Approximation of quadratic eigenvalue problem and application to damping optimization

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To my family.

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Summary

In this thesis we study the parameter dependent Hermitian quadratic eigenvalue problem (PEQP) given by $(\lambda^2(\mathbf{p})M(\mathbf{p})+\lambda(\mathbf{p})C(\mathbf{p})+K(\mathbf{p}))x(\mathbf{p})=0$, where $\mathbf{p}\in\mathbb{R}^m$ is a vector of parameters. Through this thesis matrices $M(\mathbf{p}), C(\mathbf{p}), K(\mathbf{p})\in\mathbb{R}^n$ arise from corresponding (vibrational) mechanical system represented by $M(\mathbf{p})\ddot{q}(\mathbf{p};t)+C(\mathbf{p})\dot{q}(\mathbf{p};t)+K(\mathbf{p})q(\mathbf{p};t)=0$ and represent mass, damping, and stiffness, respectively. Usually matrices $M(\mathbf{p})$, $K(\mathbf{p})$ are Hermitian positive definite and $C(\mathbf{p})$ is Hermitian positive semidefinite matrix.

After a brief introduction and problem formulation we give three approximation approaches for efficient computation of eigenpairs. These approaches preserve structure and allow computation of eigenpairs, for different sets of parameters, that is computationally efficient and at the same time they provide satisfactory relative accuracy. The first approach is based on dimension reduction, the second on first order approximation, while the third one uses modified Rayleigh quotient iterations for structured damping matrices. For the first and the third approach we need to linearize PQEP. Within the first approach we distinguish two very important cases. In the first case we consider efficient approximation for eigenvalues for the selected part of the undamped spectrum. In the second case, we consider efficient approximation of all eigenvalues. The first order approximation considered in second approach is based on Taylor's theorem and it is efficient for eigenvalue computation when the change in parameter is small enough. In the third approach we provide an efficient method of eigenvalue computation of the diagonal-plus-rank-one matrices (DPR1), and show how one can apply this method on corresponding linearized eigenvalue problem, by exploiting the structure of the damping matrix. Numerical experiments confirm efficiency and accuracy of these approahes.

Further on, we use obtained first order approximation bounds for efficient estimations of the gap functions that appear in different perturbation bounds for the quadratic eigenvalue problem. These estimations of the gap functions are based on removing perturbed quantities from them. Accuracy and efficiency of these estimations are given in numerical examples.

Last we focus on damping optimization. Two different optimization criteria are considered: minimization of total average energy, and frequency isolation, which will determine the damping matrix which ensures vibration decay is as fast as possible. While dealing with the minimization of total average energy we provide an approximation of the solution of the structured Lyapunov equation, which can be efficiently computed. Frequency isolation is eigenvalue based criterion so we use obtained eigenvalue approximations to determine the optimal damping, while the areas from which we isolate the frequencies are ellipses with centers on the imaginary axis.

Both approaches are illustrated on numerical examples.

Keywords: quadratic eigenvalue problem, vibrational mechanical systems, eigenvalue approximation, dimension reduction, first order approximation, modified RQI, estimation of gap function, damping optimization, frequency isolation, total average energy minimization

Prošireni sažetak

Kvadratni svojstveni problem se pojavio već u 30-tim godinama 20. stoljeća kada su Frazer, Duncan i Collar istraživali vibracije zakrilca u zračnim letjelicama te zajedno objavili knjigu [33], a Taussky i Lancaster su ga prvi riješili za male dimenzije [95], [44]. Osim u aerodinamici, kvadratni svojstveni problem ima brojne primjene u proučavanju dinamike mehaničkih sustava, kao što su na primjer zgrade i mostovi. Pješački Milenijski most u Londonu zatvoren je samo 2 dana nakon što je pušten u promet, zbog prejakog podrhtavanja. Podrhtavanje je posljedica rezonancije, odnosno nestabilnosti sustava do koje je došlo zato što su frekvencije sustava pod utjecajem vanjske sile, koja je djelovala na sustav (ljudi koji su hodali po mostu), bile vrlo blizu prirodnim frekvencijama sustava. Vibracije vrlo često nisu poželjne, jer u slučaju rezonancije može doći do urušavanja. Veza između kvadratnog svojstvenog problema i problema izbjegavanja rezonancije sustava je u tome što su prirodne frekvencije sustava i frekvencije sustava pod utjecajem vanjske sile rješenja kvadratnog svojstvenog problema.

Prigušenje je utjecaj na vibracijski sustav koji kao posljedicu ima ublažavanje, ograničavanje ili onemogućavanje vibracija. Prilagođavanjem viskoznosti prigušenja možemo promijeniti prirodne frekvencije sustava te na taj način izbjeći opasne vibracije, tj. možemo izbjeći frekvencije mogućih vanjskih podražaja, tj. sila koje djeluju na sustav, kao što su frekvencije kretanja pješaka po mostu, frekvencije vjetra ili potresa. Ovisno o primjeni mogu se koristiti različiti kriteriji pri optimiazciji prigušenja, kao što su minimizacija ukupne energije sustava, minimizacija amplitude vibracija u sustavu, izolacija opasnih frekvencija, itd.

U ovoj radnji proučavat će se parametarski ovisan hermitski kvadratni svojstveni problem (PQEP) koji opisuje navedene mehaničke sustave, a dan je s $(\lambda^2(\mathbf{p})M(\mathbf{p})+\lambda(\mathbf{p})C(\mathbf{p})+K(\mathbf{p}))x(\mathbf{p})=0$, pri čemu je $\mathbf{p}\in\mathbb{R}^m$ vektor parametara. Kroz radnju će se proučavati vibracijski mehanički sustav zadan diferencijalnom jednadžbom drugog reda: $M(\mathbf{p})\ddot{q}(\mathbf{p};t)+C(\mathbf{p})\dot{q}(\mathbf{p};t)+K(\mathbf{p})q(\mathbf{p};t)=0$, pri čemu matrice $M(\mathbf{p}),C(\mathbf{p}),K(\mathbf{p})$ redom predstavljaju masu, prigušenje i krutost, $q(\mathbf{p};t)$ je stanje sustava, a $\mathbf{p}$ je parametar o kojem sustav ovisi. Ukoliko se parametar nalazi samo u matrici $C(\mathbf{p})$ tada on najčešće sadrži viskoznosti prigušivača. Vrlo često su matrice $M(\mathbf{p})$, $K(\mathbf{p})$ hermitske pozitivno definite, a $C(\mathbf{p})$ je hermitska pozitivno semidefinitna.

Matricu prigušenja možemo modelirati na više različitih načina. Najčešće je zadajemo kao sumu unutarnjeg i vanjskog prigušenja, tj. $C(\mathbf{p})=C_{\text{int}}+C_{\text{ext}}(\mathbf{p})$, pri čemu matrica vanjskog prigušenja ovisi o viskoznostima i pozicijama prigušivača. Matrica unutarnjeg prigušenja se može modelirati na različite načine, ali najčešće je modeliramo kao mali postotak kritičnog prigušenja, koje je zadano s $C_{\text{crit}}=2M^{\frac{1}{2}}\sqrt{M^{-\frac{1}{2}}KM^{-\frac{1}{2}}}M^{\frac{1}{2}}$.

Nakon kratkog uvoda i formulacije problema navodimo tri različita aproksimacijska pristupa za efikasno računanje svojstvenih vrijednosti i svojstvenih vektora, koje čuvaju strukturu matrica sustava. Istovremeno navedeni aproksimacijski pristupi pružaju zadovoljavajuću relativnu točnost. Prvi pristup je baziran na redukciji dimenzije, drugi na aproksimaciji prvog reda, a treći pristup koristi modificiranu verziju iteracija Rayleighjevog kvocjenta za slučaj kada je matrica prigušenja strukturirana. Za prvi i treći pristup trebamo linearizirati PQEP. U navedenoj linearizaciji koristimo istovremenu dijagonalizaciju matrica M i K.

Prvi pristup baziran je na idejama iz radova [12] i [13], te razlikujemo dva slučaja. U prvom slučaju proučavamo efikasnu aproksimaciju svojstvenih vrijednosti za izabrni dio neprigušenog spektra, dok u drugom slučaju proučavamo efikasnu aproksimaciju svih svojstvenih vrijednosti.

Aproksimacija prvog reda koje koristimo u drugom aproksimacijom pristupu bazirana je na Taylorovom teoremu i vrlo je efikasna za računanje svojstvenih vrijednosti u slučaju kada imamo malu promjenu u parametru. Slične aproksimacije prvog reda koriste se u perturbacijskoj teoriji, npr. [14]. Jedina razlika je što se perturbacija ne vrši u parametru, kao u našem slučaju, nego u samim matricama sustava.

Treći pristup baziran je na radu [47], gdje autori predlažu modificiranu verziju iteracija Rayleighjevog kvocjenta kao efikasanu metodu računanja svojstvenih vrijednosti matrica koje su oblika: dijagonala plus matrica ranga jedan (DPR1). Pristup iz [47] svodi se na par iteracija standardnog Rayleighjevog kvocjenta te nakon toga se provodi modificirana verzija, dok se pristup opisan u Sekciji 2.3 bazira na prilagodbi duljine koraka u modificiranoj verziji iteracija Rayleighjevog kvocjenta. Također prikazujemo kako se ova metoda može primijeniti na odgovarajući linearizirani svojstveni problem iskorištavanjem strukture matrice prigušenja. Numerički primjeri potvrđuju efikasnost i točnost navedenih pristupa.

Nadalje, u Poglavlju 3 primjenjujemo aproksimacije prvog reda u perturbacijskoj teoriji kvadratnog svojstvenog problema, točnije koristimo aproksimacije prvog reda kako bismo procijenili približnu vrijednost gap funkcija koje se pojavljuju u brojnim perturbacijskom ocjenama. Ove procjene gap funkcija baziraju se na uklanjanju perturbiranih vrijednosti iz ocjene, jer želimo ocjene koje se mogu računati bez potrebe za poznavanjem perturbiranih vrijednosti. Gap funkcije koje ćemo približno procijenjivati se nalaze u perturbacijskom ocjenama iz radova [102] i [101], koje su prikazane u Sekciji 3.2.3. Točnost i efikasnost ovih približnih vrijednosti perturbacijskih ocjena su prikazane u numeričkim primjerima.

Nadalje, u Poglavlju 4 glavni fokus je na optimizaciji prigušenja pri čemu razmatramo dva različita optimizacijska kriterija koja će odrediti matricu prigušenja tako da se vibracije u sustavu primire što je prije moguće. Prvi kriterij je minimizacija ukupne prosiječne energije sustava koji povezujemo s minimizacijom traga rješenja pripadne Ljapunovljeve jednadžbe. S obzirom da optimizacija prigušenja uz navedeni kriterij zahtijeva rješavanje Ljapunovljeve jednadžbe za svaku promjenu u parametru, vrlo je bitno doći do rješenja na efikasan način. U Sekciji 4.1.1 dajemo efikasnu aproksimaciju traga rješenja Ljapunovljeve jednadžbe za slučaj kada je sustav blizu modalno priušenom sustavu, čime izbjegnemo rješavanje same jednadžbe.

Drugi razmatrani kriterij je izolacija frekvencija. Cilj izolacije frekvencija je optimizacija viskoznosti u sustavu tako da su svojstvene vrijednosti uklonjene iz opasnog područja, pri čemu opasnim područjem smatramo područje koje je blizu opasne frekvencije. Promatrat ćemo elipse s centrom na imaginarnoj osi, u opasnoj frekvenciji, kao opasno područje. Izolacija frekvencija je proučavana u [49], [32] i [69]. U [49], autori proučavaju strukture koje vibiraju na niskim frekvencijama, dok u [32], autori predlažu metodu baziranu na inverznom problemu, tj. unaprijed je zadan spektar koji želim dostići, a koji je daleko od opasnih frekvencija. Izolacija frekvencija je kriterij baziran na svojstvenim vrijednostima stoga ćemo u njemu koristiti aproksimacije svojstvenih vrijednosti prikazane u Sekciji 2.3. Oba kriterija su ilustrirana u numeričkim primjerima.

Ključne riječi: kvadratni svojstveni problem, vibracijski sustavi, aproksimacija svojstvenih vrijednosti, redukcija dimenzije, apoksimacije prvog reda, modificirani RQI, estimacija gap funkcija, optimizacija prigušenja, izolacija frekvencija, minimizacija ukupne prosječne energije

Contents

Nomenclature

In this thesis we use the following notation:

$A, B, \ldots, \Phi, \Xi, \ldots$	matrices of dimension n, if not defined differently (Latin and Greek capital letters)
$\mathbf{A}, \mathbf{B}, \ldots$	matrices of dimension $2n \times 2n$
$\mathbf{y}, \mathbf{w}, \ldots$	vectors of dimension $2n$
$[\mathbf{v}; \mathbf{v}_M; \mathbf{v}_K]$	vector of parameters where next parameter is concatenated with the previous one, (MATLAB notation)
$\mathbf{p}, \mathbf{v}, \mathbf{v}_M, \mathbf{v}_K, \ldots$	parameters (vector columns)
v, c	one dimensional parameter
$\sigma(\cdot)$	spectrum
$\mu, \overline{\mu}, \tau, \pi, \ldots$	vectors of indices
$\lambda(\cdot), \rho(\cdot)$	eigenvalue
$\widetilde{\lambda}(\cdot), \overline{\lambda}(\cdot)$	approximation of eigenvalue
$\alpha, \beta, \ldots$	scalars
$A(p, q)$	the submatrix of A obtained by intersection of rows determined with vector p and columns determined with vector q
$A(:, i)$	the ith column of matrix A
$A(i, :)$	the ith row of matrix A
$i : j$	vector of integers from i to j
$A^{\dagger}$	Moore-Penrose pseudo-inverse of a matrix
I_s	s dimensional identity matrix
$\|\| \cdot \|\|_2$	2-norm

$\oplus$	the direct sum
$\otimes$	the Kronecker product
$\kappa_2(\cdot)$	condition number of a matrix in 2-norm
$\mathcal{J}(\cdot)$	the gradient vector
$\mathcal{H}(\cdot)$	the Hessian matrix
$\mathbb{R}_+$	all positive real numbers, i.e., $\mathbb{R}_+ = \{a \in \mathbb{R} \colon a > 0\}$

CHAPTER 1

Introduction

1.1 Motivation

The quadratic eigenvalue problem first appeared in 1930's when Frazer, Duncan, and Collar researched the flutter of airplane wings and published a book [33] together. The first to solve a quadratic eigenvalue problem of small dimension were Taussky and Lancaster, see e.g. [95], [44]. Apart from aerodynamics, quadratic eigenvalue problem appears in the dynamic analysis of acoustic systems, fluid mechanics, electrical circuit simulation, modeling microelectronic mechanical systems, signal processing, and most importantly for this thesis, in the dynamic analysis of different mechanical systems, mainly buildings and bridges.

