\pi^0(b) = 0$ for this is an important assumption for homogenization results in Chapter 5.

Let $\{X(x,t)\}_{t\geq 0}$ be a diffusion process associated to second-order elliptic differential operator $\mathscr{L}$ of the form

$$\mathscr{L}f(x) = \langle b(x), \nabla f(x)\rangle + \frac{1}{2}\operatorname{Tr}\left(c(x)\nabla\nabla^{\mathrm{T}}f(x)\right).$$

Coefficients $b(x) = (b_i(x))_{i=1,\dots,\mathrm{d}}$ and $c(x) = (c_{ij}(x))_{i,j=1,\dots,\mathrm{d}}$ satisfying conditions **(D)**, **(DD)**, **(RT)**, **(J)** and can be constructed from the tempered Langevin diffusion process. For $i = 1,\dots,\mathrm{d}$ put

$$b_i(x) := 2^{-1}\sum_{j=1}^{\mathrm{d}}\partial_j c_{ij}(x) + \bar{b}_i(x)\,, \qquad x\in\mathbb{R}^\mathrm{d}\,, \tag{4.20}$$

where $\bar{b}_i(x)$ is τ-periodic, of class C^∞, does not depend on x_i, and satisfies $\int_{[0,\tau]} \bar{b}_i(x)\,\mathrm{d}x = 0$. It is then easy to see that $\pi^0(\mathrm{d}x)$ is the Lebegues measure on $\mathbb{T}^{\mathrm{d}}_\tau$ and $\pi^0(b) = 0$, and it is not hard to construct examples satisfying **(D)**, **(DD)**, **(RT)** and **(J)**.

Let $\mathrm{d} = 2$ and $\tau = (10,10)^{\mathrm{T}}$. We take diffusion coefficient $\sigma(x,y) \in C^\infty(\mathbb{R}^2)$ such that $c(x,y) = \sigma(x,y)\sigma(x,y)^{\mathrm{T}}$ is τ-periodic, positive definite on $B_{(5,5)}(3)$ and $c(x,y) \equiv 0$ for $x \in [0,10]\times[0,10] \setminus B_{(5,5)}(3)$. For example we can take

$$\sigma(x,y) := \left(\mathbb{1}_{B_{(5,5)}(3)}(x_\tau, y_\tau) e^{\frac{-1}{9-(x_\tau-5)^2-(y_\tau-5)^2}} \right) \mathbb{I}_2 .$$

It remains to choose $\bar{b}_1(x,y), \bar{b}_2(x,y) \in C^\infty(\mathbb{R}^2)$. Observe that this is enough to satisfy condition **(D)**, condition **(DD)** with $\mathscr{O} = B_{(5,5)}(3)$ and that in condition **(J)** we have $\mathscr{U} = \mathscr{O} = B_{(5,5)}(3)$.

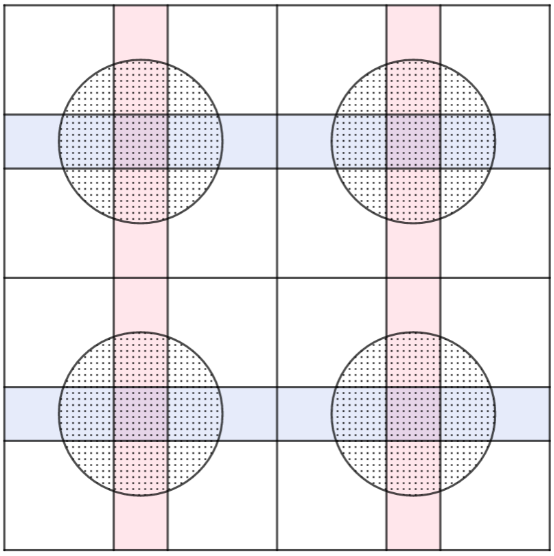

Figure 4.2: Visualization of different areas of domain for drift term b

Notice also that for conditions **(RT)** and **(J)** to be satisfied it is enough to take such $b(x), c(x)$ that there exists $t \geq 0$ such that $\mathbb{P}(\tau^x < t) = 1$ for all $(x,y) \in [0,10]^2 \setminus B_5(3)$, where $\tau^x := \inf\{t \geq 0 \colon X(x,t) \in \cup_{k\in\mathbb{Z}^2} B_{(5,5)}(3) + k\}$.

Take $\bar{b}_1(x,y) = \tilde{b}(y)$ and $\bar{b}_2(x,y) = \tilde{b}(x)$ such that $\tilde{b}(x)$ is τ-periodic and positive for $x \in [0,4]\cup[6,10]$ and on $[4,6]$ define it so that $\int_0^{10} \tilde{b}(x)\,\mathrm{d}x = 0$. For example we can take

$$\tilde{b}(x) := \begin{cases} 1, & x_\tau \in [0,4]\cup[6,10] \\ 1 - \beta e^{\frac{-1}{1-(x_\tau-5)^2}} & x_\tau \in [4,6], \end{cases}$$

where $\beta > 0$ is such that $\int_0^{10} \tilde{b}(x)\,\mathrm{d}x = 0$.

Notice that with such definition of $b(x,y)$ we have that there exists $t \geq 0$ such that for all $(x,y) \in [0,10]^2 \setminus B_5(3)$we have $\mathbb{P}(\tau^x < t) = 1$. Indeed suppose that we take (x,y) from the central white area in Figure 4.2, while we remain in white area we move at the constant speed diagonally up and to the right. We ether hit the upper right circle, right pink line or upper blue line. Since if we hit the circle we are done without loss of generality suppose that we hit the right pink line. While we are in pink area we continue moving to the right but start to go down. Therefore we ether hit the lower right circle of exit the pink line to the right between two circles. But the later is not possible because $\int_0^{10} \tilde{b}(x)\,\mathrm{d}x = 0$ and τ-periodicity of $\tilde{b}(x)$ imply that if the process moves horizontally for 10 it must vertically return to the same height (see Figure 4.3).

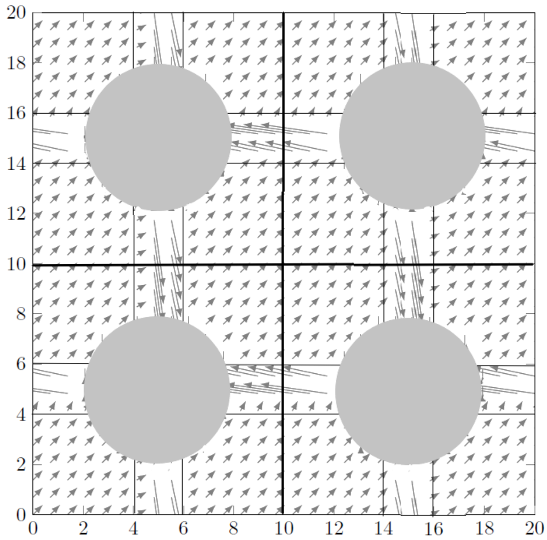

Figure 4.3: Representation of vector field b out side of supp σ

□

5. Homogenization

In this final chapter we discuss periodic homogenisation of LTPs. As mentioned in the introduction we wish to use the results from the previous chapters to solve problems related to homogenization of operators. In the first section we will see how Central limit theorems proven in Charpter 3 and Chapter 4 imply convergence of the corresponding infinitesimal generators. In the remaining chapters we prove that Central limit theorem for diffusion process (Chapter 4) implies that the solutions to the elliptic and parabolic equations corresponding to the infinitesimal generator of this diffusion process converge to the corresponding solution of homogenized equations. We do this by using the famous Feynman-Kac formula, which is formally introduced in Section 5.3 and motivation for which is given in the Section 5.2.

5.1. Convergence of operators

The following theorem is a direct consequence of [15, Theorem 7.1] and Theorem 3.4.1.

Theorem 5.1.1. *Let* $\{X_t\}_{t\geq 0}$ *be a* d*-dimensional LTP with Lévy triplet* $(b(x),c(x),\nu(x,\mathrm{d}y))$*, satisfying* ***(P)***, ***(FI)***, ***(SJ)***, ***(PE)*** *and* ***(GI)***. *If for* $f\in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}\mathscr{L}_\varepsilon f(x) =& \varepsilon^{-1}\langle b(x/\varepsilon),\nabla f(x)\rangle+2^{-1}\mathrm{Tr}\left(c(x/\varepsilon)\,\nabla\nabla^{\mathrm{T}} f(x)\right)\\ &+\varepsilon^{-2}\int_{\mathbb{R}^d}\left(f(x+\varepsilon y)-f(x)-\varepsilon\langle y,\nabla f(x)\rangle \mathbb{1}_{B_1(0)}(y)\right)\nu(x/\varepsilon,\mathrm{d}y),\end{aligned}$$

$\pi(b^*)=\int_{\mathbb{T}_\tau^{\mathrm{d}}} b(x)+\int_{B_1^c(0)} y\,\nu(x,\mathrm{d}y)\,\pi(\mathrm{d}x)$*, where* $\pi(\mathrm{d}x)$ *is a measure whose existence and uniqueness is given by Proposition 3.1.4,*

$$\Sigma=\pi\left((\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\,c\,(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)^{\mathrm{T}}+\int_{\mathbb{R}^{\mathrm{d}}}(y-\beta(\cdot+y)+\beta(\cdot))\,(y-\beta(\cdot+y)+\beta(\cdot))^{\mathrm{T}}\,\nu(\cdot,\mathrm{d}y)\right),$$

where β is a solution to (3.5) whose existence and uniqueness is given by Theorem 3.2.2, then

$$\lim_{\varepsilon\to 0}\left\|\mathscr{L}_\varepsilon f-\varepsilon^{-1}\langle\pi(b^*),\nabla f\rangle-2^{-1}\mathrm{Tr}\left(\Sigma\nabla\nabla^{\mathrm{T}}f\right)\right\|_\infty=0\,,\qquad f\in C_c^\infty(\mathbb{R}^d)\,.$$

As a consequence of [58, Theorem 1.1], [53, Theorem 17.25] and Theorem 4.3.1 or Theorem 4.5.1 we have the following.

Theorem 5.1.2. *For $f\in C_c^2(\mathbb{R}^{\mathrm{d}},\mathbb{R})\cup\{f\in C^2(\mathbb{R}^{\mathrm{d}},\mathbb{R}):\ f(x)$ is τ-periodic$\}$ let*

$$\mathscr{L}_\varepsilon f(x)\,=\,\langle a(x/\varepsilon)+\varepsilon^{-1}b(x/\varepsilon),\nabla f(x)\rangle+2^{-1}\,\mathrm{Tr}\left(c(x/\varepsilon)\,\nabla\nabla^{\mathrm{T}}f(x)\right)\,,$$

*with coefficients $a(x)=(a_i(x))_{i=1,\dots,\mathrm{d}}$, $b(x)=(b_i(x))_{i=1,\dots,\mathrm{d}}$ and $c(x)=(c_{ij}(x))_{i,j=1,\dots,\mathrm{d}}$ satisfying conditions **(D)**, **(DD)**, **(RT)** and **(J)** (or **(D)**, **(DD)**, **(RT)** if $a(x)\equiv 0$ or $b(x)\equiv b\in\mathbb{R}^{\mathrm{d}}$). Let $\pi^0(\mathrm{d}x)$ be a measure whose existence and uniqueness is given by Proposition 4.2.3, let*

$$\mathsf{c}\,=\,\pi^0\left((\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\,c\,(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)^{\mathrm{T}}\right)\qquad\text{and}\qquad\mathsf{b}\,=\,\pi^0\left((\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\,a\right)\,,\tag{5.1}$$

where β is a solution to (4.5) whose existence and uniqueness is given by Corollary 4.2.5, then

$$\lim_{\varepsilon\to 0}\left\|\mathscr{L}_\varepsilon f-\varepsilon^{-1}\langle\pi^0(b),\nabla f\rangle-2^{-1}\mathrm{Tr}\left(\mathsf{c}\nabla\nabla^{\mathrm{T}}f\right)-\langle\mathsf{b},\nabla f\rangle\right\|_\infty=0\,,$$

for $f\in C_c^2(\mathbb{R}^{\mathrm{d}},\mathbb{R})\cup\{f\in C^2(\mathbb{R}^{\mathrm{d}},\mathbb{R}):\ f(x)$ is τ-periodic$\}$.

5.2. MOTIVATION

For the motivational purpose let us, in an informal way, show a probabilistic approach to solving a Dirichlet problem for the Laplace equation. That is let $G \subseteq \mathbb{R}^{\mathrm{d}}$ be an open, bounded and convex set with C^1 boundary ∂G and $f : \partial G \to \mathbb{R}^{\mathrm{d}}$ continuous function. We are interested in a solution $u : \overline{G} \to \mathbb{R}^{\mathrm{d}}$ of an equation

$$\begin{aligned} \Delta u(x) &= 0\,, \qquad \text{for all } x \in G \\ u(x) &= f\,, \qquad \text{for all } x \in \partial G \end{aligned} \tag{5.2}$$

Using the Taylor expansion we have that the Laplacian is given by the following expression

$$\Delta u(x) = \lim_{r \to 0} \frac{\alpha}{r^{d+2}} \int_{B_r(x)} u(y) - u(x)\,\mathrm{d}y$$

To see the connection to stochastic processes consider the following random walk. Let $r > 0$ be an arbitrary small parameter and start the walk at a point $X_0 = x \in G$. At each step move from the point $X_n \in G$ to any point $X_{n+1} \in B_r(X_n)$ chosen uniformly from the ball $B_r(X_n)$. Whenever the segment from X_n to X_{n+1} crosses the boundary ∂G we stop at that point of crossing. Denote by $u_r(x)$ the expected value of function f at the point on the boundary where we have stopped the walk. If we started with a point $x \in \partial G$ then the process does not move and we have $u_r(x) = f(x)$. Otherwise, under the assumption that r is smaller then the distance between the starting point x and the boundary ∂G, we have

$$u_r(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u_r(y)\,\mathrm{d}y\,.$$

From this we conclude that

$$\int_{B_r(x)} u_r(y) - u_r(x)\,\mathrm{d}y = 0\,.$$

We observe that the function u_r “onverges” to the solution to the equation (5.2).

If instead of having X_{n+1} chosen uniformly from the ball $B_r(X_n)$ we choose X_{n+1} uniformly from $E_r(X_n)$, an ellipsoid centered at X_n. The value of r is a scale parameter that shrinks $E_r(X_n)$ to a point as $r \to 0$. In this case function u_r satisfies

$$\int_{E_r(x)} u_r(y) - u_r(x)\,\mathrm{d}y = 0\,,$$

and as $r \to 0$ it "converges" to the solution to the equation

$$\sum_{i,j=1}^{\mathrm{d}} a_{ij}\partial^{ij}u(x) = 0\,, \qquad \text{for all } x \in G$$

$$u(x) = f\,, \qquad \text{for all } x \in \partial G\,,$$

where the coefficients a_{ij} depend on the shape of $E_r(x)$. If we choose a different ellipsoid $E_r(x)$ at different point x, that would lead to x-dependent coefficients $a_{ij}(x)$.

5.3. Feynman-Kac Formula

Feynman-Kac formula establishes a link between partial differential equations and stochastic processes. It states that the solution of linear elliptic and parabolic PDEs is an expectation of certain stochastic process. Let $\mathscr{L}$ be a second-order elliptic differential operator, on the space $C^2\left(\mathbb{R}^{\mathrm{d}}\right)$, of the form

$$\mathscr{L}f(x) = \langle b(x), \nabla f(x)\rangle + \frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^{\mathrm{T}}(x)\nabla\nabla^{\mathrm{T}} f(x)\right), \tag{5.3}$$

where $b \in C\left(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}}\right)$, such that $\langle b(x) - b(y), x-y\rangle \leq c_1|x-y|^2$ for some $c_1 > 0$ and $\sigma \in C\left(\mathbb{R}^{\mathrm{d}}, \mathbb{R}^{\mathrm{d}}\right)$ such that $|\sigma(x) - \sigma(y)| \leq c_2|x-y|$ for some $c_2 > 0$.

We first consider an elliptic boundary-value problem

$$\begin{aligned} \mathscr{L}u(x) + e(x)u(x) + f(x) &= 0, \qquad \text{for all } x \in \mathscr{D} \\ u(x) &= g(x), \qquad \text{for all } x \in \partial\mathscr{D}, \end{aligned} \tag{5.4}$$

where $\mathscr{D}$ is a bounded connected domain with a boundary $\partial\mathscr{D}$ of class C^1 and $e: \mathscr{D} \to (-\infty, 0)$, $f: \mathscr{D} \to \mathbb{R}$ and $g: \partial\mathscr{D} \to \mathbb{R}$ are continuous.

Next we consider an initial-value parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t}u(x,t) + \mathscr{L}u(x,t) + e(x)u(x,t) + f(x) &= 0, \qquad \text{for all } (x,t) \in \mathbb{R}^{\mathrm{d}} \times [0,\infty) \\ u(x,0) &= g(x), \qquad \text{for all } x \in \mathbb{R}^{\mathrm{d}}, \end{aligned} \tag{5.5}$$

where $e \in C_b\left(\mathbb{R}^{\mathrm{d}}, \mathbb{R}\right)$ and $f, g \in C\left(\mathbb{R}^{\mathrm{d}}, \mathbb{R}\right)$.

In order to state Feynman-Kac formula in the most general setting we need to first define the viscosity solutions of elliptic and parabolic equations. This is a generalization that does not require a solution to be of class C^2 but only a continuous function.

Consider first the case of an elliptic equation

$$F(D^2u(x), \nabla u(x), u(x), x) = 0, \qquad \text{for all } x \in \mathscr{D} \tag{5.6}$$

$$u(x) = g(x), \qquad \text{for all } x \in \partial\mathscr{D}, \tag{5.7}$$

where $\mathscr{D} \subseteq \mathbb{R}^{\mathrm{d}}$ is an open set that satisfies the uniform exterior ball condition and whose boundary $\partial\mathscr{D}$ is of class C^1 and $F: \mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times \mathscr{D} \to \mathbb{R}$ satisfies the following two conditions

(i) F is elliptic, that is for any two symmetric matrices $A, B \in \mathbb{S}^{\mathrm{d}}$ so that $A \geq B$, we have

$$F(A, p, u, x) \leq F(B, p, u, x), \qquad \text{for all } p \in \mathbb{R}^{\mathrm{d}}, u \in \mathbb{R}, x \in \mathscr{D}.$$

(ii) F is proper, that is for any two $u, v \in \mathbb{R}$ so that $u \geq v$, we have

$$F(A, p, u, x) \geq F(A, p, v, x), \qquad \text{for all } A \in \mathbb{R}^{\mathrm{d}\times\mathrm{d}}, p \in \mathbb{R}^{\mathrm{d}}, x \in \mathscr{D}.$$

We can rewrite equation (5.4) in the form of equation (5.6) where $F : \mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times \mathscr{D} \to \mathbb{R}$ is defined as

$$F(A, p, u, x) = -\frac{1}{2} \operatorname{Tr}\left(\sigma\sigma^{\mathrm{T}}(x) A\right) - \langle b(x), p \rangle - e(x) u - f(x).$$

Note that F is proper due to the fact that $e(x) \leq 0$ for all $x \in \mathscr{D}$ and is elliptic due to the fact that $\sigma\sigma^{\mathrm{T}}(x) \geq 0$ for all $x \in \mathscr{D}$.

Definition 5.3.1. We say that $u : \mathscr{D} \to \mathbb{R}$ is

(i) a viscosity sub-solution of the equation (5.6) if u is upper semicontinuous in $\mathscr{D}$ and for every function $\varphi \in C^2(\mathscr{D}, \mathbb{R})$ such that $u - \varphi$ has a local maximum at a point $x \in \mathscr{D}$, then $F(D^2\varphi(x), \nabla\varphi(x), u(x), x) \leq 0$.