Millennium bridge in London was closed only two days after opening in 2000 due to alarming wobbling of the bridge. The wobbling of the bridge was the result of the phenomenon called resonance i.e., the amplitude of the systems vibrations were amplified because the bridge was excited by external force induced by pedestrian movements, whose frequencies became close to the natural frequencies of the bridge. A natural frequency is a frequency at which a system prefers to vibrate. Resonant vibrations can be dangerous for structures and lead to collapse, as was the case for the Tacoma Narrow bridge in 1940 (for more see [20]). This event is an example of elementary forced resonance. The bridge collapsed because winds produced aeroelastic flutter that matched the natural frequency of the bridge. Two more recent examples of undesirable vibrations are the Franjo Tuđman bridge in Dubrovnik in 2005, and the bridge over the Volga River in 2010.

The connection between vibrations in mechanical system and the quadratic eigenvalue problem is that the natural modes and frequencies of the structure are solutions of quadratic eigenvalue problem, for more see [97]. A normal mode of an vibrational mechanical system is a pattern of motion in which all parts of the system move sinusoidally with the same frequency, for more see [112].

By adjusting the damping parameters in the structure we can change natural frequencies of the system. This way we can avoid dangerous vibrations, i.e. we can avoid all possible frequencies which external force can induce. Damping is an influence within or upon an vibrational mechanical system that has the effect of reducing, restricting or preventing its oscillations. Ob-

taining the optimal parameter for a certain optimization criteria (frequency isolation, reduction of average total energy, for an overview of the criteria see [112], [73]) is a process of damping optimization.

1.2 Problem formulation

We consider the following parameter dependent quadratic eigenvalue problem (PQEP)

$$(\lambda(\mathbf{p})^2 M(\mathbf{p}) + \lambda(\mathbf{p}) C(\mathbf{p}) + K(\mathbf{p}))x(\mathbf{p}) \;=\; 0, \qquad x(\mathbf{p}) \neq 0, \tag{1.1}$$

where $M(\mathbf{p})$, $K(\mathbf{p})$ are positive definite Hermitian matrices of order n and $C(\mathbf{p})$ is a Hermitian positive semidefinite matrix of order n, for all $\mathbf{p} \in \mathbb{R}^m$, where m is number of parameters. Let $\Lambda(\mathbf{p})$ be the spectrum of (1.1) for given $\mathbf{p}$.

Such an eigenvalue problem usually arises if one studies parameter dependent mechanical systems, which is usually described with the following second-order differential equation

$$\begin{aligned} M(\mathbf{p})\ddot{q}(\mathbf{p};t) + C(\mathbf{p})\dot{q}(\mathbf{p};t) + K(\mathbf{p})q(\mathbf{p};t) = 0, \\ q(\mathbf{p};0) = q_0(\mathbf{p}) \quad \text{and} \quad \dot{q}(\mathbf{p};0) = \dot{q}_0(\mathbf{p}), \end{aligned} \tag{1.2}$$

where $q_0(\mathbf{p})$ and $\dot{q}_0(\mathbf{p})$ are initial conditions. Here matrices $M(\mathbf{p})$, $C(\mathbf{p})$, and $K(\mathbf{p})$, correspond to mass, damping, and stiffness matrices, respectively.

It is common that some parameters do not influence all matrices, i.e., $M(\mathbf{p}) = M(\mathbf{v}_M)$, $C(\mathbf{p}) = C(\mathbf{v})$ and $K(\mathbf{p}) = K(\mathbf{v}_K)$, where $\mathbf{v}_M, \mathbf{v}, \mathbf{v}_K$, are vectors of dimension s_M, s, s_K respectively. For the sake of simplicity we will use mentioned notation and we denote $\mathbf{p} = [\mathbf{v}_M; \mathbf{v}; \mathbf{v}_K]$ as a single vector, where

$$\mathbf{p}_i = \begin{cases} (\mathbf{v}_M)_j, & \text{for } i = j, j = 1, \ldots, s_M, \\ (\mathbf{v})_j, & \text{for } i = s_M + j, j = 1, \ldots, s, \\ (\mathbf{v}_K)_j, & \text{for } i = s_M + s + j, j = 1, \ldots, s_K. \end{cases}$$

PQEP (1.1) is obtained from (1.2) simply by substituting $q(\mathbf{p};t) = e^{\lambda(\mathbf{p})t}x(\mathbf{p})$.

The damping matrix can be defined in several different ways. One of the most common ways is that $C(\mathbf{p}) = C_{\text{int}} + C_{\text{ext}}(\mathbf{v})$, where C_{int} represents internal damping and only the external damping part depends on parameters $v_i > 0$ for $i = 1, \ldots, s$ (called viscosities), where $\mathbf{v} = [v_1, \ldots, v_s]^T$. Moreover, external damping can be written as

$$C_{\text{ext}}(\mathbf{v}) = v_1 C_1 + v_2 C_2 + \cdots + v_s C_s, \tag{1.3}$$

where each C_i determines the geometry of the ith damper and it has a small rank, so that $C_{\text{ext}}(\mathbf{v})$ is a semidefinite matrix in general.

Throughout this thesis, along with PQEP, we will consider the corresponding linearized parameter dependent generalized eigenvalue problem (PGEP). In damping optimization we (usually) only have the viscosity parameter in the damping matrix, i.e., $\mathbf{p} = \mathbf{v}$, which means that mass and stiffness matrices are constants, i.e., $M(\mathbf{p}) = M$ and $K(\mathbf{p}) = K$, respectively. Thus, we consider PGEP only in the case when $\mathbf{p} = \mathbf{v}$ when simultaneous diagonalization of matrices M and K does not depend on parameter $\mathbf{p}$. If we let matrices M and K depend on parameter $\mathbf{p}$ then for each change in parameter we need to do new simultaneous diagonalization which is time consuming, $O(n^3)$.

Now, let Φ be a matrix that simultaneously diagonalizes M and K, i.e., [112]

$$\Phi^T K \Phi = \Omega^2 = \operatorname{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I, \tag{1.4}$$

where

$$\omega_1 \leq \ldots \leq \omega_n \tag{1.5}$$

are eigenfrequencies of undamped mechanical system, i.e., $M\ddot{q}(t) + Kq(t) = 0$, while columns of Φ are its modes.

Then the linearized parameter dependent generalized eigenvalue problem (PGEP), which corresponds to (1.1), is obtained from

$$(\lambda^2(\mathbf{v})\Phi^T M \Phi + \lambda(\mathbf{v})\Phi^T C(\mathbf{v})\Phi + \Phi^T K \Phi)\Phi^{-1} x(\mathbf{v}) = 0, \tag{1.6}$$

i.e,

$$(\lambda^2(\mathbf{v})I + \lambda(\mathbf{v})\Phi^T C(\mathbf{v})\Phi + \Omega^2)\Phi^{-1} x(\mathbf{v}) = 0, \tag{1.7}$$

simply by substituting $y_1(\mathbf{v}) = \Omega\Phi^{-1}x(\mathbf{v})$ and $y_2(\mathbf{v}) = \lambda(\mathbf{v})\Omega^{-1}y_1(\mathbf{v})$ which gives us the following form of (1.7)

$$\begin{aligned} \Omega y_2(\mathbf{v}) &= \lambda(\mathbf{v}) y_1(\mathbf{v}), \\ -\Omega y_1(\mathbf{v}) - \Phi^T C(\mathbf{v})\Phi y_2(\mathbf{v}) &= \lambda(\mathbf{v}) y_2(\mathbf{v}), \end{aligned} \tag{1.8}$$

Then (PGEP) follows directly from (1.8) and is given by:

$$\mathbf{A}(\mathbf{v})\mathbf{y}(\mathbf{v}) = \lambda(\mathbf{v})\mathbf{y}(\mathbf{v}), \tag{1.9}$$

where

$$\mathbf{A}(\mathbf{v}) = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T C_{\text{int}}\Phi - \Phi^T C_{\text{ext}}(\mathbf{v})\Phi \end{bmatrix}, \quad \text{and} \quad \mathbf{y}(\mathbf{v}) = \begin{bmatrix} y_1(\mathbf{v}) \\ y_2(\mathbf{v}) \end{bmatrix}, \tag{1.10}$$

but now the dimension of the eigenvalue problem is doubled. Similarly we can transform (1.2) into $2n$-dimensional first-order differential equation. By setting

$$w_1(\mathbf{v};t) = \Omega\Phi^{-1}q(\mathbf{v};t), \quad w_2(\mathbf{v};t) = \Phi^{-1}\dot{q}(\mathbf{v};t), \tag{1.11}$$

system (1.2), i.e.,

$$\begin{aligned} \Phi^T M\Phi\Phi^{-1}\ddot{q}(\mathbf{v};t) + \Phi^T C(\mathbf{v})\Phi\Phi^{-1}\dot{q}(\mathbf{v};t) + \Phi^T K\Phi\Phi^{-1}q(\mathbf{v};t) = 0,\\ \Phi^{-1}\ddot{q}(\mathbf{v};t) + \Phi^T C(\mathbf{v})\Phi\Phi^{-1}\dot{q}(\mathbf{v};t) + \Omega^2\Phi^{-1}q(\mathbf{v};t) = 0, \end{aligned} \tag{1.12}$$

can be written as

$$\begin{aligned} \dot{w}_1(\mathbf{v};t) &= \Omega\Phi^T\dot{q}(\mathbf{v};t)\\ &= \Omega w_2(\mathbf{v})\\ \dot{w}_2(\mathbf{v};t) &= \Phi^{-1}\ddot{q}(\mathbf{v};t)\\ &= -\Phi^T C(\mathbf{v})\Phi\Phi^{-1}\dot{q}(\mathbf{v};t) - \Omega^2\Phi^{-1}q(\mathbf{v};t)\\ &= -\Phi^T C(\mathbf{v})\Phi w_2(\mathbf{v};t) - \Omega w_1(\mathbf{v};t) \end{aligned} \tag{1.13}$$

i.e.,

$$\dot{\mathbf{w}}(\mathbf{v};t) = \mathbf{A}(\mathbf{v})\mathbf{w}(\mathbf{v};t), \tag{1.14}$$

where

$$\mathbf{w}(\mathbf{v};t) = \begin{bmatrix} w_1(\mathbf{v};t)\\ w_2(\mathbf{v};t)\end{bmatrix}, \quad \mathbf{A}(\mathbf{v}) = \begin{bmatrix} 0 & \Omega\\ -\Omega & -\Phi^T C_{\text{int}}\Phi - \Phi^T C_{\text{ext}}(\mathbf{v})\Phi\end{bmatrix}, \tag{1.15}$$

with the solution

$$\mathbf{w}(\mathbf{v};t) = e^{\mathbf{A}(\mathbf{v})t}\,\mathbf{w}_0, \tag{1.16}$$

where

$$\mathbf{w}_0 = \begin{bmatrix} w_1(\mathbf{v};0)\\ w_2(\mathbf{v};0)\end{bmatrix} = \begin{bmatrix} \Omega\Phi^{-1}q(\mathbf{v};0)\\ \Phi^{-1}\dot{q}(\mathbf{v};0)\end{bmatrix} = \begin{bmatrix} \Omega\Phi^{-1}q_0\\ \Phi^{-1}\dot{q}_0\end{bmatrix} \text{ is the initial data.} \tag{1.17}$$

Furthermore, for the mechanical system given by (1.2) it can be shown that matrix $\mathbf{A}$ is (asymptotically) stable (Hurwitz), see [112], [19]. We say that the matrix $\mathbf{A}$ is stable if all eigenvalues of $\mathbf{A}$ from (1.9) are in the left half of the complex plane, i.e.,

$$\text{Re}(\lambda(\mathbf{v})) < 0, \quad \forall\,\lambda(\mathbf{v}) \in \Lambda(\mathbf{v}).$$

The damping part that contains damping positions and viscosities is denoted by $\Phi^T C_{\text{ext}}(\mathbf{v})\Phi$, where Φ is given in (1.4) and $C_{\text{ext}}(\mathbf{v})$ is given by (1.3).

The internal damping C_{int} can be modeled in different ways. The usual assumption on internal damping is that it corresponds to the so-called modally damped case. Modally damped systems are systems that satisfies the following identity:

$$MK^{-1}C(\mathbf{v}) \;=\; C(\mathbf{v})K^{-1}M, \tag{1.18}$$

for more see [23], [112]. In [112, Theorem 2.3] author shows that so-called commuting condition (1.18) is equivalent to the fact that all three matrices M, $C(\mathbf{v})$, K can be simultaneously diagonalized.

Furthermore, in [113] authors shown that identity (1.18) is equal to commutation of matrices Ω and $\Phi^T C(\mathbf{v})\Phi$, i.e.

$$\Omega\Phi^T C(\mathbf{v})\Phi = \Phi^T C(\mathbf{v})\Phi\Omega, \tag{1.19}$$

and this is equivalent with the fact that matrix Φ diagonalizes the matrix C_{int}. The usual assumption is that internal damping is a small multiple of the critical damping, i.e., in the case of critical damping

$$C_{\text{int}} = \alpha_c C_{\text{crit}}, \tag{1.20}$$

$$C_{\text{crit}} = 2M^{\frac{1}{2}}\sqrt{M^{-\frac{1}{2}}KM^{-\frac{1}{2}}}M^{\frac{1}{2}} \tag{1.21}$$

see, e.g., [107], [8], [112], [57], [1]. In this thesis we are focused on internal damping which is small multiple of critical damping, then $\Phi^T C_{\text{int}}\Phi = 2\alpha_c\Omega$ holds, but our results can be easily extended to other cases of internal damping that correspond to the modally damped case, such as proportional damping (Rayleigh damping) $C_{\text{int}} = \alpha M + \beta K$, where $\Phi^T C_{\text{int}}\Phi = \alpha I + \beta\Omega^2$ holds. More details regarding the considered model can be found in [110] where author considers corresponding inverse quadratic eigenvalue problem.