(ii) a viscosity super-solution of the equation (5.6) if u is lower semicontinuous in $\mathscr{D}$ and for every function $\varphi \in C^2(\mathscr{D}, \mathbb{R})$ such that $u - \varphi$ has a local minimum at a point $x \in \mathscr{D}$, then $F(D^2\varphi(x), \nabla\varphi(x), u(x), x) \geq 0$.

(iii) a viscosity solution of the equation (5.6) if it is a viscosity sub-solution and a super-solution.

Note that $u \in C^2(\mathscr{D}, \mathbb{R})$ is a classical solution of equation (5.6) if and only if it is a solution in the viscosity sense. To see this first suppose it is a classical solution and take $\varphi \in C^2(\mathscr{D}, \mathbb{R})$ such that $u - \varphi$ has a local maximum at a point $x \in \mathscr{D}$. Then $\nabla u(x) = \nabla\varphi(x)$ and $D^2 u(x) \leq D^2\varphi(x)$. From this and the fact that F is elliptic we get

$$F(D^2\varphi(x), \nabla\varphi(x), u(x), x) = F(D^2\varphi(x), \nabla u(x), u(x), x) \leq F(D^2 u(x), \nabla u(x), u(x), x) = 0.$$

This implies that u is a sub-solution and we analogously prove that it is also a super-solution. Let $u \in C^2(\mathscr{D}, \mathbb{R})$ be a solution of the equation (5.6) in the viscosity sense then by taking $\varphi = u$ we see that it is also a classical solution.

Next step is to include the boundary condition (5.7) into the definition of the viscosity solution of an elliptic equation.

Definition 5.3.2. Let $u : \overline{\mathscr{D}} \to \mathbb{R}$ and let $y \in \overline{\mathscr{D}}$.

(i) We say that $(X, p) \in \mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}}$ is in $J^+u(y)$ if there exists a sequence $(X_n, p_n, y_n)_{n\in\mathbb{N}}$ in $\mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \overline{\mathscr{D}}$ such that $(X_n, p_n, u(y_n), y_n) \overset{(n\to\infty)}{\longrightarrow} (X, p, u(y), y)$ and

$$\limsup_{\mathscr{D}\ni z\to y_n} \frac{u(z) - u(y_n) - \langle p_n, z - y_n\rangle - \frac{1}{2}\langle X_n(z - y_n), z - y_n\rangle}{|z - y_n|^2} \leq 0 \quad \text{for all } n \in \mathbb{N}.$$

We say that u is a viscosity sub-solution of (5.6) and (5.7) if u is upper semicontinuous in $\overline{\mathscr{D}}$ and

- $F(X, p, u(x), x) \leq 0$ for $x \in \mathscr{D}$ and $(X, p) \in J^+u(x)$
- $F(X, p, u(x), x) \leq 0$ or $u(x) \leq g(x)$ for $x \in \partial\mathscr{D}$ and $(X, p) \in J^+u(x)$

(ii) We say that $(X, p) \in \mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}}$ is in $J^-u(y)$ if there exists a sequence $(X_n, p_n, y_n)_{n\in\mathbb{N}}$ in $\mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \overline{\mathscr{D}}$ such that $(X_n, p_n, u(y_n), y_n) \overset{(n\to\infty)}{\longrightarrow} (X, p, u(y), y)$ and

$$\liminf_{\mathscr{D}\ni z\to y_n} \frac{u(z) - u(y_n) - \langle p_n, z - y_n\rangle - \frac{1}{2}\langle X_n(z - y_n), z - y_n\rangle}{|z - y_n|^2} \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

We say that u is a viscosity super-solution of (5.6) and (5.7) if u is lower semicontinuous in $\overline{\mathscr{D}}$ and

- $F(X, p, u(x), x) \geq 0$ for $x \in \mathscr{D}$ and $(X, p) \in J^-u(x)$
- $F(X, p, u(x), x) \geq 0$ or $u(x) \geq g(x)$ for $x \in \partial\mathscr{D}$ and $(X, p) \in J^-u(x)$

(iii) We say that $u \in C(\overline{\mathscr{D}}, \mathbb{R})$ is a viscosity solution of (5.6) and (5.7) if it is both viscosity sub-solution and super-solution.

Now we are ready to state the Feynman-Kac Formula in this general setting. According to [76, Theorem 3.49]

Theorem 5.3.3. *Let $\{D(x,t)\}_{t\geq 0}$ be a diffusion process associated to a second-order elliptic differential operator $\mathscr{L}$ given in equation (5.3). Define the stopping time*

$$\tau^x := \inf\left\{t \geq 0, D(x,t) \notin \overline{\mathscr{D}}\right\}$$

and assume that

$$\Lambda := \{x \in \partial\mathscr{D} : \mathbb{P}(\tau^x > 0) = 0\}$$

is a (topologically) closed set. Then

$$u(x) := \mathbb{E}\left[g\left(D(x,\tau^x)\right) \mathrm{e}^{\int_0^{\tau^x} e(D(x,s))\,\mathrm{d}s} + \int_0^{\tau} f\left(D(x,s)\right) \mathrm{e}^{\int_0^{s} e(D(x,u))\,\mathrm{d}u}\,\mathrm{d}s\right]$$

is continuous on $\overline{\mathscr{D}}$ and is a unique continuous viscosity solution to (5.4)*.*

We proceed with similar reasoning in the case of parabolic equation

$$\frac{\partial u}{\partial t}(x,t) = F(D^2u(x,t), \nabla u(x,t), u(x,t), x, t)\,, \qquad \text{for all } (x,t) \in \mathbb{R}^{\mathrm{d}} \times [0,\infty) \tag{5.8}$$

$$u(x,0) = g(x)\,, \qquad \text{for all } x \in \mathbb{R}^{\mathrm{d}}. \tag{5.9}$$

where $F : \mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times [0,\infty) \to \mathbb{R}$ satisfies the following two conditions

(i) F is elliptic, that is for any two symmetric matrices $A, B \in \mathbb{S}^{\mathrm{d}}$ so that $A \geq B$, we have

$$F(A,p,u,x,t) \leq F(B,p,u,x,t)\,, \qquad \text{for all } p \in \mathbb{R}^{\mathrm{d}}, u \in \mathbb{R}, x \in \mathbb{R}^{\mathrm{d}}, t \in [0,\infty)\,.$$

(ii) F is proper, that is for any two $u, v \in \mathbb{R}$ so that $u \geq v$, we have

$$F(A,p,u,x,t) \geq F(A,p,v,x,t)\,, \qquad \text{for all } A \in \mathbb{R}^{\mathrm{d}\times\mathrm{d}}, p \in \mathbb{R}^{\mathrm{d}}, x \in \mathbb{R}^{\mathrm{d}}, t \in [0,\infty)\,.$$

We can rewrite equation (5.5) in the form of equation (5.8) where $F : \mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times [0,\infty) \to \mathbb{R}$ is defined as

$$F\left(A,p,u,x,t\right) = -\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^{\mathrm{T}}(x)A\right) - \langle b(x), p\rangle - e(x)u - f(x)\,.$$

Definition 5.3.4. Let $u : \mathbb{R}^{\mathrm{d}} \times [0,\infty) \to \mathbb{R}$ and let $(y,s) \in \mathbb{R}^{\mathrm{d}} \times [0,\infty)$.

(i) We say that $(X,p,q) \in \mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$ is in $P^{+}u(y,s)$ if there exists $(X_n, p_n, q_n, y_n, s_n)_{n\in\mathbb{N}}$, a sequence in $\mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times [0,\infty)$ such that $(X_n, p_n, q_n, u(y_n,s_n), y_n, s_n) \overset{(n\to\infty)}{\longrightarrow} (X,p,q,u(y),y,s)$ and for all $n \in \mathbb{N}$

$$\limsup_{\substack{(z,r)\to(y_n,s_n)\\ \mathbb{R}^{\mathrm{d}}\times[0,\infty)}} \frac{u(z,r) - u(y_n,s_n) - q_n(r-s_n) - \langle p_n, z-y_n\rangle - \frac{1}{2}\langle X_n(z-y_n), z-y_n\rangle}{|r-s_n| + |z-y_n|^2} \leq 0\,.$$

We say that u is a viscosity sub-solution of (5.8) and (5.9) if u is upper semicontinuous in $\mathbb{R}^{\mathrm{d}} \times [0,\infty)$ and

- $F(X,p,u(x,t),x,t)+q \le 0$ for $(x,t) \in \mathbb{R}^{\mathrm{d}} \times [0,\infty)$ and $(X,p,q) \in P^{+}u(x,t)$
- $u(x,0) \le g(x)$ for all $x \in \mathbb{R}^{\mathrm{d}}$.

(i) We say that $(X,p,q) \in \mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$ is in $P^{-}u(y,s)$ if there exists $(X_n,p_n,q_n,y_n,s_n)_{n\in\mathbb{N}}$, a sequence in $\mathbb{S}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times [0,\infty)$ such that $(X_n,p_n,q_n,u(y_n,s_n),y_n,s_n) \overset{(n\to\infty)}{\longrightarrow} (X,p,q,u(y),y,s)$ and for all $n \in \mathbb{N}$

$$\liminf_{\substack{(z,r)\to(y_n,s_n)\\ \mathbb{R}^{\mathrm{d}}\times[0,\infty)}} \frac{u(z,r)-u(y_n,s_n)-q_n(r-s_n)-\langle p_n, z-y_n\rangle - \frac{1}{2}\langle X_n(z-y_n), z-y_n\rangle}{|r-s_n|+|z-y_n|^2} \ge 0\,.$$

We say that u is a viscosity super-solution of (5.8) and (5.9) if u is lower semicontinuous in $\mathbb{R}^{\mathrm{d}} \times [0,\infty)$ and

- $F(X,p,u(x,t),x,t)+q \ge 0$ for $(x,t) \in \mathbb{R}^{\mathrm{d}} \times [0,\infty)$ and $(X,p,q) \in P^{-}u(x,t)$
- $u(x,0) \ge g(x)$ for all $x \in \mathbb{R}^{\mathrm{d}}$.

(iii) We say that $u \in C(\mathbb{R}^{\mathrm{d}} \times [0,\infty),\mathbb{R})$ is a viscosity solution of (5.8) and (5.9) if it is both viscosity sub-solution and super-solution.

Notice that if $u \in C^2\left(\mathbb{R}^{\mathrm{d}},\mathbb{R}\right)$ is a classical solution to (5.5) then for $\lambda > 0$ function $v(x,t) := u(x,t)\mathrm{e}^{\lambda t}$ is a solution tu

$$\frac{\partial v}{\partial t}v(x,t)+\mathscr{L}v(x,t)+(e(x)-\lambda)\,v(x,t)+f(x)=0\,, \qquad \text{for all } (x,t) \in \mathbb{R}^{\mathrm{d}} \times [0,\infty)$$

$$v(x,0)=g(x)\,, \qquad \text{for all } x \in \mathbb{R}^{\mathrm{d}}\,.$$

Indeed, we have

$$\begin{aligned}
&\frac{\partial v}{\partial t}(x,t)+\mathscr{L}v(x,t)+(e(x)-\lambda)\,v(x,t)+f(x) = \\
&\mathrm{e}^{\lambda t}\frac{\partial u}{\partial t}(x,t)+\lambda\mathrm{e}^{\lambda t}u(x,t)+\mathrm{e}^{\lambda t}\mathscr{L}u(x,t)+(e(x)-\lambda)\,\mathrm{e}^{\lambda t}u(x,t)+f(x) = \\
&\mathrm{e}^{\lambda t}\left(\frac{\partial u}{\partial t}u(x,t)+\mathscr{L}u(x,t)+e(x)u(x,t)+f(x)\right) = 0\,, \qquad \text{for all } (x,t) \in \mathbb{R}^{\mathrm{d}} \times [0,\infty) \\
&v(x,0) = u(x,0) = g(x)\,, \qquad \text{for all } x \in \mathbb{R}^{\mathrm{d}}\,.
\end{aligned}$$

The same holds for viscosity solutions and therefore, since again F is proper if $e \le 0$, in the case of parabolic equation we can assume that e is only bounded.

Now we are ready to state the Feynman-Kac Formula for parabolic equation in this general setting. According to [76, Theorem 3.43], (see also [75, Remark 2.5])

Theorem 5.3.5. *Let $\{D(x,t)\}_{t\geq 0}$ be a diffusion process associated to a second-order elliptic differential operator $\mathscr{L}$ given in equation (5.3) where coefficients b and σ are also bounded. Assume additionally that $f, g \in C(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$ are such that*

$$|f(x)| + |g(x)| \leq K\,(1 + |x|^{\kappa})$$

for some $\kappa, K > 0$ and all $x \in \mathbb{R}^{\mathrm{d}}$.Then

$$u(x,t) \coloneqq \mathbb{E}\left[g\,(D(x,t))\,\mathrm{e}^{\int_0^t e(D(x,s))\,\mathrm{d}s} + \int_0^t f\,(D(x,s))\,\mathrm{e}^{\int_0^s e(D(x,u))\,\mathrm{d}u}\,\mathrm{d}s\right]$$

is a continuous function of $(x,t) \in \mathbb{R}^{\mathrm{d}} \times [0,\infty)$ which grows at most polynomially at infinity, and it is the unique viscosity solution of (5.5).

5.4. Elliptic Boundary-value Problem

In this section we assume that $\{X^{\varepsilon}(x,t)\}_{t\geq 0}$ is a diffusion process associated to a second-order elliptic differential operator of the form

$$\mathscr{L}^{\varepsilon} f(x) = \langle a(x/\varepsilon)+\varepsilon^{-1}b(x/\varepsilon), \nabla f(x)\rangle + \frac{1}{2}\operatorname{Tr}\left(c(x/\varepsilon)\nabla\nabla^{\mathrm{T}} f(x)\right),$$

with coefficients $a(x)=(a_i(x))_{i=1,\dots,\mathrm{d}}$, $b(x)=(b_i(x))_{i=1,\dots,\mathrm{d}}$ and $c(x)=(c_{ij}(x))_{i,j=1,\dots,\mathrm{d}}$ satisfying **(D)**-**(J)** (or **(D)**-**(RT)** if $c(x)\equiv 0$ or $b(x)\equiv 0$) and we additionally assume that $\pi^0(b)=0$, where $\pi^0(\mathrm{d}x)$ is a measure whose existence and uniqueness is given by Proposition 4.2.3.

Let $\mathscr{D}$ is an open bounded subset of $\mathbb{R}^{\mathrm{d}}$, regular in the sense that there exists a function $\phi\in C_b^2(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ such that

- $\mathscr{D}=\{x:\phi(x)<0\}$ and
- $|\nabla\phi(x)|\geq\delta>0$, for all $x\in\partial\mathscr{D}$.

Further, let $e\colon\mathscr{D}\to(-\infty,-\alpha]$, $\alpha>0$, $f\colon\mathscr{D}\to\mathbb{R}$ and $g\colon\partial\mathscr{D}\to\mathbb{R}$ be continuous. Define a stopping time $\tau_x^{\varepsilon}:=\inf\{t\geq 0\colon X^{\varepsilon}(x,t)\notin\overline{\mathscr{D}}\}$ and let

$$\Lambda^{\varepsilon} := \{x\in\partial\mathscr{D}\colon \mathbb{P}(\tau_x^{\varepsilon}>0)=0\},$$

Assume that Λ^{ε} is a (topologically) closed set for all $\varepsilon\in(0,\varepsilon_0]$. Then, according to Theorem 5.3.3,

$$u^{\varepsilon}(x) := \mathbb{E}\left[g\left(X^{\varepsilon}(x,\tau_x^{\varepsilon})\right)\mathrm{e}^{\int_0^{\tau_x^{\varepsilon}} e(X^{\varepsilon}(x,s)/\varepsilon)\,\mathrm{d}s}+\int_0^{\tau_x^{\varepsilon}} f\left(X^{\varepsilon}(x,s)\right)\mathrm{e}^{\int_0^{s} e(X^{\varepsilon}(x,u)/\varepsilon)\,\mathrm{d}u}\,\mathrm{d}s\right]$$

is a unique continuous viscosity solution to

$$\begin{aligned}\mathscr{L}^{\varepsilon}u^{\varepsilon}(x)+e(x/\varepsilon)\,u^{\varepsilon}(x)+f(x) &= 0, \qquad x\in\mathscr{D},\\ u^{\varepsilon}(x) &= g(x), \qquad x\in\partial\mathscr{D},\end{aligned}$$

Theorem 5.4.1. *In addition to the above assumptions assume that $e(x)$ is τ-periodic. Let $\{W^{\mathsf{b},\mathsf{c}}(x,t)\}_{t\geq 0}$ be a d-dimensional Brownian motion determined by drift vector b and covariance matrix c given in (5.1). Let $\tau_x^0:=\inf\{t\geq 0\colon W^{\mathsf{b},\mathsf{c}}(x,t)\notin\overline{\mathscr{D}}\}$ and assume* *that* $\Lambda^0 := \{x \in \partial\mathscr{D} \colon \mathbb{P}(\tau_x^0 > 0) = 0\}$ *is a (topologically) closed set and assume that* $(\nabla\phi(x))^{\mathrm{T}}\mathsf{c}\nabla\phi(x) > 0$ *for all* $x \in \partial\mathscr{D}$. *If*

$$u^0(x) := \mathbb{E}\left[g\left(W^{\mathsf{b},\mathsf{c}}(x,\tau_x^0)\right)\mathrm{e}^{\pi^0(e)\tau_x^0} + \int_0^{\tau_x^0} f\left(W^{\mathsf{b},\mathsf{c}}(x,s)\right)\mathrm{e}^{\pi^0(e)s}\mathrm{d}s\right]$$

is a unique continuous viscosity solution to

$$\langle \mathsf{b}, \nabla u^0\rangle(x) + 2^{-1}\mathrm{Tr}\left(\mathsf{c}\nabla\nabla^{\mathrm{T}}u^0\right)(x) + \pi^0(e)\,u^0(x) + f(x) = 0\,, \qquad x \in \mathscr{D}\,,$$
$$u^0(x) = g(x)\,, \qquad x \in \partial\mathscr{D}\,,$$

given by Theorem 5.3.3, then

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = u^0(x) \qquad \forall x \in \mathscr{D}\,.$$

Proof. We follow the approach from [12, Theorem 3.4.5]. Define

$$\xi^\varepsilon(0,t) := \int_0^t e\left(X^\varepsilon(x,s)/\varepsilon\right)\mathrm{d}s = \varepsilon^2\int_0^{\varepsilon^{-2}t} e\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s\,,$$

where we used $\tilde{X}^\varepsilon(x,t) = \varepsilon^{-1}X^\varepsilon(\varepsilon x,\varepsilon^2 t)$. Since e is τ-periodic analogously as in the proof of Theorem 4.5.1 we see that

$$\int_0^t e\left(X^\varepsilon(x,s)/\varepsilon\right)\mathrm{d}s \xrightarrow[\varepsilon\to 0]{\mathrm{L}^2(\mathbb{P})} \pi^0(e)\,t\,.$$

Set $\xi(0,t) := \pi^0(e)\,t$. This together with the fact that the process $\{\xi^\varepsilon(0,t)\}_{t\geq 0}$ is tight, due to [50, Theorem VI.3.21], implies that $\{\xi^\varepsilon(0,t)\}_{t\geq 0} \xRightarrow[\varepsilon\to 0]{(\mathrm{d})} \{\xi(0,t)\}_{t\geq 0}$. Since $\{\xi(0,t)\}_{t\geq 0}$ is a constant in the space $C([0,\infty),\mathbb{R})$ using Theorem 4.5.1 we conclude

$$\{(X^\varepsilon,\xi^\varepsilon)(x_0,t)\}_{t\geq 0} \xRightarrow[\varepsilon\to 0]{(\mathrm{d})} \{(W^{\mathsf{a},\mathsf{b}},\xi)(x_0,t)\}_{t\geq 0}\,, \tag{5.10}$$

where x_0 denotes a $(\mathrm{d}+1)$-dimensional vector with first d coordinates being equal to x and the last coordinate being 0. Let $\psi := C([0,\infty),\mathbb{R}^{\mathrm{d}}) \times C([0,\infty),\mathbb{R})$ and provide it with Borel σ-algebra generated by the sets $\left\{(W^{\mathsf{a},\mathsf{b}},\xi)(x,\cdot) \in \psi \mid (W^{\mathsf{a},\mathsf{b}},\xi)(x,s) \in B\right\}$ where $s \in [0,\infty), B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}+1})$. The processes $\{(X^\varepsilon,\xi^\varepsilon)(x,t)\}_{t\geq 0}$ and $\{(W^{\mathsf{a},\mathsf{b}},\xi)(x,t)\}_{t\geq 0}$ introduce on ψ probability measures $\mu_x^\varepsilon(B) = \mathbb{P}((X^\varepsilon,\xi^\varepsilon)(x,\cdot) \in B)$, $B \in \mathscr{B}(\psi)$ and $\mu_x(B) = \mathbb{P}_x\left((W^{\mathsf{a},\mathsf{b}},\xi)(x,\cdot) \in B\right)$, $B \in \mathscr{B}(\psi)$.