Damping optimization was widely studied in the last few decades. In the most general context, the problem is for given mass and stiffness to determine the damping matrix that ensures evanescence of unwanted vibrations, i.e., we want to determine the optimal dampers' positions and viscosities. This requires proper optimization criterion and the choice of proper criterion strongly depends on applications, but there are some cases where particular criteria were used. For example, in [67] the question of placement of damping elements was investigated, while in [52] the problem of periodic optimal control, which maximizes energy dissipation, was considered. An overview of different damping optimization criteria can be found, e.g., in [112] and [73].

For the non-stationary case, which means that the system is additionally excited, damping optimization was also studied in [105] and [58], where the authors derived explicit formulas for objective functions for particular types of mechanical systems, while in [78] it was shown how to compute eigenfrequencies of structures composed of a series of inclined cables. In the case

of multiple input multiple output systems one can use standard system norm, such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$ system norms as a criterion, as it was considered in [18], [99], [8] and [74].

Furthermore, regarding the non-stationary case one can minimize the objective function:

$$\frac{1}{\tau}\int_0^\tau \|q(\mathbf{v};t)\|^2 dt\,, \tag{1.22}$$

which is usually called the "average displacement amplitude". This approach was studied in [51] where authors compare the heuristics from [105], [98] and [11] with the mixed-integer programming approach from [50]. All approaches give us, at the same time, both optimal positions and optimal viscosities of the dampers, but viscosities are chosen among predetermined candidates.

Our problem described by (1.2) corresponds to stationary case for which one can consider different optimization criteria. One of the criteria is minimization of the total energy of the system, i.e.,

$$\int_0^\infty E(\mathbf{v};t)\,\mathrm{d}t \to \min, \tag{1.23}$$

where $E(\mathbf{v};t)$ is the total energy of the system at given time t

$$E(\mathbf{v};t) = \frac{1}{2}\dot{q}(\mathbf{v};t)^T M\dot{q}(\mathbf{v};t) + \frac{1}{2}q(\mathbf{v};t)^T Kq(\mathbf{v};t), \tag{1.24}$$

i.e., it is a sum of kinetic energy and elastic potential energy. Further on,

$$\begin{aligned}
\|\mathbf{w}(\mathbf{v};t)\|^2 &= \begin{bmatrix} w_1^T(\mathbf{v};t)\, w_2^T(\mathbf{v};t)\end{bmatrix}\begin{bmatrix} w_1(\mathbf{v};t)\\ w_2(\mathbf{v};t)\end{bmatrix} && (1.25)\\
&= w_1(\mathbf{v};t)^T w_1(\mathbf{v};t) + w_2(\mathbf{v};t)^T w_2(\mathbf{v};t) && (1.26)\\
&= (\Omega\Phi^{-1}q(\mathbf{v};t)^T\Omega\Phi^{-1}q(\mathbf{v};t) + (\Phi^{-1}\dot{q}(\mathbf{v};t))^T\Phi^{-1}\dot{q}(\mathbf{v};t) && (1.27)\\
&= q(\mathbf{v};t)^T\Phi^{-T}\Omega^2\Phi^{-1}q(\mathbf{v};t) + \dot{q}(\mathbf{v};t)^T\Phi^{-T}\Phi^{-1}\dot{q}(\mathbf{v};t) && (1.28)\\
&= \dot{q}(\mathbf{v};t)^T M\dot{q}(\mathbf{v};t) + q(\mathbf{v};t)^T Kq(\mathbf{v};t) && (1.29)\\
&= 2E(\mathbf{v};t), && (1.30)
\end{aligned}$$

i.e., the square of Euclidian norm of this phase space representation of the solution of system (1.2) equals twice the total energy of the system. Phase space is a space in which all possible states of a system are represented by their position and momentum, i.e., each possible state is represented by an unique point in phase space. Matrix $\mathbf{A}$ from (2.64) is called phase space matrix of the system (1.2). Matrix $\mathbf{A}$ from (1.14) is not the only phase space matrix, but all phase space matrices are unitary equivalent, see [113].

Since criterion (1.23) depends on the initial condition, the simplest way to correct this is to take the average of (1.23) over all initial states of the unit total energy and a given frequency

range, i.e.,

$$\int\limits_{\|w_0\|=1} \int\limits_{0}^{\infty} E(\mathbf{v};t)\mathrm{d}t\mathrm{d}\sigma, \tag{1.31}$$

where σ is some probability measure on the unit sphere in $\mathbb{R}^{2n}$. It can be shown that this average corresponds to the trace of the solution of the corresponding Lyapunov equation [111], [73].

Therefore, the minimization of the total energy (1.23) is equivalent to the minimization of the trace of the solution of the corresponding Lyapunov equation. more details can be found in [73], [100], [106], [113], [22].

It can be shown that the criterion of the minimization of the total energy (1.23) is equivalent to

$$\mathrm{tr}(\mathbf{Z}\mathbf{X}_{\Phi}(\mathbf{v})) \to \min, \tag{1.32}$$

where $\mathbf{X}_{\Phi}(\mathbf{v})$ is the solution of the following Lyapunov equation

$$\mathbf{A}^T(\mathbf{v})\mathbf{X}_{\Phi}(\mathbf{v}) + \mathbf{X}_{\Phi}(\mathbf{v})\mathbf{A}(\mathbf{v}) = -\mathbf{I}, \tag{1.33}$$

and $\mathbf{Z}$ is a symmetric positive semidefinite matrix that determines which undamped eigenfrequencies have to be damped. Furthermore, $\mathbf{Z}$ may be normalized to have a unit trace, for more detail see [73]. Furthermore, it is easy to show that

$$\mathrm{tr}(\mathbf{Z}\mathbf{X}_{\Phi}(\mathbf{v})) = \mathrm{tr}(\mathbf{Y}(\mathbf{v})),$$

where $\mathbf{Y}(\mathbf{v})$ is a solution of the so-called dual Lyapunov equation

$$\mathbf{A}(\mathbf{v})\mathbf{Y}(\mathbf{v}) + \mathbf{Y}(\mathbf{v})\mathbf{A}^T(\mathbf{v}) = -\mathbf{Z}. \tag{1.34}$$

The structure of the matrix $\mathbf{Z}$ has been studied in detail in [73] and some of these results are presented in [100].

Throughout this thesis we will assume that the matrix $\mathbf{Z}$ has the following form

$$\mathbf{Z} = \begin{bmatrix} 0_{t_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{t_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{t_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{t_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{t_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{t_3} \end{bmatrix}, \tag{1.35}$$

where I_{t_2} is the t_2-dimensional identity matrix, and 0_{t_i} is the t_i-dimensional ($i = 1,3$) zero matrix, with t_1 and t_3 defined such that the eigenfrequencies from (1.5) smaller than ω_{t_1} and greater than

$\omega_{t_1+t_2}$ are not dangerous (observe that $t_3 = n - t_1 - t_2$).

There exist several methods for solving the Lyapunov equation. Standard direct approaches for solving the Lyapunov equation use Schur form, such as Bartels–Stewart algorithm [7] and Hammarling algorithm [40], [55], an overview of the direct approaches can be found in [3]. On the other hand, the structured Lyapunov equation can be solved by iterative approaches, such as, sign function method [53], [10], [30], projection-type method based on Krylov subspaces [89] and several versions of ADI (alternating direction implicit) method [79], [62].

Criterion based on minimization of total energy was considered in [22], [12], [13], [11] [100], [111], [106], [107], [113], [19]. Approximations of the trace of the solution of the Lyapunov equation (1.34) based on modal eigenvectors and dimension reduction was considered in [12] and [13]. These approaches provide an efficient calculation of the objective function, [12] in the case where parameter is one-dimensional and [13] in case of multidimensional parameter and low-rank right hand side of the Lyapunov equation (1.34).

The algorithm presented in [111], [100] or [107] explicitly calculates the trace of the solution of the corresponding Lyapunov equation and they consider the case with one or more dampers with the same viscosity.

In [111] author gave an efficient algorithm for computation of optimal viscosity v (dimension of the parameter is one), where $C(v) = vcc^T$, i.e., $C_{\text{int}} = 0$ and rank $C(v) = 1$. The author also gave an explicit formula for the trace of the corresponding Lyapunov equation. In [100], the author considers the case where rank $C(v) > 1$, but the dimension of the parameter is again one, and gave an efficient algorithm which derives a formula for the trace of the solution of the Lyapunov equation, based on Bartels-Stewart given in [7]. Similarly, but for low-rank right hand side of the Lyapunov equation (1.34), in [107] authors gave an efficient algorithm which computes the trace of the solution of Lyapunov equation, but this time it is based on CF-ADI (Cholesky factor ADI) given in [62]. On the other hand, a more general case with the damping matrix

$$C(\mathbf{v}) = C_{\text{int}} + C_0 \text{diag}(v_1, \ldots, v_s) C_0^T$$

has been considered in [19] and it is a Newton-type algorithm. As shown in [107], the algorithm proposed in [19] can produce a poor result due to the problems with determination of the starting point.

The existence and the uniqueness of the global minimum, if the damping varies over the set of all possible positive definite matrices was considered in [113] and [22]. In [106] authors showed that the trace of a corresponding Lyapunov equation can be represented as a rational function of viscosity.

Note that the solution of the Lyapunov equation (1.34) is a function of several variables, damper positions and corresponding viscosities. Thus, simultaneous optimization of damper positions and viscosity can be computationally very demanding. Since up to date an efficient general algorithm for the optimization of damping does not exist, i.e., available algorithms opti-

mize only viscosities of dampers, not their positions, although for estimation of optimal viscosity for given damper positions in [11] authors propose the algorithm based on the dimension reduction of the corresponding Lyapunov equation, given in [12] and [13], while for the optimization of damper positions they proposed two heuristics. Both algorithms unfortunately do not have bounds for their accuracy.

On the other hand, we consider eigenvalue-based criteria, which also play a very important role, e.g., in [75], [39], [30]. One example of an eigenvalue-based criterion is the spectral abscissa criterion that minimizes the maximal real part of all eigenvalues, i.e.,

$$\min_{\mathbf{v}\in\mathbb{R}^s_+} \alpha_{\mathrm{MCK}}(\mathbf{v}), \quad \text{where} \quad \alpha_{\mathrm{MCK}}(\mathbf{v}) = \max_{\lambda(\mathbf{v})\in\Lambda(\mathbf{v})} \mathrm{Re}(\lambda(\mathbf{v})), \tag{1.36}$$

which is connected to the rate of decay of the total energy. Since it is known that $E(\mathbf{v};t) \leq \kappa E(\mathbf{v};0)e^{2\eta t}$ for some finite $\kappa > 0$ and $\eta < 0$ independently of the chosen initial data, we define the decay rate as a function of $C(\mathbf{v})$, i.e.,

$$\eta(C(\mathbf{v})) = \min\{\eta : \exists\kappa.\,\text{s.t.}\, E(\mathbf{v};t) \leq \kappa E(\mathbf{v};0)e^{2\eta t}, \forall t > 0\}, \tag{1.37}$$

where $E(\mathbf{v};t)$ is total energy defined in (1.24). In [23] author shows that decay rate of total energy is spectral abscissa, i.e.,

$$\eta(C(\mathbf{v})) = \max_{\lambda(\mathbf{v})\in\Lambda(\mathbf{v})} \mathrm{Re}(\lambda(\mathbf{v})).$$

This means that spectral abscissa is considered as a criterion in order to judge vibrations and their decay as in [34],[23].

Similar criterion i.e., also based on corresponding eigenvalues requires minimization of the objective function

$$\max_{\lambda(\mathbf{v})\in\Lambda(\mathbf{v})} \frac{\mathrm{Re}(\lambda(\mathbf{v}))}{|\lambda(\mathbf{v})|}.$$

This criterion is designed to minimize the number of oscillations before the system comes to rest. Objective functions sometimes depend on additional constraints. For example one can consider a constraint based on

$$|\max_{\lambda(\mathbf{v})\in\Lambda(\mathbf{v})} \mathrm{Im}(\lambda(\mathbf{v}))| \leq a,$$

for some positive number a (see e.g. [19, 70, 73]). All these eigenvalue-based criteria are independent of initial condition of the system.

Eigenvalues $\lambda(\mathbf{v})$ correspond to a natural frequencies of the system (1.2) for viscosity vector $\mathbf{v}$, i.e., these are the frequencies on which the system prefers to vibrate. Vibrations can be increased if the system is excited by an external force whose frequencies are close to its natural frequencies. All frequencies of the external forces which can significantly excite the system are called dangerous frequencies (e.g., frequency induced by pedestrian movements, wind induced

frequency, frequencies induced by earthquakes etc.).

The frequency isolation problem corresponds to the problem of parameter optimization in such a way that the new system has no eigenvalues close to the dangerous frequencies (dangerous areas). This problem has been previously studied in [49] and [32]. In [49], the author proposed a Newton-type method for structures vibrating at low frequencies, while in [32], the authors proposed a less costly inverse eigenvalue method: a target spectrum away from the resonance band is fixed in advance. The frequency isolation problem in undamped vibrational systems was considered in [69], where authors consider intervals as the dangerous areas. We would like to use frequency isolation in a more general case which means that we vary parameters also in the external damping matrix. Further more, we will use ellipses with their centers on the imaginary axis as dangerous areas, where these centers are dangerous frequencies of the undamped system.

As one can see, eigenvalues play a very important role in damping optimization and there are different approaches for their computation: direct and iterative methods. Also one can differ the methods in those that solve the quadratic eigenvalue problem (QEP) directly and those that work with its linearized form. Most of the methods that deal directly with QEP are based on Newton's method, such as methods from [56], [84], [80], [88] or [30], where author uses Aberth–Ehrlich method to find roots of the corresponding characteristic polynomial. As we already said there are also methods that work on the linearized form of QEP and once we obtain the linearized form we can apply one of many direct methods for solving generalized eigenvalue problems, such as the QZ algorithm, for more on QZ algorithm see [68], [37] or the QR algorithm with column pivoting [37]. On the other hand there are iterative methods for solving the linearized form of QEP, e.g., Lanczos [60], Arnoldi [4], rational Krylov [85] or Jacobi–Davidson method [90]. There exists several Arnoldi type methods, e.g., IRA (Implicitly Restarted Arnoldi) [61], SOAR (Second Order Arnoldi) [5], TOAR (Two level orthogonal Arnoldi) [31]. A nice overview of all methods is given in [66], [87], [97].

Additionally, there are methods that approximate eigenvalues, e.g., methods presented in [75], [94], [112, Chapter 19].

1.3 Organization of the thesis

In this section we present the organization of the thesis. This thesis is based on works that have either published or been submitted for publication:

[104] Truhar, N., Tomljanović, Z., Puvača, M. (2018), Approximation of damped quadratic eigenvalue problem by dimension reduction, *Applied mathematics and computation*, 374, 40-53.

[82] Puvača, M., Truhar, N., Tomljanović, Z. (2019), Efficient Approximation of Novel Residual Bounds for Parameter Dependent Quadratic Eigenvalue Problem, *submitted in Journal of Computational and Applied Mathematics.*

[103] Truhar, N., Tomljanović, Z., Puvača, M. (2017), An Efficient Approximation For Optimal Damping In Mechanical Systems, *International journal of numerical analysis and modeling*, 14(2), 201-217.

[81] Puvača, M., Jakovčević Stor, N., Tomljanović, Z., Mitchell T. (2020), Frequency-weighted damping via non smooth optimization and fast computation of QEPs with low rank updates, *in preparation*

In Chapter 2 we present three different approximation approaches of the parameter dependant quadratic eigenvalue problem (PQEP). The first one is based on dimension reduction, and this result is available in [104]. The second approach is based on first order approximations (result is available in [82]), and the third approach is based on computation of eigenvalues of diagonal plus rank one matrices (DPR1), i.e., it is based on exploiting the structure of the damping matrix. This result is available in [81].

In Chapter 3 we present an application of approximations of parameter dependant quadratic eigenvalue problem (PQEP) in perturbation theory, i.e., we use error bounds of the approximations from Section 2.2 to obtain the efficient estimation of perturbation bounds, that do not contain perturbed eigenvalues and eigenvectors in the bound itself. Some results from Chapter 3 are also available in [82].

In Chapter 4 we deal with damping optimization and different optimization criteria, one of the criteria is minimization of total average energy and the other one is frequency isolation, i.e., it is eigenvalue-based. For the second criterion we will use approximations of eigenvalues of PQEP given in Section 2.3. We will distinguish two different criteria for frequency isolation, first one is the minimization of spectral abscissa when there are no eigenvalues in fixed ellipses. The second criterion is maximization of the major axes of these ellipses when there are no eigenvalues in them and spectral abscissa is less then a given tolerance. This result is available in [81]. The result in Chapter 4 based on minimization of total average energy is available in [103].

We state PQEP as in (1.1) with parameter $\mathbf{p}$ in all matrices $M(\mathbf{p})$, $C(\mathbf{p})$, $K(\mathbf{p})$, since in Chapter 3 we discus perturbation theory of PQEP where perturbation can be in all three matrices. Also, since in Chapter 3 we use error bounds of the approximations from Section 2.2 so in that section parameter $\mathbf{p}$ will also be in all there matrices. On the other hand in all other sections we are mainly focused on damping optimization, thus we will have parameter only in damping matrix, i.e., $\mathbf{p} = \mathbf{v}$ ($\mathbf{v}$ corresponds to viscosity parameter in damping matrix).

CHAPTER 2

Efficient approximations of PQEP

In this chapter we present three different approximation approaches of the parameter dependant quadratic eigenvalue problem (PQEP) (1.1).

Since our goal is the application of eigenvalues in damping optimization where we need to vary parameters to obtain the optimal one, the main issues we are addressing in this chapter are: "How one can efficiently calculate the eigenvalues $\lambda_i(\mathbf{p})$ (for all $i = 1,\dots,2n$, or just for one important part of the spectrum) for a large variety of parameters $\mathbf{p}$?", "How one can efficiently calculate eigenvalues $\lambda_i(\mathbf{p})$, for all $i = 1,\dots,2n$, if we already have eigenvalues $\lambda_i(\mathbf{p}^0)$ calculated, for all $i = 1,\dots,2n$, and the change in parameter is small enough, i.e., $\|\mathbf{p}-\mathbf{p}^0\| \leq \varepsilon_{\mathbf{p}^0}$, for given tolerance $\varepsilon_{\mathbf{p}^0}$?" and "How can we exploit the structure of the damping matrix to speed up the computation of eigenvalues for different parameter $\mathbf{p}$?"

As we already mentioned, the above problem is related to the efficient solution of various problems connected with damped mechanical systems (1.2), such as

1. efficient calculation of approximations of all eigenvalues or just of selected "most important undamped eigenfrequencies" with corresponding error bounds,

2. efficient damping optimization where optimization criteria is minimization of spectral abscissa, i.e., minimization of the (penalty) function $\alpha_{\mathrm{MCK}}(\mathbf{p}) = \max\limits_{i=1,\dots,2n} \mathrm{Re}(\lambda_i(\mathbf{p}))$, where $\lambda(\mathbf{p})$ are eigenvalues from eigenvalue problem (1.1), or frequency isolation or some other spectrum dependant penalty functions.

This chapter is mainly devoted to the problems from item 1. A similar problem of efficient calculation of eigenvalues $\lambda_i(\mathbf{v})$, depending only on parameter of viscosity, has been considered in [75] and [94]. In [75] authors present two different approximations, one for the case when $0 \leq v_i \ll 1$, and one for $v_i \gg 1$, $i = 1,\dots,s$, while in [94] only the case $v_i \gg 1$ has been considered.

In [112, Chapter 19] and [75] the authors consider the approximation of eigenvalues $\lambda_i(\mathbf{v})$ for $0 < v_i \ll 1$ using the approach based on so-called modal approximation.

Approximations that will be presented in this chapter will be used in Chapter 4 considering item 2.

This chapter is organized as follows. Section 2.1 related to approximations of eigenvalues by dimension reduction is divided into three subsections, where in Subsection 2.1.2 we calculate all eigenvalues and in Subsection 2.1.1 we calculate just selected part of the eigenvalues. The main result of Subsections 2.1.1 and 2.1.2 are a certain generalization of the results from [75] and [112] that holds for $v_i \ll 1$ on the set of parameters of modest magnitudes. We present an approach that can be used for the efficient calculation of the whole spectrum or just a part of the spectrum for problem (1.1), where parameters v_i, $i = 1, \ldots, s$, are of modest magnitude. However, the notion of modest magnitude for viscosity parameters depends strongly on applications.

To the best of authors' knowledge, the calculation of all eigenvalues with available algorithms will not be always fully satisfactory for such a large parameter space. On the other hand, if one is interested in calculation of just a few eigenvalues, some of the existing approaches like [38] can be superior to our approach.

The approximations of eigenvalues that are presented in the next section are based mainly on the ideas from papers [12] and [13], where the authors have considered damping optimization based on the minimization of the total average energy.

Moreover, each of these subsections will provide error bounds based on the standard Gershgorin type of bounds (see e.g. [37], [92]) and perturbation bounds like in [63], for given approximations.

In Subsection 2.1.3 we illustrate, the performance of the approximations presented in Subsections 2.1.1 and 2.1.2, and the quality of the given bounds.

In Section 2.2 we present a first order approximation based on Taylor's theorem and corresponding error bound. This section answers the question of efficient computation of eigenvalues when the change in viscosity is small enough. In Subsection 2.2.1 we compare this first order approximation with the first order approximation that appears in perturbation theory. Also, in this section we illustrate accuracy of these approximations and error bounds.

In Section 2.3 we present an approach to efficient computation of eigenvalues and eigenvectors in a case when we have the special structure of a matrix, i.e., if a matrix is diagonal plus rank one (DPR1). Efficiency of this approach is illustrated in examples in Subsection 2.3.1.

2.1 Approximations by dimension reduction

Since our goal is to apply our approximations to damping optimization where only parameter is viscosity $\mathbf{v}$, in this section we will assume that mass and stiffness matrices are independent of parameter $\mathbf{p}$ and that parameter $\mathbf{p}$ represents only viscosity of the damping matrix, i.e., $\mathbf{p} = \mathbf{v}$, which means that $M(\mathbf{p}) = M$ and $K(\mathbf{p}) = K$.

In this section, instead of analyzing PQEP (1.1), we consider the corresponding linearized parameter dependent generalized eigenvalue problem (PGEP) (1.9).

Also, in this section we avoid writing the dependence of matrices and eigenvalues (and other quantities of interest) on viscosity parameters for the sake of easier notation. This dependence

is assumed and it will only be sometimes written explicitly for emphasis.

The PGEP is given by (1.9) where

$$\mathbf{A}(\mathbf{v}) = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T C_{\text{int}} \Phi - \hat{C}(\mathbf{v}) \end{bmatrix}.$$

Here the damping part that contains damping positions and viscosities is denoted by $\hat{C}(\mathbf{v}) = \Phi^T C_{\text{ext}}(\mathbf{v})\Phi$, where Φ is given in (1.4) and $C_{\text{ext}}(\mathbf{v})$ is given by (1.3).

We distinguish two very important cases. In the first case we consider efficient approximation for eigenvalues for the selected part of the undamped spectrum. Within the second case, we consider efficient approximation for all eigenvalues. The procedures for approximations and corresponding error bounds are given in the next two subsections.

2.1.1 The approximation of the selected part of the eigenvalues

Before we present the main result in this subsection, we briefly present results on dimension reduction from [13], which allow us to efficiently approximate selected eigenvalues of the parameter dependent generalized eigenvalue problem (PGEP). Here by dimension reduction we mean that we replace the original problem containing matrices of dimension n by a problem with matrices that have much smaller dimension.

In order to introduce this approximation, first we consider matrix $\mathbf{P}$, i.e, the perfect shuffle permutation, defined by

$$\mathbf{P} \quad = \quad [e_1, e_{n+1}, e_2, e_{n+2}, \ldots, e_n, e_{2n}], \tag{2.1}$$

where e_i denotes i-th canonical vector. The perfect shuffle permutation is the permutation that splits a set of even cardinality into two sets of equal cardinality and interleaves them, precisely, it maps $k \mapsto \begin{cases} 2k-1, & \text{if } k \leq n \\ 2(k-n), & \text{if } k > n. \end{cases}$, for more see e.g. [13]. Then instead of the PGEP from (1.9), we consider a permuted one of the form

$$\mathbf{P}^T \mathbf{A} \mathbf{P} \mathbf{P}^T y = \lambda \mathbf{P}^T y, \tag{2.2}$$

where $\mathbf{A}$ is given by (1.10).

Our approach is based on dimension reduction of the PGEP (2.2), for that purpose, we construct an approximation of the matrix $\mathbf{P}^T\mathbf{A}\mathbf{P}$. In order to obtain an appropriate approximation, first we need to introduce an additional permutation denoted by matrix $\widehat{\mathbf{P}}$. The additional permutation takes into account the magnitude of elements of matrix $\widehat{C}(\mathbf{v}) = \Phi^T C_{\text{ext}}(\mathbf{v})\Phi$, which plays important role in our approximation. Moreover, the elements of the matrix Φ are closely related to displacements of the corresponding modes and very often one can expect big differences in magnitude of elements in matrix $\widehat{C}(\mathbf{v})$ (for more details on this approximation see [12], [13],

[35]).

Let the vectors $\mu \in \mathbb{N}^r$ and $\overline{\mu} \in \mathbb{N}^{n-r}$ be chosen such that the following conditions hold:

i) $\mu \cup \overline{\mu} = \{1,2,\ldots,n\}$.

ii) μ is the vector of indices of dimension r ($l \leq r$), where the first l elements of μ correspond to the indices of eigenfrequencies of our interest (for example, eigenfrequencies which have to be damped or excluded form some interval).

iii) $\overline{\mu}$ and μ are index vectors such that $\max_{i,j} |\widehat{C}(\overline{\mu}(i), \mu(j))| \leq \texttt{tol}$, for a given tolerance `tol`.

The vectors $\mu \in \mathbb{N}^r$ and $\overline{\mu} \in \mathbb{N}^{n-r}$ should be chosen such that r is as small as possible for given parameters l and `tol`. A strategy for determining $\mu, \overline{\mu}$ is discussed below.

Let us define a vector $\tau \in \mathbb{N}^n$ by $\tau(i) = \mu(i)$ for $i = 1,\ldots,r$ and $\tau(i) = \overline{\mu}(i-r)$ for $i = r+1,\ldots,n$.

The matrix $\mathbf{P}$ is the perfect shuffle permutation matrix and $\widehat{\mathbf{P}} = I(:,\tau) \otimes I_2$. Now for these permutations it holds

$$\mathbf{A}_P = \begin{bmatrix} 0 & \omega_{\tau(1)} & 0 & 0 & \ldots & 0 & 0 \\ -\omega_{\tau(1)} & -2\alpha_c\omega_{\tau(1)} - \hat{c}_{\tau(1),\tau(1)} & 0 & -\hat{c}_{\tau(1),\tau(2)} & \ldots & 0 & -\hat{c}_{\tau(1),\tau(n)} \\ 0 & 0 & 0 & \omega_{\tau(2)} & \ldots & 0 & 0 \\ 0 & -\hat{c}_{\tau(2),\tau(1)} & -\omega_{\tau(2)} & -2\alpha_c\omega_{\tau(2)} - \hat{c}_{\tau(2),\tau(2)} & \ldots & 0 & -c_{\tau(2),\tau(n)} \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -\hat{c}_{\tau(n-1),\tau(1)} & 0 & -\hat{c}_{\tau(n-1),\tau(2)} & \ldots & 0 & -\hat{c}_{\tau(n-1),\tau(n)} \\ 0 & 0 & 0 & 0 & \ldots & 0 & \omega_{\tau(n)} \\ 0 & -\hat{c}_{\tau(n),\tau(1)} & 0 & -\hat{c}_{\tau(n),\tau(2)} & \ldots & -\omega_{\tau(n)} & -2\alpha_c\omega_{\tau(n)} - \hat{c}_{\tau(n),\tau(n)} \end{bmatrix}, \tag{2.3}$$

with $\mathbf{A}_P = \widehat{\mathbf{P}}^T\mathbf{P}^T\mathbf{A}\mathbf{P}\widehat{\mathbf{P}}$, for more see [12], [13] .