We now define functional $F : \psi \to \mathbb{R} \cup \{\infty\}$

$$F(y,z) = \begin{cases} g(y(\tau(y)))\mathrm{e}^{z(\tau(y))} + \int_0^{\tau(y)} f(y(t))\mathrm{e}^{z(t)}\,\mathrm{d}t, & \text{if } \tau(y) < \infty \text{ and } z(t) \leq -\alpha t, \forall t \geq 0 \\ \int_0^\infty f(y(t))\mathrm{e}^{z(t)}\,\mathrm{d}t, & \text{if } \tau(y) = \infty \text{ and } z(t) \leq -\alpha t, \forall t \geq 0 \\ \infty, & \text{otherwise} \end{cases}$$

where $\tau(y) = \inf\{t : t \geq 0, y(t) \notin \overline{\mathscr{D}}\}$. Clearly we have

$$u^{\varepsilon}(x) = \mathbb{E}\left[F\left(\{(X^{\varepsilon}, \xi^{\varepsilon})(x_0,t)\}_{t\geq 0}\right)\right] \quad \text{and} \quad u(x) = \mathbb{E}\left[F\left(\{(W^{\mathsf{a},\mathsf{b}}, \xi)(x_0,t)\}_{t\geq 0}\right)\right],$$

which explains why we define $F(y,z)$ in this manner. Next, we can assert the following properties

(i) F is measurable and bounded a.s. with respect to μ_x^{ε} and μ_x

(ii) F is continuous a.s. with respect to μ_x.

To see that property (i) holds note that if $z(t) \leq -\alpha t$, for all $t \geq 0$ then

$$\begin{aligned} |F(y,z)| &= \left| g(y(\tau(y)))\mathrm{e}^{z(\tau(y))} + \int_0^{\tau(y)} f(y(t))\mathrm{e}^{z(t)}\mathrm{d}t \right| \leq \|g\|_{\infty} + \|f\|_{\infty} \int_0^{\infty} \mathrm{e}^{-\alpha t}\,\mathrm{d}t \\ &= \|g\|_{\infty} + \|f\|_{\infty}\frac{1}{\alpha} < \infty. \end{aligned}$$

Due to the definition of processes $\{\xi^{\varepsilon}(0,t)\}_{t\geq 0}$, $\{\xi(0,t)\}_{t\geq 0}$ and the fact that for all $x \in \mathbb{R}^{\mathrm{d}}$ we have $e(x) \leq -\alpha$ property (i) follows.

To see property (ii), we need to check that if $\{y_n\}_{n\in\mathbb{N}}$ converges to y uniformly on compact intervals, then

$$\lim_{n\to\infty} \mathrm{F}(y_n, \zeta) = \mathrm{F}(y, \zeta) \tag{5.11}$$

Recall that $\zeta(0,t) \leq -\alpha t$ for all $t \geq 0$. The relation in (5.11) will follow from the proof of [12, Lemma 3.4.3] where they prove statement (4.95) that is, if $y_n \to y$ uniformly on compact intervals and $(\nabla\phi(x))^{\mathrm{T}}\mathsf{c}\nabla\phi(x) > 0$ for all $x \in \partial\mathscr{D}$ (see the definition of the function $\beta(t)$ in [12, pp. 412]), then $\tau(y_n) \to \tau(y)$. The small difference to our case is that they define $\tau(y) = \inf\{t : t \geq 0, y(t) \notin \mathscr{D}\}$, but this makes no difference since in the proof of the same lemma they also show that these two stopping times coincide a.s.

To see that τ being continuous implies (5.11) first note that from this statement we trivially get that $\int_0^{\tau(y_n)} f(y(t))\mathrm{e}^{\zeta(0,t)}\mathrm{d}t \to \int_0^{\tau(y)} f(y(t))\mathrm{e}^{\zeta(0,t)}\mathrm{d}t$ and the rest we conclude as follows.

If $\tau(y) = \infty$, then $\lim_{n\to\infty}\tau(y_n) = \infty$ then, since g is bounded and $\zeta(0,t) \leq -\alpha t$ for all $t \geq 0$, we have $\lim_{n\to\infty} g(y_n(\tau(y_n)))\mathrm{e}^{\zeta(0,\tau(y_n))} = 0$.

If $\tau(y) < \infty$, then there exists $T > 0$ such that $\tau(y_n) \in [0,T]$ for all $n \geq n_y$ and therefore

$$\begin{aligned}
&\left| g\left(y_n\left(\tau(y_n)\right)\right) \mathrm{e}^{\zeta(0,\tau(y_n))} - g\left(y\left(\tau(y)\right)\right) \mathrm{e}^{\zeta(0,\tau(y))} \right| \\
&\leq \|g\|_\infty \left| \mathrm{e}^{\zeta(0,\tau(y_n))} - \mathrm{e}^{\zeta(0,\tau(y))} \right| + \mathrm{e}^{\zeta(0,\tau(y))} \left| g\left(y_n\left(\tau(y_n)\right)\right) - g\left(y\left(\tau(y_n)\right)\right) \right| \\
&\quad + \mathrm{e}^{\zeta(0,\tau(y))} \left| g\left(y\left(\tau(y_n)\right)\right) - g\left(y\left(\tau(y)\right)\right) \right| \\
&\leq \|g\|_\infty \left| \mathrm{e}^{\zeta(0,\tau(y_n))} - \mathrm{e}^{\zeta(0,\tau(y))} \right| + \mathrm{e}^{\zeta(0,\tau(y))} \sup_{0 \leq t \leq T} \left| g\left(y_n(t)\right) - g\left(y(t)\right) \right| \\
&\quad + \mathrm{e}^{\zeta(0,\tau(y))} \left| g\left(y\left(\tau(y_n)\right)\right) - g\left(y\left(\tau(y)\right)\right) \right| .
\end{aligned}$$

Clearly, the first and last terms in the above inequality tend to zero as n tends to infinity. Suppose that $\limsup_{n\to\infty} \sup_{0\leq t\leq T} |g(y_n(t)) - g(y(t))| > 0$. Then there exist $\varepsilon > 0$ and sequences $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and $\{t_k\}_{k\in\mathbb{N}} \subseteq [0,T]$, such that $\lim_{k\to\infty} t_k = t \in [0,T]$ and $|g(y_{n_k}(t_k)) - g(y(t_k))| > \varepsilon$ for all $k \in \mathbb{N}$. However, since $\lim_{k\to\infty} g(y(t_k)) = g(y(t))$, and

$$\begin{aligned}
\lim_{k\to\infty} |y_{n_k}(t_k) - y(t)| &\leq \lim_{k\to\infty} |y_{n_k}(t_k) - y(t_k)| + \lim_{k\to\infty} |y(t_k) - y(t)| \\
&\leq \lim_{k\to\infty} \sup_{0\leq s\leq T} |y_{n_k}(s) - y(s)| + \lim_{k\to\infty} |y(t_k) - y(t)| \\
&= 0 ,
\end{aligned}$$

this is not possible. From this we conclude that

$$\lim_{n\to\infty} \left| g\left(y_n\left(\tau(y_n)\right)\right) \mathrm{e}^{\zeta(0,\tau(y_n))} - g\left(y\left(\tau(y)\right)\right) \mathrm{e}^{\zeta(0,\tau(y))} \right| = 0 ,$$

which proves the assertion. ∎

5.5. Initial-value Parabolic Problem

In this section we again assume that $\{X_t^\varepsilon\}_{t\geq 0}$ is a diffusion process associated to a second-order elliptic differential operator of the form

$$\mathscr{L}^\varepsilon f(x) = \langle a(x/\varepsilon) + \varepsilon^{-1} b(x/\varepsilon), \nabla f(x)\rangle + \frac{1}{2}\mathrm{Tr}\left(c(x/\varepsilon)\nabla\nabla^{\mathrm{T}} f(x)\right),$$

with coefficients $a(x) = (a_i(x))_{i=1,\dots,\mathrm{d}}$, $b(x) = (b_i(x))_{i=1,\dots,\mathrm{d}}$ and $c(x) = (c_{ij}(x))_{i,j=1,\dots,\mathrm{d}}$ satisfying **(D)**-**(J)** (or **(D)**-**(RT)** if $c(x)\equiv 0$ or $b(x)\equiv 0$) and, again, we additionally assume that $\pi^0(b) = 0$, where $\pi^0(\mathrm{d}x)$ is a measure whose existence and uniqueness is given by Proposition 4.2.3.