We are interested in dimension reduction which allows us to approximate eigenvalues efficiently, thus we define an approximation matrix with

$$\widetilde{\mathbf{A}}_P = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \tag{2.4}$$

where

$$A_{11} = \mathbf{A}_P(1:2r, 1:2r) \quad \text{and} \quad A_{22} = \mathbf{A}_P(2r+1:2n, 2r+1:2n) \tag{2.5}$$

for $\mathbf{A}_P$ given as in (2.3).

By this approach we reduce dimension from $2n$ to $2r$, since we compute eigenvalues of A_{11} instead of $\mathbf{A}_P$. The parameter r is called the reduced dimension. Note that the more elements of $\widehat{C}$ are small by magnitude, the smaller the reduced dimension is.

Therefore, our approach is efficient when we have large amount of small elements in the matrix $\widehat{C}$. This is often the case in non homogeneous systems, for more see [12], [13], [35]. On

the other hand, when this is not the case, our approach will not be that efficient, e.g., in cases with Gyroscopic system [6], [97].

Vectors μ and $\overline{\mu}$ can be calculated adaptively using Algorithm 1, which can be found in [13, Algorithm 1], where the indices that correspond to the selected part are included in the vector μ based on important (significant) undamped eigenfrequencies of our interest (for example, those which have to be damped). These indices usually correspond to a certain percentage of the undamped spectra, e.g. a certain percentage of the smallest undamped eigenfrequencies or undamped eigenfrequencies within some interval. In order to achieve an efficient approximation, we ensure that all elements of the matrix $\widehat{C}$ to be omitted are smaller up to the chosen tolerance, i.e., we have that $\max_{i,j} |\widehat{C}(\mu(i),\overline{\mu}(j))| < \texttt{tol}$, which will have a direct impact on the corresponding error bound.

Algorithm 1: Construction of μ and $\overline{\mu}$

Require: `tol`;
$v_i, C_i, i = 1,\ldots,s$ – viscosity and the external geometry matrix for the ith damper;
$\mu(1),\mu(2),\ldots,\mu(l)$ – indices of the eigenfrequencies which have to be damped;
Ensure: $\mu,\overline{\mu}$

1: $\mu = [\mu(1),\mu(2),\ldots,\mu(l)]$
2: Determine vector $\overline{\mu}$ such that $\mu \cup \overline{\mu} = \{1,2,\ldots,n\}$
3: $T = 1$
4: $\widehat{C} = \Phi^T(v_1C_1 + v_2C_2 + \cdots + v_sC_s)\Phi$,
5: **while** $T = 1$ **do**
6: $M = \max_{ij} |\widehat{C}(\mu(i),\overline{\mu}(j))|$
7: **if** $M >$ `tol` **then**
8: Determine indices i_0, j_0 such that $M = |\hat{c}_{i_0,j_0}|$, while ensuring that j_0 is not used before and i_0 contained in μ
9: $\mu = [\mu, j_0]$
10: Determine vector $\overline{\mu}$ such that $\mu \cup \overline{\mu} = \{1,2,\ldots,n\}$
11: **else**
12: $T = 0$
13: **end if**
14: **end while**

Once we have obtained vectors μ and $\overline{\mu}$, we can introduce Algorithm 2 for calculation of eigenvalue approximations.

The following subsubsection provides two error bounds for the approximations obtained by Algorithm 2.

2.1.1.1 Error bound for approximation of selected eigenfrequencies

The bounds for approximations of selected eigenfrequencies can be derived using several different approaches usually used for error estimation in the eigenvalue approximation. One approach is based on the standard Gershgorin type of bounds, like, e.g., in [37] and [92]. Another

Algorithm 2: Approximation of eigenvalues that correspond to selected eigenfrequencies

Require: α_c, Φ – such that $\Phi^T K\Phi = \Omega^2 = \mathrm{diag}(\omega_1^2,\ldots,\omega_n^2)$ and $\Phi^T M\Phi = I$;
v_i, C_i, $i = 1,\ldots,s$ – viscosity and the external geometry matrix for the ith damper;
$\mu(1),\mu(2),\ldots,\mu(l)$ – indices of the eigenfrequencies which have to be damped;
`tol` – tolerance needed for Algorithm 1.

Ensure: approximation of eigenvalues

1: Determine vectors $\overline{\mu} \in \mathbb{N}^{n-r}$ and $\mu \in \mathbb{N}^r$ using Algorithm 1,
2: $\Omega_r = \mathrm{diag}(\omega_{\mu(1)}, \omega_{\mu(2)}, \ldots, \omega_{\mu(r)})$,
3: $\widehat{C} = \Phi^T(v_1C_1 + v_2C_2 + \cdots + v_sC_s)\Phi$,
4: Calculate all eigenvalues of matrix $\begin{bmatrix} 0 & \Omega_r \\ -\Omega_r & -2\alpha_c\Omega_r - \hat{C}(\mu,\mu) \end{bmatrix}$.

approach is based on perturbation bounds from [63].

First, we present a bound based on the bound from [63, Theorem 4.1]. In order to derive the error bound, here we consider matrix

$$\mathbf{A}_P = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{2.6}$$

that corresponds to block partition of matrix (2.3), where A_{11} is block of dimension $2r$. Moreover, let $\widetilde{\mathbf{A}}_P$ correspond to the perturbed equation (2.4).

We establish a bound on $|\widetilde{\lambda} - \lambda|$, where λ is an eigenvalue of $\mathbf{A}_P$ and $\widetilde{\lambda}$ is an eigenvalue of $\widetilde{\mathbf{A}}_P$.

Theorem 2.1 Let $\mathbf{A}_P$ and $\widetilde{\mathbf{A}}_P$ be as in (2.6) and (2.4) (i.e., $\widetilde{\mathbf{A}}_P$ is obtained by Algorithm 2). Furthermore, let matrices $\mathbf{A}_P$ and $\widetilde{\mathbf{A}}_P$ be diagonalizable. If $\widetilde{\lambda}_i$ is an eigenvalue of A_{11}, and if

$$\eta_2(\widetilde{\lambda}_i) = \min_{\rho \in \sigma(A_{22})} |\widetilde{\lambda}_i - \rho| > 0\,,$$

then $\mathbf{A}_P$ has an (exact) eigenvalue $\lambda_{\pi(i)}$ such that

$$|\widetilde{\lambda}_i - \lambda_{\pi(i)}| \leq \kappa_2(\mathbf{X}) \frac{\|A_{12}\|_2 \|A_{21}\|_2}{\eta_2(\widetilde{\lambda}_i)}, \tag{2.7}$$

where matrix $\mathbf{X}$ is a matrix that diagonalizes $\widetilde{\mathbf{A}}_P$.

Proof. The proof follows directly from [63, Theorem 4.1], where the bound for diagonalizable non-Hermitian pencils $\mathbf{A} - \lambda\mathbf{B}$, has been presented, by setting $\mathbf{B} = \mathbf{I}$ and $\delta\mathbf{B} = 0$. □

The application of bound (2.7) makes sense if $\kappa_2(\mathbf{X})$ has the modest magnitude and if gap $\eta_2(\widetilde{\lambda}_i)$ can be calculated efficiently and is not too small. For the purpose of clarifying this, we present the following remark.

Remark 2.1. Since throughout this section we are interested in calculation of the approximation of the part of the spectrum $(\widetilde{\lambda}_i,\, i=1,\ldots,r)$, we can assume that $\mathbf{X}$ can be written as

$$\mathbf{X} = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{pmatrix} X_{11} & \\ & I \end{pmatrix} \end{array},$$

where X_{11} diagonalizes A_{11}. Then it follows from (2.7) in Theorem 2.1 that

$$|\widetilde{\lambda}_i - \lambda_{\pi(i)}| \leq \kappa_2(X_{11}) \frac{\|A_{12}\|_2 \|A_{21}\|_2}{\eta_2(\widetilde{\lambda}_i)}, \tag{2.8}$$

where

$$\eta_2(\widetilde{\lambda}_i) = \min_{\rho \in \sigma(\widetilde{A}_{22})} |\widetilde{\lambda}_i - \rho| > 0\,.$$

On the other hand, one can notice that in general calculation of the gap $\eta_2(\widetilde{\lambda}_i)$ can be demanding, especially if $r \ll n$, i.e., when A_{22} has "significant" dimension.

If that is the case, bound (2.7) or (2.8) can be applied using the simple estimation for $\eta_2(\widetilde{\lambda}_i)$ based on the approximation of the spectrum of the matrix A_{22}, which can be obtained easily. As we explain in more detail in the next section, one example of a matrix with the "significant" dimension is A_{22} of the following form

$$A_{22} = \bigoplus_{i=r+1}^{n} \Psi_{\tau(i)} + E_{22}, \quad \text{where} \quad \Psi_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -\gamma_i - \hat{c}_{ii} \end{bmatrix},$$

and where $\|E_{22}\|$ has a modest magnitude. The spectrum of the matrix A_{22} can be easily approximated with eigenvalues of Ψ_i. Here parameter γ_i depends on internal damping. In particular, if internal damping is a small multiple of critical damping defined in (1.20), then $\gamma_i = 2\alpha_c\omega_i$.

We would like to note that bounds (2.7) and (2.8) can be useful if the gap $\eta_2(\widetilde{\lambda}_i)$ can be estimated efficiently and accurately, which in general will not be possible. Thus, in what follows we present an error bound in terms of Gershgorin eigenvalue bounds. Similarly to the above, we assume that we have determined index vectors $\overline{\mu}$ and μ such that $\max_{ij} |\widehat{C}(\overline{\mu}(i), \mu(j))| \leq \texttt{tol}$ for a given tolerance `tol`. Then, in order to apply the Gershgorin bound, we diagonalize the block A_{11}, and for this we assume that all eigenvalues from the block A_{11} are simple. In most cases, this is true. But even if the eigenvalues are not simple, we can still obtain the approximations, but without this error bound. Let X_{11} be such that

$$A_{11} = X_{11}\Lambda_{11}X_{11}^{-1}, \tag{2.9}$$

where A_{11} is given by (2.5). Here, diagonal elements of the matrix $\Lambda_{11} = \operatorname{diag}(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{2r})$

provide approximations of desired eigenvalues given by Algorithm 2. Then, using the block diagonal matrix

$$\mathbf{X}_1 = \begin{bmatrix} X_{11} & 0 \\ 0 & I \end{bmatrix}$$

we obtain

$$\mathbf{X}_1^{-1}\mathbf{A}_P\mathbf{X}_1 = \begin{bmatrix} \Lambda_{11} & X_{11}^{-1}A_{12} \\ A_{21}X_{11} & A_{22} \end{bmatrix}.$$

In order to obtain an error bound that separately provides an error for each eigenvalue, we apply the Gershgorin theorem (see, e.g. [37], [92]). Here, we use a row version of the Gershgorin bound; thus, for each i there exists an index $\pi(i)$ such that the following bound holds:

$$|\widetilde{\lambda}_i - \lambda_{\pi(i)}(\mathbf{A}_P)| \leq \sum_{j=1}^{2n-2r} |(X_{11}^{-1}A_{12})_{ij}|, \tag{2.10}$$

for $i = 1, \ldots, 2r$, where A_{12} and X_{11} are given by (2.6) and (2.9), respectively.

Now, once we have the two error bounds, a legitimate question is which of the bounds, (2.8) or (2.10), is better? As we will show in Subsection 2.1.3, sometimes bound (2.8) is better and sometimes it is (2.10), thus the best option would be to take the minimum of both derived bounds for the given eigenvalue approximation.

2.1.2 The approximation of all eigenvalues

As before, we apply the perfect shuffle permutation matrix $\mathbf{P}$ and add an additional permutation that allows us to approximate all eigenvalues. In order to achieve that, we need to determine all elements from the matrix $\widehat{C} = \Phi^T C_{\text{ext}}\Phi$ with absolute value larger than a given tolerance. Thus, apart from the permutations from the previous section, here we determine vectors $\mu \in \mathbb{N}^r$ and $\overline{\mu} \in \mathbb{N}^{n-r}$ chosen such that the following conditions hold:

i) $\mu \cup \overline{\mu} = \{1, 2, \ldots, n\}$.

ii) $\overline{\mu}$ and μ are index vectors such that $\max_{ij} |\widehat{C}(\overline{\mu}(i), \mu(j))| \leq \texttt{tol}$ for a given tolerance `tol`.

iii) $\overline{\mu}$ and μ are index vectors such that $\max_{ij} |\widehat{C}_{\text{off}}(i,j)| \leq \texttt{tol}$ for a given tolerance `tol`, where matrix $\widehat{C}_{\text{off}} \in \mathbb{R}^{(n-r)\times(n-r)}$ contains off-diagonal elements from matrix $\widehat{C}(\overline{\mu}, \overline{\mu})$, i.e., it is given by

$$\widehat{C}_{\text{off}}(i,j) = \begin{cases} \widehat{C}(\overline{\mu}(i), \overline{\mu}(j)), & i \neq j, \\ 0, & i = j. \end{cases}$$

Here we also try to obtain vectors $\mu \in \mathbb{N}^r$ and $\overline{\mu} \in \mathbb{N}^{n-r}$, such that r is as small as possible, but since we are interested in all eigenvalues we will obtain the reduced dimension which is always at least as large as in the case of approximation of the important (significant) part of the spectrum.

Moreover, since we have more demanding conditions for the determination of the first r indices, it is more likely that the final reduced dimension is larger than in the previous section. On the other hand, we are able to approximate all eigenvalues, so it is natural to expect that the bigger reduced dimension is needed.

Now, similarly to the previous section, we define a vector $\tau \in \mathbb{N}^n$ by $\tau(i) = \mu(i)$ for $i = 1,\dots,r$ and $\tau(i) = \overline{\mu}(i-r)$ for $i = r+1,\dots,n$. With this permutation, our aim is to approximate all eigenvalues of the matrix $\mathbf{A}_P$ from (2.3).

In this case, we approximate the matrix $\mathbf{A}_P$ by the matrix

$$\overline{\mathbf{A}}_P = \begin{bmatrix} \overline{A}_{11} & 0 \\ 0 & \overline{A}_{22} \end{bmatrix},$$

with

$$\overline{A}_{11} = \mathbf{A}_P(1:2r, 1:2r), \tag{2.11}$$

$$\overline{A}_{22} = \bigoplus_{i=r+1}^{n} \Psi_{\tau(i)},$$

$$\Psi_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -\gamma_i - \hat{c}_{ii} \end{bmatrix}. \tag{2.12}$$

Due to the block structure of our approximated matrix $\overline{\mathbf{A}}_P$, in order to efficiently determine approximation of all eigenvalues, we need to determine eigenvalues of the matrix $\overline{A}_{11}$, while eigenvalues of the matrix $\overline{A}_{22}$ can be calculated by a formula. Thus, the matrix $\Psi_{\tau(i)}$ given by (2.12) has eigenvalues

$$\begin{aligned} \widetilde{\lambda}_{2i-1} &= \frac{-\gamma_i - \hat{c}_{ii} - \sqrt{(\gamma_i + \hat{c}_{ii})^2 - 4\omega_i^2}}{2} \quad \text{for } i = r+1,\dots,n, \\ \widetilde{\lambda}_{2i} &= \frac{-\gamma_i - \hat{c}_{ii} + \sqrt{(\gamma_i + \hat{c}_{ii})^2 - 4\omega_i^2}}{2} \quad \text{for } i = r+1,\dots,n, \end{aligned} \tag{2.13}$$

where $\hat{c}_{ii} = \sum_{i=1}^{s} v_i \Phi(:,i)^T C_i \Phi(:,i)$ and $\gamma_i = 2\alpha_c \omega_i$.

All of the above-mentioned is summarized in Algorithm 3, which calculates approximation of all eigenvalues.

In the next subsubsection we provide an error bound for the approximations given by Algorithm 3.

We would like to emphasize that in general the new approach (summarized in Algorithm 3) improves the approximation technique studied in [75] and [112] (derived specifically for a small viscosity $\mathbf{v} \ll 1$). More precisely, using Algorithm 3 with a tolerance `tol` large enough, one can obtain approximations of the same quality as in [75] and [112]. On the other hand, a smaller tolerance `tol` leads to the reduced dimension $r > 0$, that further ensures a better approximation

Algorithm 3: Approximation of all eigenvalues

Require: α_c, Φ – such that $\Phi^T K\Phi = \Omega^2 = \mathrm{diag}(\omega_1^2,\ldots,\omega_n^2)$ and $\Phi^T M\Phi = I$;
v_i, C_i, $i = 1,\ldots,s$ – viscosity and the external geometry matrix for the ith damper;
`tol` – tolerance needed for determination of the vector τ.

Ensure: approximation of eigenvalues

1: Each index $i \in \{1,\ldots,n\}$ such that $|\hat{c}_{ij}| >$ `tol` for some index $j \neq i$, is included in vector of indices μ. The number of elements in μ determines r and $\overline{\mu}$ is such that $\mu \cup \overline{\mu} = \{1,2,\ldots,n\}$.

2: $\Omega_r = \mathrm{diag}(\omega_{\mu(1)}, \omega_{\mu(2)}, \ldots, \omega_{\mu(r)})$,

3: $\widehat{C} = \Phi^T(v_1C_1 + v_2C_2 + \cdots + v_sC_s)\Phi$,

4: Calculate all eigenvalues of matrix

$$\begin{bmatrix} 0 & \Omega_r \\ -\Omega_r & -2\alpha_c\Omega_r - \hat{C}(p,p) \end{bmatrix},$$

5: Calculate other eigenvalues using formulas (2.13).

than the one from [75] and [112], as we will illustrate in the numerical experiments.

The benefits of the new approximation technique over the approach, which has been derived specifically for small viscosities v_i in [75] and [112], are illustrated in the Subsection 2.1.3, especially Figure 2.3 illustrates the eigenvalue behavior obtained by formulas (2.13) as well as the approximations that were also obtained by Algorithm 3 for the tolerance `tol` large enough.

2.1.2.1 Error bound for approximation of all eigenvalues

In this subsubsection, we present error bounds for the eigenvalue approximations made by Algorithm 3 in the sense of Gershgorin. We assume that for the fixed viscosity $\mathbf{v}$ we have determined index vectors $\overline{\mu}$ and μ as in Algorithm 3.

Similar to (2.3) our matrix of interest can be written in the block diagonal form as:

$$\mathbf{A}_P = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix},$$

where $\overline{A}_{11}$ is a matrix of dimension $2r \times 2r$. Now, our approximation corresponds to a matrix

$$\overline{\mathbf{A}}_P = \begin{bmatrix} \overline{A}_{11} & 0 \\ 0 & \oplus_{i=r+1}^{n} \Psi_{\tau(i)} \end{bmatrix},$$

where Ψ_i is given by (2.12).

As in the previous section, we assume that all eigenvalues of $\overline{\mathbf{A}}_P$ are simple.

In order to apply the Gershgorin bound we need to diagonalize all diagonal blocks of the

matrix $\overline{\mathbf{A}}_P$. Thus, we first diagonalize the block $\overline{A}_{11}$, i.e., we calculate the matrix X_{11} such that

$$\overline{A}_{11} = X_{11}\mathrm{diag}(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{2r})X_{11}^{-1}, \tag{2.14}$$

where elements

$$\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{2r} \tag{2.15}$$

provide the approximations of true eigenvalues.

Moreover, we need to diagonalize all two-by-two matrices $\Psi_{\tau(i)}$ for $i = r+1,\ldots,n$. For that, we assume that $(\gamma_{\tau(i)} + \hat{c}_{\tau(i)\tau(i)})^2 - 4\omega_{\tau(i)}^2 \neq 0$ for $i = r+1,\ldots,n$; thus there exist matrices $Y_{r+1,r+1},\ldots,Y_{n,n}$ such that

$$\Psi_{\tau(i)} = Y_{ii}\mathrm{diag}(\widetilde{\lambda}_{2i-1},\widetilde{\lambda}_{2i})Y_{ii}^{-1}, \quad i = r+1,\ldots,n,$$

where $\Psi_{\tau(i)}$ and $\widetilde{\lambda}_{2i-1}$, $\widetilde{\lambda}_{2i}$ are given in (2.12) and (2.13), respectively.

Then, using a block diagonal matrix

$$\widehat{\mathbf{X}} = \begin{bmatrix} X_{11} & 0 \\ 0 & \widehat{X}_{22} \end{bmatrix} \quad \text{with} \quad \widehat{X}_{22} = \left(\bigoplus_{l=r+1}^{n} Y_{ll} \right) \tag{2.16}$$

we obtain

$$\widehat{\mathbf{X}}^{-1}\overline{\mathbf{A}}_P\widehat{\mathbf{X}} = \begin{bmatrix} \Lambda_{11} & X_{11}^{-1}\overline{A}_{12}\widehat{X}_{22} \\ \widehat{X}_{22}^{-1}\overline{A}_{21}X_{11} & A_Y \end{bmatrix}$$

with

$$A_Y = \widehat{X}_{22}^{-1}\overline{A}_{22}\widehat{X}_{22}, \tag{2.17}$$

where $\widehat{X}_{22}$ is given by (2.16).

Similarly to the previous section, we apply the Gershgorin theorem. To the upper and the lower diagonal block we apply a row version of the Gershgorin bound; thus, for each $i \in \{1,2,\ldots,2n\}$ there exists index $\pi(i) \in \{1,2,\ldots,2n\}$ such that the following bound holds:

$$\begin{aligned}
|\widetilde{\lambda}_i - \lambda_{\pi(i)}(\mathbf{A}_P)| &\leq \sum_{j=r+1}^{2n} \left| (X_{11}^{-1}\overline{A}_{12}\widehat{X}_{22})_{ij} \right|, \quad i = 1,\ldots,2r, \\
|\widetilde{\lambda}_{2i-1} - \lambda_{\pi(2i-1)}(\mathbf{A}_P)| &\leq \sum_{j=1}^{2r} \left| \left(\widehat{X}_{22}^{-1}\overline{A}_{21}X_{11} \right)_{2i-2r-1,j} \right| \\
&\quad + \sum_{\substack{j=1, \\ j\neq 2i-2r-1}}^{n-2r} |(A_Y)_{2i-2r-1,j}|, \quad i = r+1,\ldots,n, \\
|\widetilde{\lambda}_{2i} - \lambda_{\pi(2i)}(\mathbf{A}_P)| &\leq \sum_{j=1}^{2r} \left| \left(\widehat{X}_{22}^{-1}\overline{A}_{21}X_{11} \right)_{2i-2r,j} \right|
\end{aligned} \tag{2.18}$$

$$+ \sum_{\substack{j=1 \\ j\neq 2i-2r}}^{n-2r} \left|(A_Y)_{2i-2r,j}\right|, \quad i = r+1,\ldots,n,$$

where r is the number of elements in the vector μ and A_Y and $\widehat{X}_{22}$ are given by (2.17), (2.16), respectively.

In what follows we will present another bound of similar type (Gershgorin type bound) from [94], which will be used for the comparison in the next subsection. In [94] author considers following quadratic eigenvalue problem

$$(\lambda^2(v)M + \lambda(v)vC + K)x(v) = 0, \tag{2.19}$$

where v is a positive real number that represents viscosity. In order to apply Gershgorin type bound, (2.19) is linearized in the following way

$$\begin{bmatrix} & B^{\frac{1}{2}} \\ -B^{\frac{1}{2}} & -vA \end{bmatrix} z = \lambda z, \tag{2.20}$$

where $A = M^{-\frac{1}{2}}CM^{-\frac{1}{2}}$, $B = M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$ and $\operatorname{rank} C = \hat{r}$. Let $\rho_1, \rho_2, \ldots, \rho_{\hat{r}}$ be nonzero eigenvalues of $-vA$, then [94, Lema 3.3.1] gives Gershgorin type bound. For each $i \in \{1,2,\ldots,2n\}$ there exists index $\pi(i) \in \{1,2,\ldots,2n\}$ such that the following bounds hold

$$\begin{aligned} |\widetilde{\lambda}_i| &\leq ||B||_2^{\frac{1}{2}}, & i &= 1,\ldots,\hat{r}, \\ |\widetilde{\lambda}_i - \rho_{\pi(i)}| &\leq ||B||_2^{\frac{1}{2}}, & i &= \hat{r}+1,\ldots,2n, \end{aligned} \tag{2.21}$$

where

$$0, \rho_1, \rho_2, \ldots, \rho_{\hat{r}} \tag{2.22}$$

are approximations of eigenvalues of (2.19) where multiplicity of eigenvalue 0 is $2n - \hat{r}$.

2.1.3 Numerical experiments

In this subsection, we present two examples. In these examples we will compare our approximations with exact eigenvalues obtained by MATLAB's function `eig` in double precision, for more on `eig` see [37],[27].

In the first example, we consider the eigenvalue behavior for all eigenvalues considered in Subsection 2.1.2 and we compare our bounds given in (2.18) with the bounds given in (2.19). In the second example we illustrate the quality of eigenvalue approximation given by Algorithm 2, where we are interested in the behavior only of the part of the undamped eigenfrequencies.

Example 2.1 We consider an n-mass oscillator or oscillator ladder with two dampers, which describes the mechanical system of n masses and $n+1$ springs, shown in Figure 2.1. Similar

models were considered e.g. in [12], [13], [107] and [112].

For this mechanical system, the mathematical model is given by (1.1), where the mass and stiffness matrices are

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n),$$
$$K = \begin{pmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n & \\ & & & -k_n & k_n+k_{n+1} \end{pmatrix}.$$

Mass and stiffness are given by the following configuration

$$n = 1000; \quad k_i = 1, \quad i = 1, \ldots, n+1; \qquad m_i = \begin{cases} 1200-2i, & i = 1, \ldots, 200, \\ 4i, & i = 201, \ldots, n. \end{cases}$$

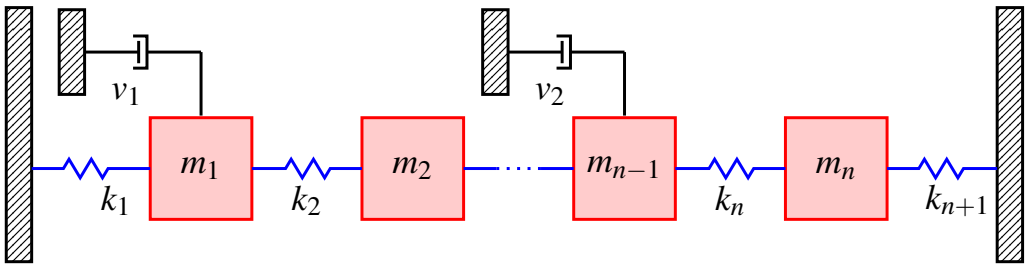

Figure 2.1: n mass oscillator

We present two cases: in case a) we show the quality of the obtained approximation and the eigenvalue behavior while we vary viscosity parameters and in case b) we compare our approximations given by (2.13) to the approximations given by (2.22) and our bound given in (2.18) to the bound given in (2.21).

Case a)

The damping matrix is $C(\mathbf{v}) = C_{\text{int}} + C_{\text{ext}}(\mathbf{v})$, where the internal damping C_{int} is defined as in (1.20) with $\alpha_c = 0.02$. We consider two dampers of different viscosities, i.e., $\mathbf{v} = [v_1, v_2]$; thus external damping introduced in (1.3) is defined by $C_{\text{ext}}(\mathbf{v}) = v_1 e_i e_i^T + v_2 e_j e_j^T$, where $1 \leq i < j \leq n$. We can consider different damping positions, but in order to illustrate benefits of our approximation technique, while we vary viscosities, we fix damping positions to $i = 600$ and $j = 900$, with viscosities which vary over the feasible interval.

For the purpose of an easier illustration of the eigenvalue behavior, we consider the following

configurations of viscosities:

$$(v_1, v_2) = \left(\frac{v}{4}, v\right), \quad v = 1, 2, \ldots, 100. \tag{2.23}$$

For the tolerance needed in Algorithm 3 we use $\texttt{tol} = 5 \cdot 10^{-4}$.