Let $d, e \in C_b(\mathbb{R}^{\mathrm{d}},\mathbb{R})$, and let $f, g \in C(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ be such that

$$|f(x)| + |g(x)| \leq K(1+|x|^\kappa) \tag{5.12}$$

for some $\kappa, K > 0$ and all $x\in\mathbb{R}^{\mathrm{d}}$. Then, according to Theorem 5.3.5 for any $\varepsilon > 0$,

$$\begin{aligned} u^\varepsilon(x,t) := \mathbb{E}\Big[&g(X^\varepsilon(x,t))\, \mathrm{e}^{\int_0^t\left(\varepsilon^{-1}d(X^\varepsilon(x,s)/\varepsilon)+e(X^\varepsilon(x,s)/\varepsilon)\right)\mathrm{d}s} + \\ &\int_0^t f(X^\varepsilon(x,s))\, \mathrm{e}^{\int_0^s\left(\varepsilon^{-1}d(X^\varepsilon(x,u)/\varepsilon)+e(X^\varepsilon(x,u)/\varepsilon)\right)\mathrm{d}u}\,\mathrm{d}s\Big] \end{aligned}$$

is a viscosity solution to

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t}u(x,t) &= \mathscr{L}^\varepsilon u^\varepsilon(x,t) + \left(\varepsilon^{-1}d(x/\varepsilon) + e(x/\varepsilon)\right)u^\varepsilon(x,t) + f(x), \ \forall (x,t)\in\mathbb{R}^{\mathrm{d}}\times[0,\infty) \\ u^\varepsilon(x,0) &= g(x), \qquad \forall x\in\mathbb{R}^{\mathrm{d}}. \end{aligned}$$

Assume further that $d\in C^1(\mathbb{R}^{\mathrm{d}},\mathbb{R})$, is τ-periodic and such that $\pi^0(d) = 0$ (otherwise we can just replace $d(x)$ by $d(x) - \pi^0(d)$). Then according to Proposition 4.4.1, conditions **(D)**-**(J)** imply that

$$\delta(x) := -\int_0^\infty \tilde{P}_t^0 d(x)\,\mathrm{d}t, \qquad x\in\mathbb{R}^{\mathrm{d}},$$

is well defined, τ-periodic, continuously differentiable and $\delta\in\mathscr{D}_{\tilde{\mathscr{A}}^0}$ and $\tilde{\mathscr{A}}^0\delta(x) = d(x)$.

Theorem 5.5.1. *In addition to the above assumptions, assume $d\in C^2(\mathbb{R}^{\mathrm{d}},\mathbb{R})$ and e is τ-periodic. Let $\{\mathsf{W}_t^{\bar{\mathsf{b}},\mathsf{c}}\}_{t\geq 0}$ be a* d*-dimensional Brownian motion determined by drift vector* $\bar{\mathsf{b}} := \mathsf{b} - \pi^0((\mathbb{I}_{\mathrm{d}} - \mathrm{D}\beta)\,c\nabla\delta)$ *and covariance matrix* c *given in* (5.1)*. Let* $\mathsf{e}:\mathbb{R}^{\mathrm{d}}\to\mathbb{R}$ *be given by* $\mathsf{e} = 2^{-1}(\nabla\delta)^{\mathrm{T}} c\nabla\delta + e - (\nabla\delta)^{\mathrm{T}} a$*. If*

$$u^0(t,x) := \mathbb{E}\left[g(W^{\bar{\mathsf{b}},\mathsf{c}}(x,t))\,\mathrm{e}^{\pi^0(\mathsf{e})t} + \int_0^t f(W^{\bar{\mathsf{b}},\mathsf{c}}(x,s))\,\mathrm{e}^{\pi^0(\mathsf{e})s}\mathrm{d}s\right]$$

is a unique continuous viscosity solution to

$$\partial_t u^0(x,t) = \langle \bar{\mathsf{b}}, \nabla u^0\rangle(x,t) + 2^{-1}\mathrm{Tr}\left(\mathsf{c}\nabla\nabla^{\mathrm{T}} u^0\right)(x,t) + \pi^0(\mathsf{e})\, u^0(x,t) + f(x)\,,\ \forall (x,t)\in\mathbb{R}^{\mathrm{d}}\times[0,\infty)\,,$$
$$u^0(x,0) = g(x)\,, \qquad \forall x\in\mathbb{R}^{\mathrm{d}}\,,$$

given by Theorem 5.3.5, then

$$\lim_{\varepsilon\to 0} u^\varepsilon(t,x) = u^0(t,x) \qquad \forall\,(t,x)\in[0,\infty)\times\mathbb{R}^{\mathrm{d}}\,.$$

Proof. We first assume that $f\equiv 0$ and show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[g\left(X^\varepsilon(x,t)\right)\mathrm{e}^{\int_0^t\left(\varepsilon^{-1}d(X^\varepsilon(x,s)/\varepsilon)+e(X^\varepsilon(x,s)/\varepsilon)\right)\mathrm{d}s}\right] = \mathbb{E}\left[g(W^{\bar{\mathsf{b}},\mathsf{c}}(x,t))\right]\mathrm{e}^{\pi^0(\mathsf{e})\,t}\,.$$

Clearly, $\varepsilon^{-1}\int_0^t d\left(X^\varepsilon(x,s)/\varepsilon\right)\mathrm{d}s$ is the most problematic term in the above representation of $u^\varepsilon(x,t)$. By using $\tilde{X}^\varepsilon(x,t) = \varepsilon^{-1}X^\varepsilon(\varepsilon x,\varepsilon^2 t)$, $t\geq 0$ we can write it as

$$\varepsilon^{-1}\int_0^t d\left(X^\varepsilon(x,s)/\varepsilon\right)\mathrm{d}s = \varepsilon\int_0^{\varepsilon^{-2}t} d\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s\,.$$

From Lemma 4.4.2 applied to function d and the fact that $\pi^0(d)=0$, for every $t\geq 0$ we get

$$\begin{aligned}\delta\left(\tilde{X}^\varepsilon(x,t)\right) &= \delta(x) + \int_0^t d\left(\tilde{X}^\varepsilon(x,s)\right)\mathrm{d}s + \varepsilon\int_0^t\langle\nabla\delta,a\rangle\left(\tilde{X}^\varepsilon(x,s)\right)\mathrm{d}s\\ &\quad+\int_0^t(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^\varepsilon(x,s)\right)\mathrm{d}W_s^\varepsilon\,,\end{aligned}$$

and therefore we can express the term $\varepsilon^{-1}\int_0^t d\left(X^\varepsilon(x,s)/\varepsilon\right)\mathrm{d}s$ using function δ

$$\begin{aligned}\varepsilon^{-1}\int_0^t d\left(X^\varepsilon(x,s)/\varepsilon\right)\mathrm{d}s &= \varepsilon\delta\left(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right) - \varepsilon\delta(x/\varepsilon) - \varepsilon^2\int_0^{\varepsilon^{-2}t}\langle\nabla\delta,a\rangle\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s\\ &\quad-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon\,.\end{aligned}$$

Which implies

$$\begin{aligned}u^\varepsilon(x,t) = \mathbb{E}\Big[&g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\\ &\mathrm{e}^{\varepsilon\delta\left(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)-\varepsilon\delta(x/\varepsilon)-\varepsilon^2\int_0^{\varepsilon^{-2}t}(\langle\nabla\delta,a\rangle-e)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon}\Big]\,.\end{aligned}$$

Since δ is bounded we know that $\mathrm{e}^{\varepsilon\delta\left(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)-\varepsilon\delta(x/\varepsilon)}\xrightarrow[\varepsilon\to 0]{\mathrm{L}^p(\mathbb{P})} 1$, for $p\geq 1$. For this reason it is reasonable to suspect that $u^\varepsilon(x,t)$ converges as $\varepsilon\to 0$ if, and only if,

$$\mathbb{E}\left[g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\mathrm{e}^{-\varepsilon^2\int_0^{\varepsilon^{-2}t}(\langle\nabla\delta,a\rangle-e)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon}\right]$$

converges, and if this is the case that the limit is the same. To see this formally we will use Cauchy inequality,

$$\left|u^\varepsilon(x,t) - \mathbb{E}\left[g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\, \mathrm{e}^{-\varepsilon^2\int_0^{\varepsilon^{-2}t}(\langle\nabla\delta,a\rangle-e)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon}\right]\right|$$
$$\leq \mathbb{E}\left[\left|g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\right|^2\right]^{1/2}\mathbb{E}\left[\left|\mathrm{e}^{\varepsilon\delta\left(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)-\varepsilon\delta(x/\varepsilon)}-1\right|^4\right]^{1/4}$$
$$\mathbb{E}\left[\mathrm{e}^{-8\varepsilon^2\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^{\mathrm{T}}c\nabla\delta\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-4\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon}\right]^{1/4}\mathrm{e}^{t\|2\nabla\delta^{\mathrm{T}}c\nabla\delta-\langle\nabla\delta,a\rangle+e\|_\infty}$$
$$= \mathbb{E}\left[\left|g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\right|^2\right]^{1/2}\mathbb{E}\left[\left|\mathrm{e}^{\varepsilon\delta\left(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)-\varepsilon\delta(x/\varepsilon)}-1\right|^4\right]^{1/4}\mathrm{e}^{t\|2\nabla\delta^{\mathrm{T}}c\nabla\delta-\langle\nabla\delta,a\rangle+e\|_\infty},$$

where in the last equality we have used Example 2.6.7 (the exponential martingale). It remains to see that $\mathbb{E}\left[\left|g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\right|^2\right]$ is uniformly bounded for ε on finite intervals. By assumption (5.12),

$$\mathbb{E}\left[\left|g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\right|^2\right] \leq 2K^2\left(1+\mathbb{E}\left[\left|\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right|^{2\kappa}\right]\right). \tag{5.13}$$

By combining (2.32) and Lemma 4.4.2 applied to function b we have

$$\begin{aligned}\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2) &= x+\varepsilon\beta\left(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)-\varepsilon\beta\left(\tilde{X}_0^\varepsilon\right)+\varepsilon^2\int_0^{\varepsilon^{-2}t}(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)a\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s\\ &\quad+\varepsilon\int_0^{\varepsilon^{-2}t}(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\sigma\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon.\end{aligned} \tag{5.14}$$

Without loss of generality we may assume that $\kappa\in\mathbb{N}$. Thus,

$$|\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)|^{2\kappa} \leq \bar{K}\left(|x|^{2\kappa}+\varepsilon^{2\kappa}+t^{2\kappa}+\varepsilon^{2\kappa}\left|\int_0^{\varepsilon^{-2}t}(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\sigma\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon\right|^{2\kappa}\right),$$

for some $\bar{K}>0$ which does not depend on ε. From [109, Theorem 1] we conclude

$$\mathbb{E}\left[\left|\int_0^{\varepsilon^{-2}t}(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\sigma\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon\right|^{2\kappa}\right] \leq C\left(\int_0^{\varepsilon^{-2}t}\|(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\sigma\|_\infty^2\,\mathrm{d}s\right)^\kappa \leq C'\varepsilon^{-2\kappa}t^\kappa$$

and therefore

$$\mathbb{E}\left[|\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)|^{2\kappa}\right] \leq \tilde{K}\left(|x|^{2\kappa}+\varepsilon^{2\kappa}+t^{2\kappa}+t^\kappa\right), \tag{5.15}$$

for some $\tilde{K}>0$ which does not depend on ε. By combining (5.13) and (5.15) we get the desired boundedness. Thus, we need to study the convergence, as $\varepsilon\to 0$, of

$$\mathbb{E}\left[g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\mathrm{e}^{-\varepsilon^2\int_0^{\varepsilon^{-2}t}(\langle\nabla\delta,a\rangle-e)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon}\right]. \tag{5.16}$$

Now we will see why the function $\mathsf{e} = 2^{-1}\,(\nabla\delta)^{\mathrm{T}}\,c\,\nabla\delta + e - (\nabla\delta)^{\mathrm{T}}\,a$ appears in the limit. Notice that $\mathrm{e}^{-\frac{\varepsilon^2}{2}\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^{\mathrm{T}}c\nabla\delta\right)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}W_s^{\varepsilon}}$ is (again by Example 2.6.7) a strictly positive martingale for which we will be able to use Girsanov theorem and therefore prefer it as opposed to the exponential term in (5.16). Since

$$\mathrm{e}^{-\varepsilon^2\int_0^{\varepsilon^{-2}t}(\langle\nabla\delta,a\rangle-e)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}W_s^{\varepsilon}} =$$
$$\mathrm{e}^{\varepsilon^2\int_0^{\varepsilon^{-2}t}\mathsf{e}\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s}\cdot\mathrm{e}^{-\frac{\varepsilon^2}{2}\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^{\mathrm{T}}c\nabla\delta\right)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}W_s^{\varepsilon}},$$

using Cauchy inequality, we get

$$\begin{aligned}&\Bigg|\mathbb{E}\left[g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\mathrm{e}^{-\varepsilon^2\int_0^{\varepsilon^{-2}t}(\langle\nabla\delta,a\rangle-e)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}W_s^{\varepsilon}}\right]-\\&\quad\mathrm{e}^{\pi^0(\mathsf{e})\,t}\,\mathbb{E}\left[g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\mathrm{e}^{-\frac{\varepsilon^2}{2}\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^{\mathrm{T}}c\nabla\delta\right)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}W_s^{\varepsilon}}\right]\Bigg|\\&\leq\mathbb{E}\left[\left|g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\right|^2\right]^{1/2}\mathbb{E}\left[\left|\mathrm{e}^{\varepsilon^2\int_0^{\varepsilon^{-2}t}\mathsf{e}\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s}-\mathrm{e}^{\pi^0(\mathsf{e})\,t}\right|^4\right]^{1/4}\\&\quad\mathbb{E}\left[\mathrm{e}^{-2\varepsilon^2\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^{\mathrm{T}}c\nabla\delta\right)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s-4\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}W_s^{\varepsilon}}\right]^{1/4}.\end{aligned}\tag{5.17}$$

From (5.13) and (5.15) the first term on the right-hand side in (5.17) is uniformly bounded for ε on finite intervals.

For the second term, since e is τ-periodic, analogously as in the proof of Theorem 4.5.1, we see that

$$\varepsilon^2\int_0^{\varepsilon^{-2}t}\mathsf{e}\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\,\mathrm{d}s\xrightarrow[\varepsilon\to0]{\mathrm{L}^2(\mathbb{P})}\pi^0\,(\mathsf{e})\,t\,.$$

Consequently, Skorohod representation theorem and dominated convergence theorem imply that the second term on the right-hand side in (5.17) converges to zero as $\varepsilon\to0$.

For the third term on the right-hand side in (5.17) (again by Example 2.6.7) we have

$$\mathbb{E}\left[\mathrm{e}^{-2\varepsilon^2\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^{\mathrm{T}}c\nabla\delta\right)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s-4\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}W_s^{\varepsilon}}\right]\leq\mathrm{e}^{6t\|\nabla\delta^{\mathrm{T}}c\nabla\delta\|_\infty}.$$

Thus, $u^{\varepsilon}(x,t)$ converges as $\varepsilon\to0$ if, and only if,

$$\mathrm{e}^{\pi^0(\mathsf{e})\,t}\,\mathbb{E}\left[g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\mathrm{e}^{-\frac{\varepsilon^2}{2}\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^{\mathrm{T}}c\nabla\delta\right)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s)\right)\mathrm{d}W_s^{\varepsilon}}\right]$$

converges, and if this is the case the limit is the same. Now the Girsanov theorem implies the existence of a measure $\mathbb{P}^{\varepsilon}(\mathrm{d}\omega)$ such that the Radon–Nikodym derivative $\frac{\mathrm{d}\mathbb{P}^{\varepsilon}}{\mathrm{d}\mathbb{P}}(x)$

satisfies

$$\mathbb{E}\left[\frac{\mathrm{d}\mathbb{P}^\varepsilon}{\mathrm{d}\mathbb{P}}(x)|\mathscr{F}_{\varepsilon^{-2}t}\right] = \mathrm{e}^{-\frac{\varepsilon^2}{2}\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^{\mathrm{T}}a\nabla\delta\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon}, \qquad t\geq 0\,.$$

Clearly,

$$\begin{aligned}\mathbb{E}^\varepsilon\left[g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\frac{\mathrm{d}\mathbb{P}^\varepsilon}{\mathrm{d}\mathbb{P}}(x)|\mathscr{F}_{\varepsilon^{-2}t}\right]\right]\\ \mathbb{E}\left[g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\right.&\left.\mathrm{e}^{-\frac{\varepsilon^2}{2}\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^{\mathrm{T}}a\nabla\delta\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon}\right].\end{aligned}$$

Thus, it remains to prove that

$$\lim_{\varepsilon\to 0}\mathbb{E}^\varepsilon\left[g\left(\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)\right] = \mathbb{E}\left[g\left(W^{\bar{\mathrm{b}},\mathrm{c}}(x,t)\right)\right] \qquad \forall t\geq 0\,.$$

We will do this by first proving that process $\{\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\}_{t\geq 0}$ converges in low with respect to $\mathbb{P}^\varepsilon$ to $\{W^{\bar{\mathrm{b}},\mathrm{c}}(x,t)\}_{t\geq 0}$. Due to boundedness of β, $\{\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\}_{t\geq 0}$ converges in law with respect to $\mathbb{P}^\varepsilon$ if, and only if

$$\{\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)-\varepsilon\beta(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2))+\varepsilon\beta(x/\varepsilon)\}_{t\geq 0} \tag{5.18}$$

converges in law with respect to $\mathbb{P}^\varepsilon$, and if this is the case the limit is the same. The corollary to Girsanov theorem implies that $\tilde{W}_t^\varepsilon := W_t^\varepsilon + \varepsilon\int_0^t \sigma^{\mathrm{T}}\nabla\delta\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s$, $t\geq 0$, is a $\mathbb{P}^\varepsilon$-Brownian motion. This together with representation in (5.14) we have

$$\begin{aligned}\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2) =& x+\varepsilon\beta\left(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)-\varepsilon\beta(x/\varepsilon)\\ &+\varepsilon^2\int_0^{\varepsilon^{-2}t}((\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)a-(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\,c\,\nabla\delta)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s\\ &+\varepsilon\int_0^{\varepsilon^{-2}t}(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\sigma\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}\tilde{W}_s^\varepsilon\,.\end{aligned}$$

Clearly, process in (5.18) is a semimartingale with bounded variation and predictable quadratic covariation parts

$$\left\{\varepsilon^2\int_0^{\varepsilon^{-2}t}((\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)a-(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\,c\,\nabla\delta)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s\right\}_{t\geq 0},$$

and

$$\left\{\varepsilon^2\int_0^{\varepsilon^{-2}t}(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)\,c\,(\mathbb{I}_{\mathrm{d}}-\mathrm{D}\beta)^{\mathrm{T}}\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s\right\}_{t\geq 0}.$$

respectively. From Theorem 2.6.16 we know that in order to show converges in law with respect to $\mathbb{P}^\varepsilon$ of process $\{\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)-\varepsilon\beta(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2))+\varepsilon\beta(x/\varepsilon)\}_{t\geq 0}$ to $\{W^{\bar{\mathsf{b}},\mathsf{c}}(x,t)\}_{t\geq 0}$ it suffices to see that

$$\varepsilon^2\int_0^{\varepsilon^{-2}t}\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)a-(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,\nabla\delta\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s \xrightarrow[\varepsilon\to 0]{\mathbb{P}^\varepsilon} \bar{\mathsf{b}}t\,,$$

and

$$\varepsilon^2\int_0^{\varepsilon^{-2}t}(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\;c\;(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)^\mathrm{T}\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s \xrightarrow[\varepsilon\to 0]{\mathbb{P}^\varepsilon} \mathsf{c}t$$

for all $t\geq 0$. We will do this by proving the convergence in $L^1(\mathbb{P}^\varepsilon)$ for the bounded variation and an analogous relation holds for the predictable quadratic covariation part. Using Cauchy inequality we conclude

$$\begin{aligned}
&\mathbb{E}^\varepsilon\left[\varepsilon^2\left|\int_0^{\varepsilon^{-2}t}\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)a-(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,\nabla\delta-\bar{\mathsf{b}}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right|\right]^2 = \\
&\mathbb{E}\left[\varepsilon^2\left|\int_0^{\varepsilon^{-2}t}\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)a-(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,\nabla\delta-\bar{\mathsf{b}}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right|\right.\\
&\qquad\left.\mathrm{e}^{-\frac{\varepsilon^2}{2}\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^\mathrm{T}c\nabla\delta\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-\varepsilon\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^\mathrm{T}\sigma\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon}\right]^2 \leq \\
&\varepsilon^4\mathbb{E}\left[\left(\int_0^{\varepsilon^{-2}t}\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)a-(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,\nabla\delta-\bar{\mathsf{b}}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right)^\mathrm{T}\right.\\
&\qquad\left.\left(\int_0^{\varepsilon^{-2}t}\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)a-(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,\nabla\delta-\bar{\mathsf{b}}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s\right)\right]\\
&\quad\mathbb{E}\left[\mathrm{e}^{-2\varepsilon^2\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^\mathrm{T}c\nabla\delta\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}s-2\varepsilon\int_0^{\varepsilon^{-2}t}\left(\nabla\delta^\mathrm{T}\sigma\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\mathrm{d}W_s^\varepsilon}\right]\mathrm{e}^{t\|\nabla\delta^\mathrm{T}a\nabla\delta\|_\infty}\,.
\end{aligned}$$