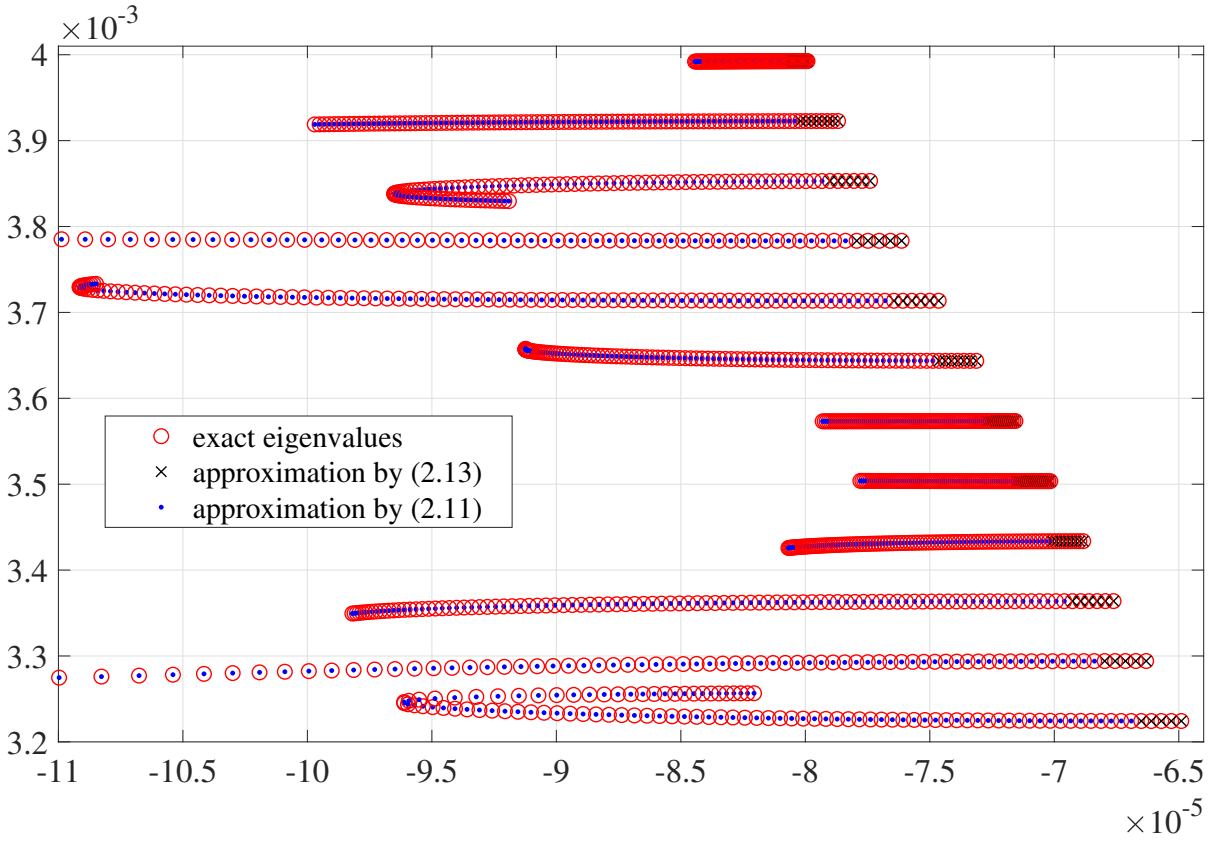

Figure 2.2: Eigenvalue behavior for Example 2.1

First, in Figure 2.2 we illustrate the eigenvalue behavior when parameter v varies as it is given by (2.23). The exact eigenvalues are denoted by red circles, on the other hand, the approximations obtained by formulas (2.13) are denoted by black x-es, while the approximations obtained by solving a small eigenvalue problem defined by matrix $\overline{A}_{11}$ from (2.11) are denoted by blue dots. Here we should note that we plot the eigenvalue behavior only for small parts of considered eigenvalues, since otherwise it would be hard to follow all eigenvalues in one figure.

In Figure 2.3, we illustrate the benefits of our approximation technique over the approach derived specifically for a small viscosity v, which was studied in [75] and [112]. We plot the eigenvalue behavior obtained by using modal approximation from [112]. The following approximation can also be obtained by Algorithm 3 by setting large enough tolerance `tol`. We can conclude that modal approximation does not capture the main behaviour of eigenvalues, while our approach achieves eigenvalue tracking efficiently.

Figure 2.4 shows the relative errors for the eigenvalues and the upper bounds for the relative errors for the viscosity $v = 10$, where the reduced dimension is $r = 416$. We plot relative errors

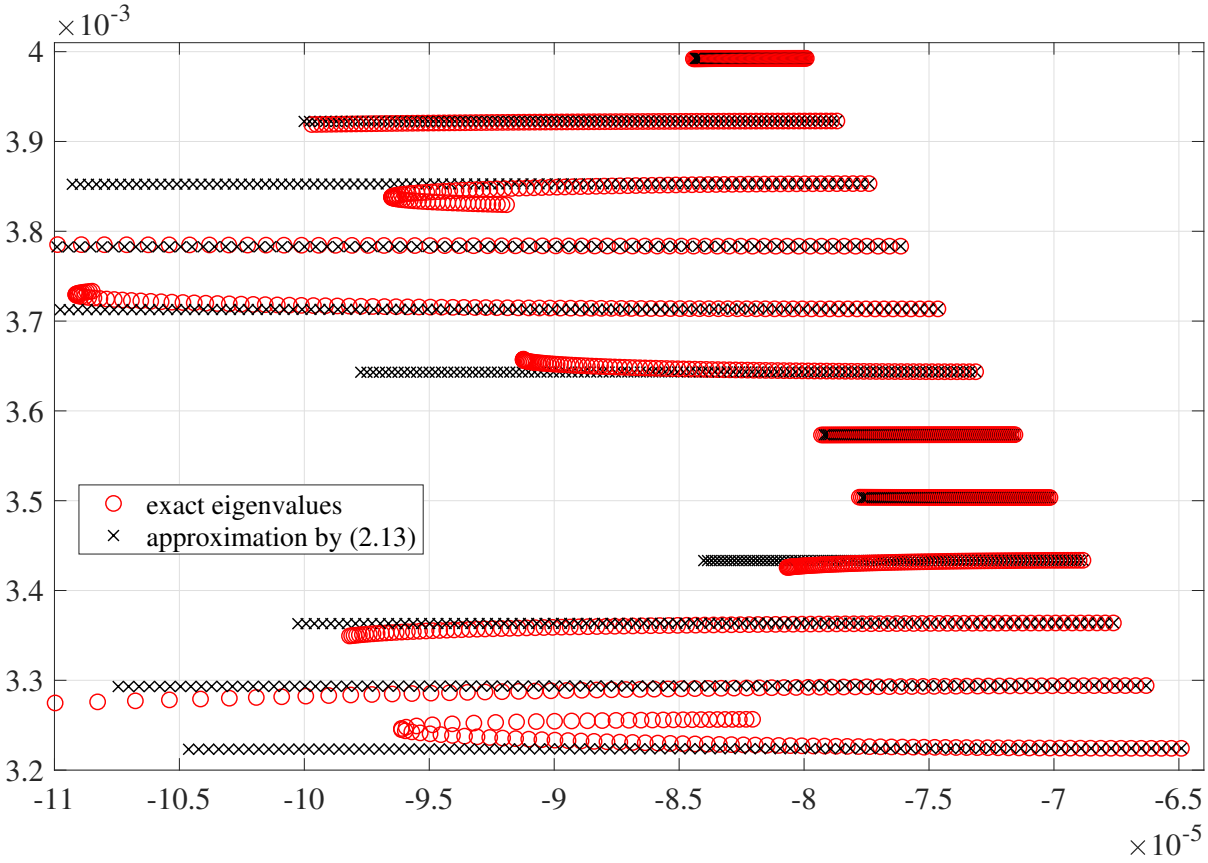

Figure 2.3: Eigenvalue behavior for Example 2.1 for the approximation obtained using only formulas (2.13)

greater than 10^{-12} only for a half eigenvalues, because they are all symmetric with respect to real axis, which implies the errors are also symmetric.

In order to illustrate the magnitude of the reduced dimension while we vary parameter v as in (2.23), we show reduced dimension r in Figure 2.5, where r varies from $r = 0$ (meaning that we use only formulas (2.13)) to $r = 733$, which is 73.3% of the full dimension.

Case b)

In this case the damping matrix contains only external damping $C(\mathbf{v}) = C_{\text{ext}}(\mathbf{v})$, so we can compare our bounds given in (2.18) with the bounds given in (2.21).

We consider two blocks of dampers with different viscosities, i.e., $\mathbf{v} = [v_1, v_2]$; thus external damping is defined by $C_{\text{ext}}(\mathbf{v}) = v_1(e_i e_i^T + \cdots + e_{i+49} e_{i+49}^T) + v_2(e_j e_j^T + \cdots + e_{j+49} e_{j+49}^T)$, where $1 \leq i < j \leq n$. We could consider different positions of blocks, but in order to illustrate benefits of our approximation technique while we vary viscosities, we fix block positions to $i = 600$ and $j = 900$ with viscosities which vary over the feasible interval.

For the purpose of an easier illustration of the eigenvalue behavior, we consider the following

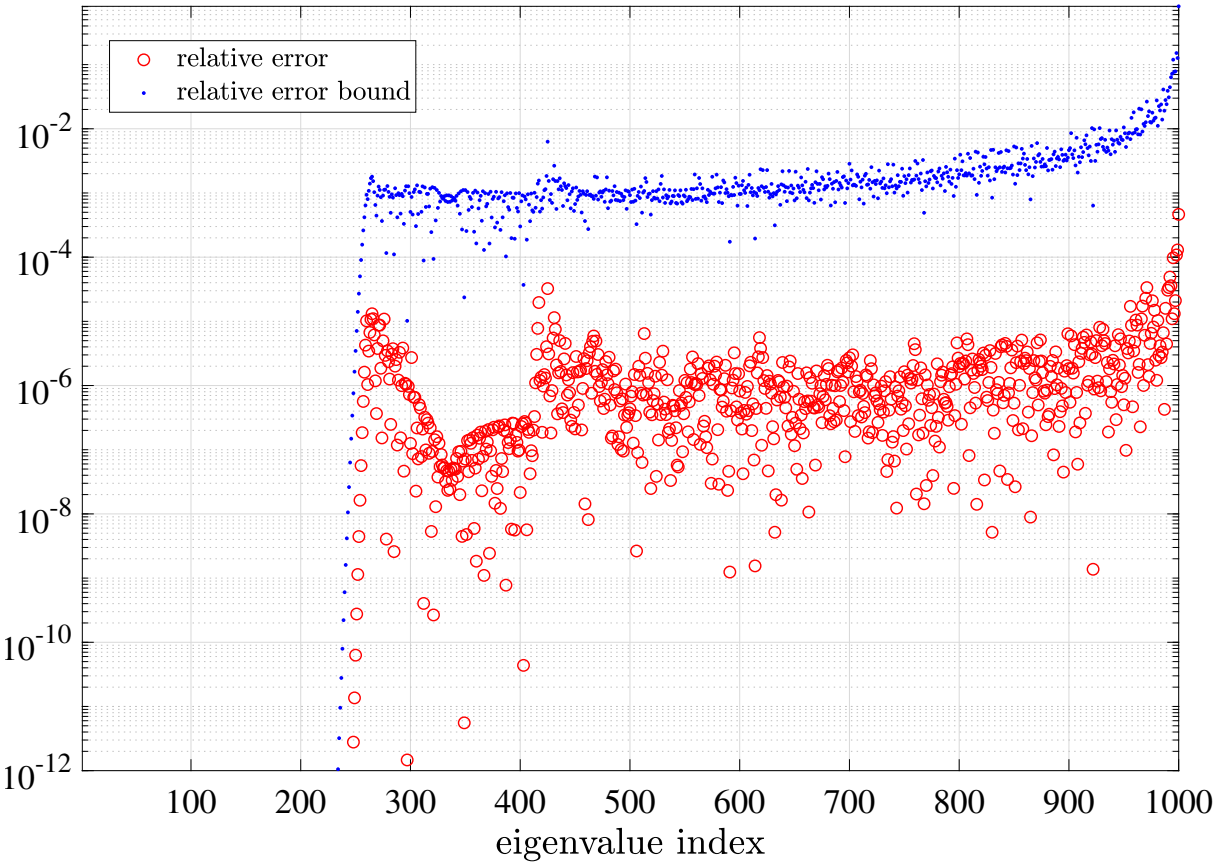

Figure 2.4: Relative error for Example 2.1 for $\nu = 10$, $r = 416$.

configurations of viscosities:

$$(\nu_1, \nu_2) = \left(\frac{\nu}{4}, \nu\right), \quad \nu = 1, 2, \ldots, 100. \tag{2.24}$$

For the tolerance needed in Algorithm 3 we use $\texttt{tol} = 10^{-5}$.

Figure 2.6 shows the relative errors for the approximations of eigenvalues given in (2.13) and (2.15) and the corresponding upper bound given in (2.18) for the relative errors, furthermore, it shows the relative errors for the approximations of eigenvalues given in (2.22) and corresponding upper bound given in (2.21) for the relative error, for the viscosity $\nu = 1000$. For this viscosity, the reduced dimension for the above given tolerance was $r = 750$. We plot all relative errors greater than 10^{-12} for all real eigenvalues (from 901 to 1100 on Figure 2.6) and only the half of the errors for complex eigenvalues (from 1 to 900 on Figure 2.6), due to the symmetry.

In the previous example, we have considered the eigenvalue behavior of all eigenvalues. In the next example, we illustrate our approach on the case when one is interested in calculation of a part of the spectrum.