Since (again by Example 2.6.7) the expectation in the last line is one, we precede as in the proof of Theorem 4.5.1 to conclude that

$$\varepsilon^2\int_0^{\varepsilon^{-2}t}\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)a-(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,\nabla\delta-\bar{\mathsf{b}}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s \xrightarrow[\varepsilon\to 0]{\mathrm{L}^2(\mathbb{P})} 0\,,$$

which implies

$$\varepsilon^2\int_0^{\varepsilon^{-2}t}\left((\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)a-(\mathbb{I}_\mathrm{d}-\mathrm{D}\beta)\,c\,\nabla\delta-\bar{\mathsf{b}}\right)\left(\tilde{X}^\varepsilon(x/\varepsilon,s)\right)\,\mathrm{d}s \xrightarrow[\varepsilon\to 0]{\mathrm{L}^1(\mathbb{P}^\varepsilon)} 0\,.$$

Analogous result holds for the predictable quadratic covariation part. Thus, process $\{\varepsilon\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\}_{t\geq 0}$ converges in low with respect to $\mathbb{P}^\varepsilon$ to $\{W^{\bar{\mathsf{b}},\mathsf{c}}(x,t)\}_{t\geq 0}$. Continuous mapping theorem implies that $g\left(\tilde{X}^\varepsilon(x/\varepsilon,t/\varepsilon^2)\right)$ converges in low with respect to $\mathbb{P}^\varepsilon$

to $g\left(W^{\bar{\mathsf{b}},\mathsf{c}}\right)$ for every $t \geq 0$. From Skorohod representation theorem and Fatou's lemma (without loss of generality we may assume that g is non-negative) we conclude

$$\liminf_{\varepsilon\to 0} \mathbb{E}^{\varepsilon}\left[g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\right] \geq \mathbb{E}\left[g\left(W^{\bar{\mathsf{b}},\mathsf{c}}(x,t)\right)\right] \qquad \forall t \geq 0\,.$$

To prove the reverse inequality we proceed as follows. For any $t \geq 0$ we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\mathbb{E}^{\varepsilon}\left[g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\right] \leq \\ &\limsup_{m\to\infty}\limsup_{\varepsilon\to 0}\mathbb{E}^{\varepsilon}\left[(g\wedge m)\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\right] + \\ &\limsup_{m\to\infty}\limsup_{\varepsilon\to 0}\mathbb{E}^{\varepsilon}\left[g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\mathbb{1}_{\{g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\geq m\}}\right].\end{aligned}$$

Skorohod representation theorem and dominated convergence theorem imply

$$\begin{aligned}&\limsup_{m\to\infty}\limsup_{\varepsilon\to 0}\mathbb{E}^{\varepsilon}\left[(g\wedge m)\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\right] \leq \\ &\limsup_{m\to\infty}\mathbb{E}\big[(g\wedge m)\big(W^{\bar{\mathsf{b}},\mathsf{c}}(x,t)\big)\big] = \mathbb{E}\big[g\big(W^{\bar{\mathsf{b}},\mathsf{c}}(x,t)\big)\big]\,.\end{aligned}$$

Cauchy inequality and Markov inequality imply

$$\begin{aligned}&\mathbb{E}^{\varepsilon}\left[g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\mathbb{1}_{\{g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\geq m\}}\right] \\ &\leq \mathbb{E}^{\varepsilon}\left[\left|g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\right|^2\right]^{1/2}\left(\mathbb{P}^{\varepsilon}\left(g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\geq m\right)\right)^{1/2} \\ &\leq \frac{1}{m}\mathbb{E}^{\varepsilon}\left[\left|g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\right|^2\right].\end{aligned}$$

As in (5.15) we get

$$\mathbb{E}^{\varepsilon}\left[g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\right] \leq \hat{K}(1+|x|^{2\kappa}+\varepsilon^{2\kappa}+t^{2\kappa}+t^{\kappa})$$

for some $\hat{K} > 0$ which does not depend on ε. This proves that

$$\liminf_{\varepsilon\to 0} \mathbb{E}^{\varepsilon}\left[g\left(\varepsilon\tilde{X}^{\varepsilon}(x/\varepsilon,t/\varepsilon^2)\right)\right] \geq \mathbb{E}\left[g\left(W^{\bar{\mathsf{b}},\mathsf{c}}(x,t)\right)\right] \qquad \forall t \geq 0\,.$$

which completes the first part of the proof.

We next assume that $g \equiv 0$ and show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^t f\left(X^{\varepsilon}(x,s)\right)\mathrm{e}^{\int_0^s\left(\varepsilon^{-1}d(X^{\varepsilon}(x,u)/\varepsilon)+e(X^{\varepsilon}(x,u)/\varepsilon)\right)\mathrm{d}u}\mathrm{d}s\right] = \mathbb{E}\left[\int_0^t f(W^{\bar{\mathsf{b}},\mathsf{c}}(x,s))\,\mathrm{e}^{\pi^0(\mathsf{e})s}\mathrm{d}s\right].$$

From the first case we see that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[f\left(X^{\varepsilon}(x,s)\right)\mathrm{e}^{\int_0^s\left(\varepsilon^{-1}d(X^{\varepsilon}(x,u)/\varepsilon)+e(X^{\varepsilon}(x,u)/\varepsilon)\right)\mathrm{d}u}\right] = \mathbb{E}\left[f(W^{\bar{\mathsf{b}},\mathsf{c}}(x,s))\,\mathrm{e}^{\pi^0(\mathsf{e})s}\right], \qquad \forall s \geq 0\,,$$

and

$$
\begin{aligned}
&\mathbb{E}\left[f\left(X^{\varepsilon}(x,s)\right)\mathrm{e}^{\int_0^s\left(\varepsilon^{-1}d(X^{\varepsilon}(x,u)/\varepsilon)+e(X^{\varepsilon}(x,u)/\varepsilon)\right)\mathrm{d}u}\right] \leq \\
&\mathbb{E}\left[\left|f\left(X^{\varepsilon}(x,s)\right)\right|^2\right]^{1/2}\mathbb{E}\left[\mathrm{e}^{2\int_0^s\left(\varepsilon^{-1}\left(d(X^{\varepsilon}(x,u)/\varepsilon)-\pi^0(d)\right)+e(X^{\varepsilon}(x,u)/\varepsilon)\right)\mathrm{d}u}\right]^{1/2} \leq \\
&\check{K}\left(1+|x|^{\kappa}+\varepsilon^{\kappa}+s^{\kappa}\right)\cdot \\
&\mathbb{E}\left[\mathrm{e}^{2\varepsilon\delta\left(\tilde{X}^{\varepsilon}(x/\varepsilon,s/\varepsilon^2)\right)-2\varepsilon\delta(x/\varepsilon)-2\varepsilon^2\int_0^{\varepsilon^{-2}s}(\langle\nabla\delta,a\rangle-e)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,u)\right)\mathrm{d}u-2\varepsilon\int_0^{\varepsilon^{-2}s}(\nabla\delta^{\mathrm{T}}\sigma)\left(\tilde{X}^{\varepsilon}(x/\varepsilon,u)\right)\mathrm{d}W_u^{\varepsilon}}\right]^{1/2} \\
&\leq \check{K}\left(1+|x|^{\kappa}+\varepsilon^{\kappa}+s^{\kappa}\right)\mathrm{e}^{2\varepsilon\|\delta\|_{\infty}+\|\langle\nabla\delta,a\rangle-e-\nabla\delta^{\mathrm{T}}c\nabla\delta\|_{\infty}s}.
\end{aligned}
$$

The result now follows from the dominated convergence theorem.

■

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CURRICULUM VITAE

Ivana Valentić was born on 17th of June 1991 in Zagreb, Croatia, where she finished primary and secondary school. During secondary school she attended mathematical competitions, winning Bronze Medal at the Middle European Mathematical Olympiad.

In 2010 she started studies at Department of Mathematics, Faculty of Science, University of Zagreb, where she obtained her master's degree in 2015 under supervision of Prof. dr sc. Bojan Basrak. During her studies she won a Third Prize at International Mathematical Competition and Rector's Award for paper *Applications of random dyadic systems to decompositions of functions and operators*, with co-author Mario Stipčić and advisor Assoc. Prof. dr. sc. Vjekoslav Kovać.

In 2015 she enrolled in the PhD program in Mathematics at the University of Zagreb under supervision of Assoc. Prof. dr. sc. Nikola Sandrić and became a member of the Zagreb Probability theory seminar. In 2016 she started working as a research and teaching assistant at the same faculty. She attended eleven conferences, summer schools and workshops, where she gave one talk and presented two posters. She had a short scientific visits, during which she gave a talk, to mathematical department at TU Dresden, Germany as part of a MSES-DAAD bilateral project with Germany *Random time-change and jump processes* (project leaders: Prof. dr. sc. René Schilling and Assoc. Prof. dr. sc. Nikola Sandrić). She also participated in the project *Stochastic stability and potential theory of Markov processes*, founded by Croatian Science Foundation, (project leader: Assoc. Prof. dr. sc. Nikola Sandrić).

She co-authored one paper:

- N. Sandrić, I. Valentić, and J. Wang. *Periodic homogenization of a Lévy-type process with small jumps*, Journal of Evolution Equations, 2020.