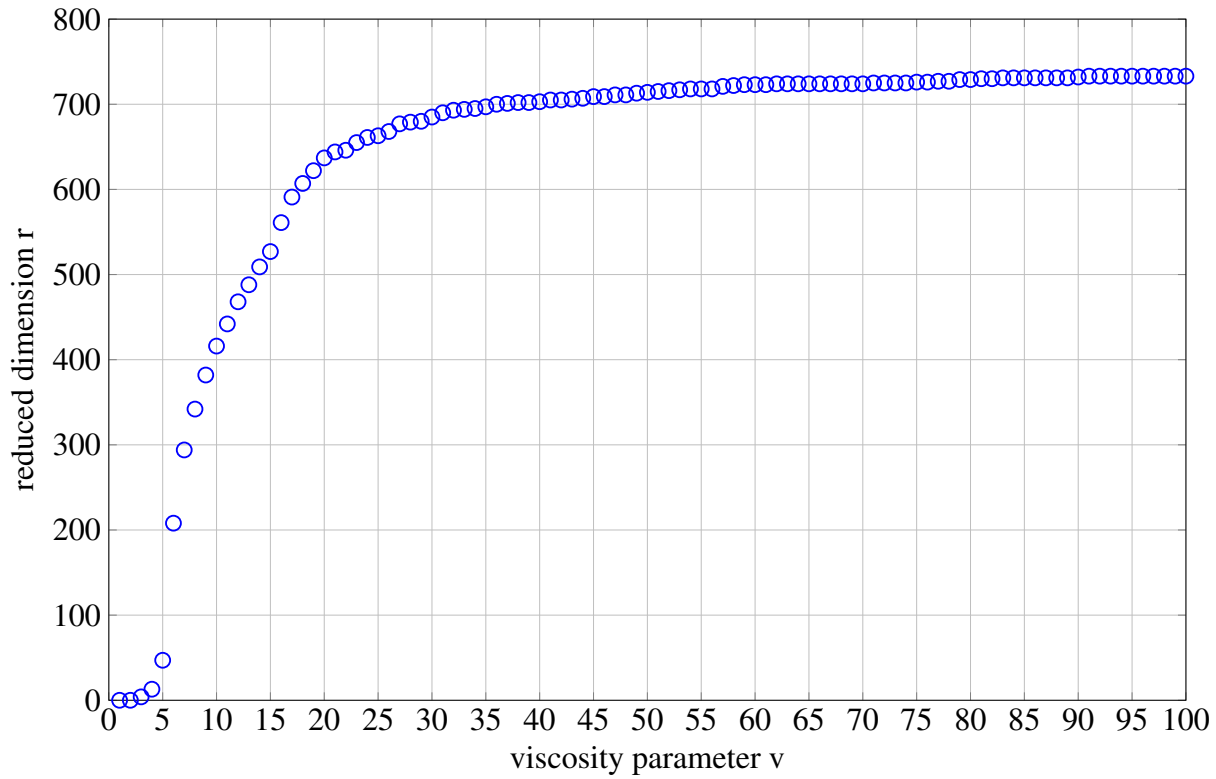

Figure 2.5: Reduction dimensions r for Example 2.1 for $v = 1, 2, 3, \ldots, 100$

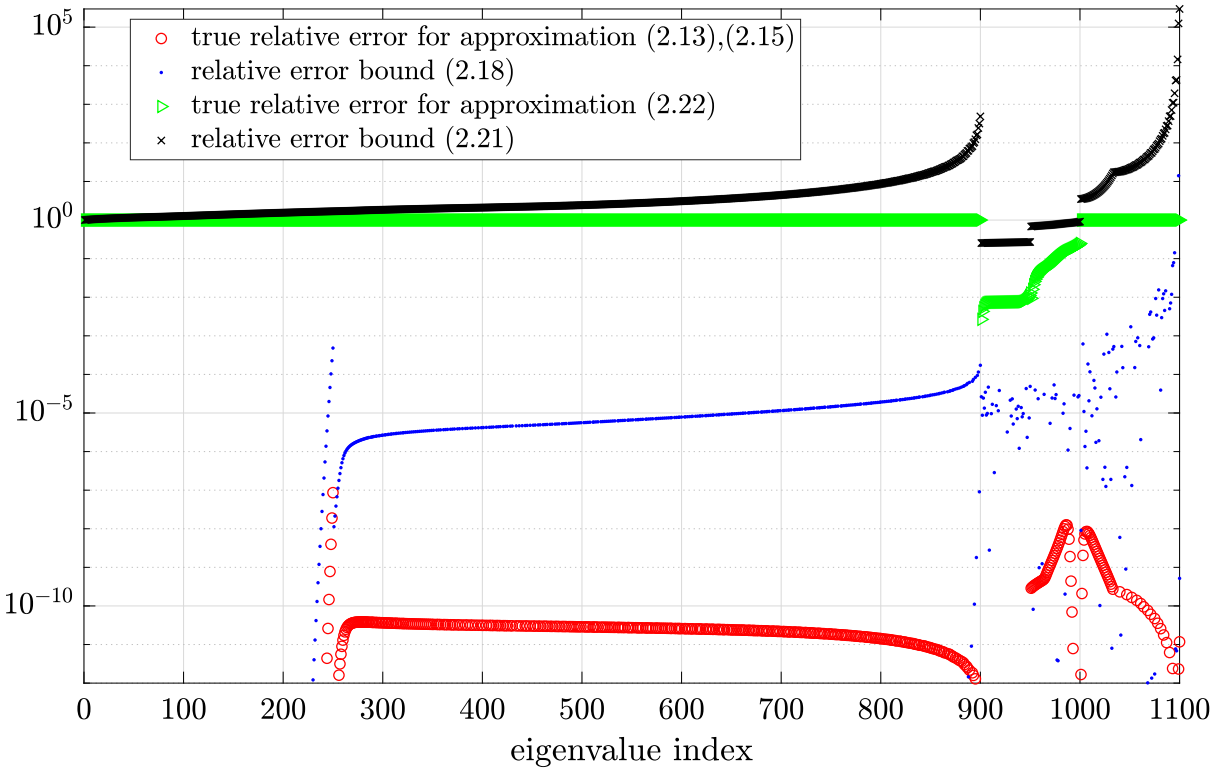

Figure 2.6: Relative error for Case a) in Example 2.1 for $v = 1000$, $r = 750$.

Example 2.2 We consider a mechanical system shown in Figure 2.7 with three rows of d masses, which gives $3d+1$ masses and $3d+4$ springs. Here we consider three dampers of the same viscosity. In each row damper acts only on one mass. Each row has springs of the same stiffness equal to k_1, k_2, k_3, respectively. The last mass m_{3d+1} is connected to the fixed base with the spring with stiffness k_4.

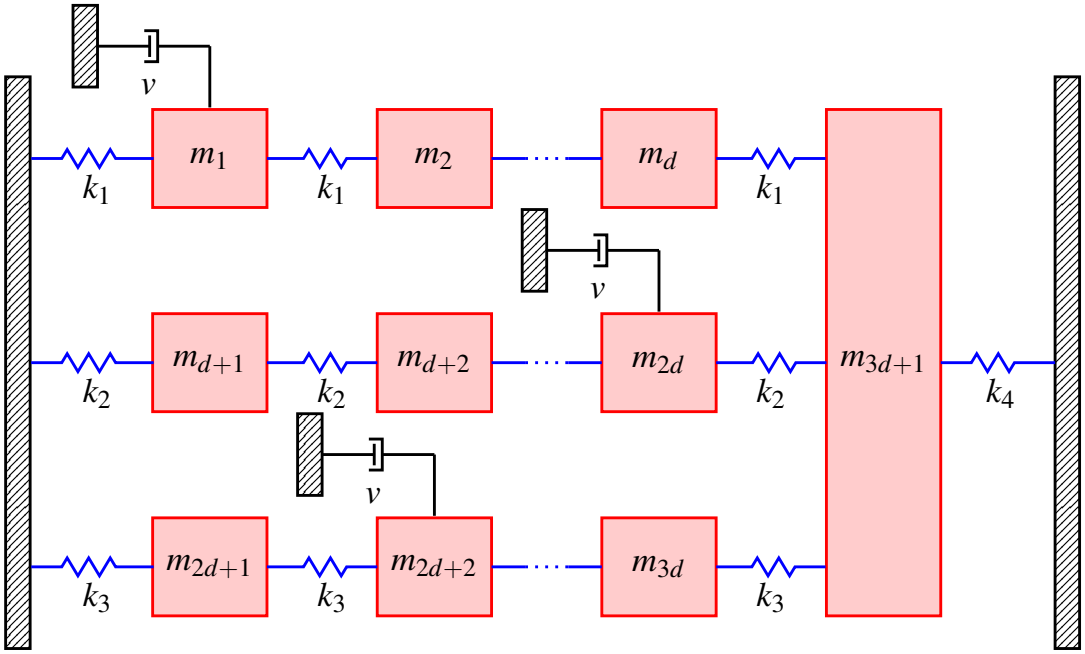

Figure 2.7: $3d+1$ mass oscillator

A mathematical model for the considered vibrational system is given by (1.2), where the mass matrix is

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n).$$

The stiffness matrix is defined as

$$K = \begin{bmatrix} K_{11} & & & -\kappa_1 \\ & K_{22} & & -\kappa_2 \\ & & K_{33} & -\kappa_3 \\ -\kappa_1^T & -\kappa_2^T & -\kappa_3^T & k_1+k_2+k_3+k_4 \end{bmatrix},$$

where

$$K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \kappa_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_i \end{bmatrix}, \quad i = 1,2,3.$$

We consider the following configuration

$$\begin{aligned} &d = 400, \quad n = 3d+1 = 1201, \\ &m_i = i, \quad i = 1,\dots,n, \\ &k_1 = 1, \quad k_2 = 20, \quad k_3 = 40, \quad k_4 = 50. \end{aligned}$$

Similarly to the previous example, for the sake of easier illustration of the obtained results, we fix damper positions. We set them to $840, 960, 1200$. Since the three dampers have the same viscosity v, the damping matrix is equal to

$$C(v) = C_{\text{int}} + C_{\text{ext}}(v), \quad \text{with,} \quad C_{\text{ext}}(v) = ve_{840}e_{840}^T + ve_{960}e_{960}^T + ve_{1200}e_{1200}^T,$$

where internal damping is given by (1.20) with $\alpha_c = 0.001$. In this example, in Algorithm 2, we use tolerance $\texttt{tol} = 10^{-4}$.

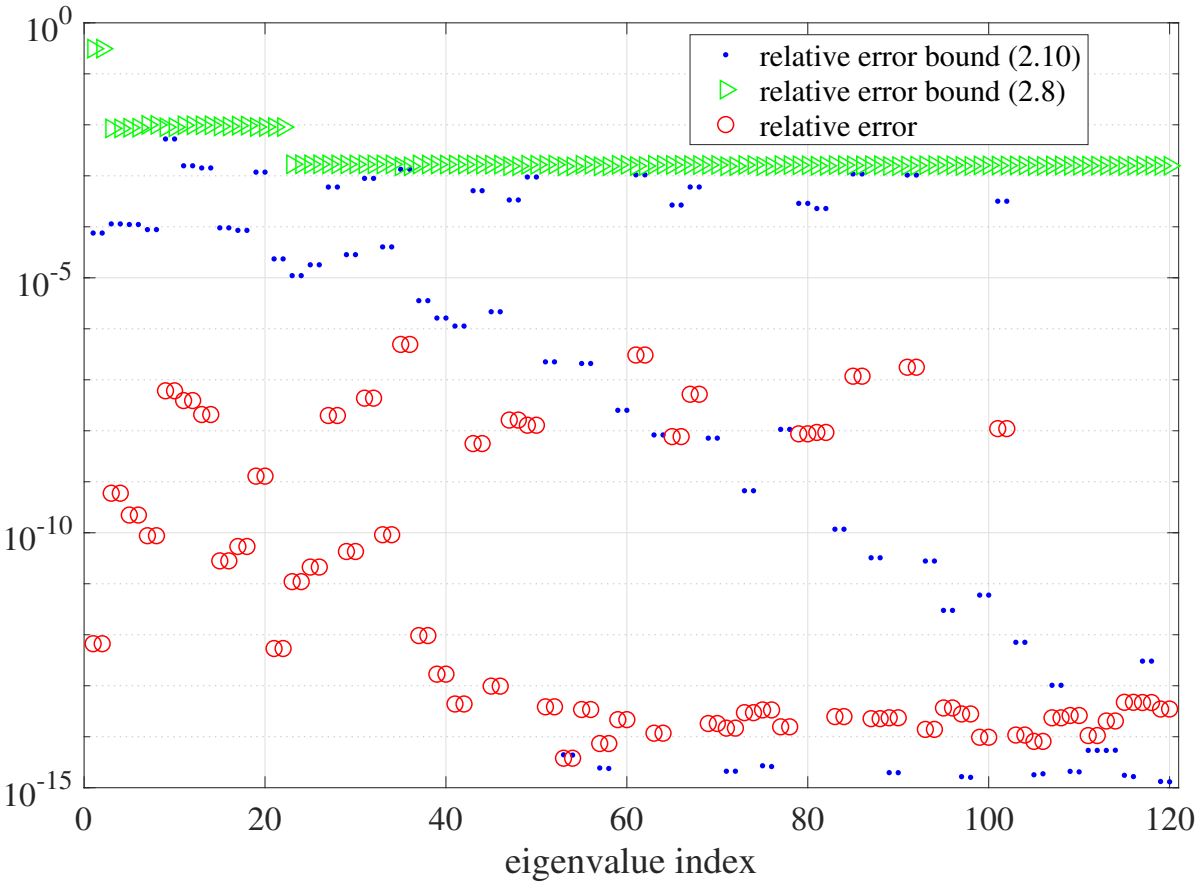

Figure 2.8: Relative error for Example 2.2 for $v = 1.9$, $r = 60$.

We illustrate the quality of eigenvalue approximation given by Algorithm 2, where we are interested in the behavior (in damping) only of the part of the undamped eigenfrequencies larger than 0.1 and smaller than 0.11. In this example this means that we need to consider the

eigenvalue behavior of only 49 undamped eigenfrequencies which gives the parameter $l = 49$ in Algorithm 2. Undamped eigenfrequencies ω_i for $i = 335, \ldots, 383$, where ω_i from (1.4) are sorted in increasing order and their indices define vector μ required in Algorithm 2.

Figure 2.8 contains the relative errors and the bounds derived in Subsection 2.1.1.1. In particular, red circles denote true relative errors with respect to eigenvalues calculated by MATLAB's function `eig`. Blue dots show the relative errors for the bound given by (2.10) while the error bound given by (2.8) is presented by green triangles. The figure shows the quality of the derived upper bound for the fixed viscosity $v = 1.9$, where Algorithm 1 returned the reduced dimension $r = 60$. Thus, instead of only 98 eigenvalues, that we would like to track, we have obtained approximations for all together, 120 eigenvalues.

As one can see from Figure 2.8, there exist eigenvalues for which bound (2.10) is better, but there also exist eigenvalues for which bound (2.8) gives a better estimation for the error. In general, this can vary as we change the viscosity v, so the best option would be to take the minimum of both derived upper bounds. Moreover, there are cases where the bound (2.10) is less then the error, which is the result of the tolerance taken into account in Algorithm 2.

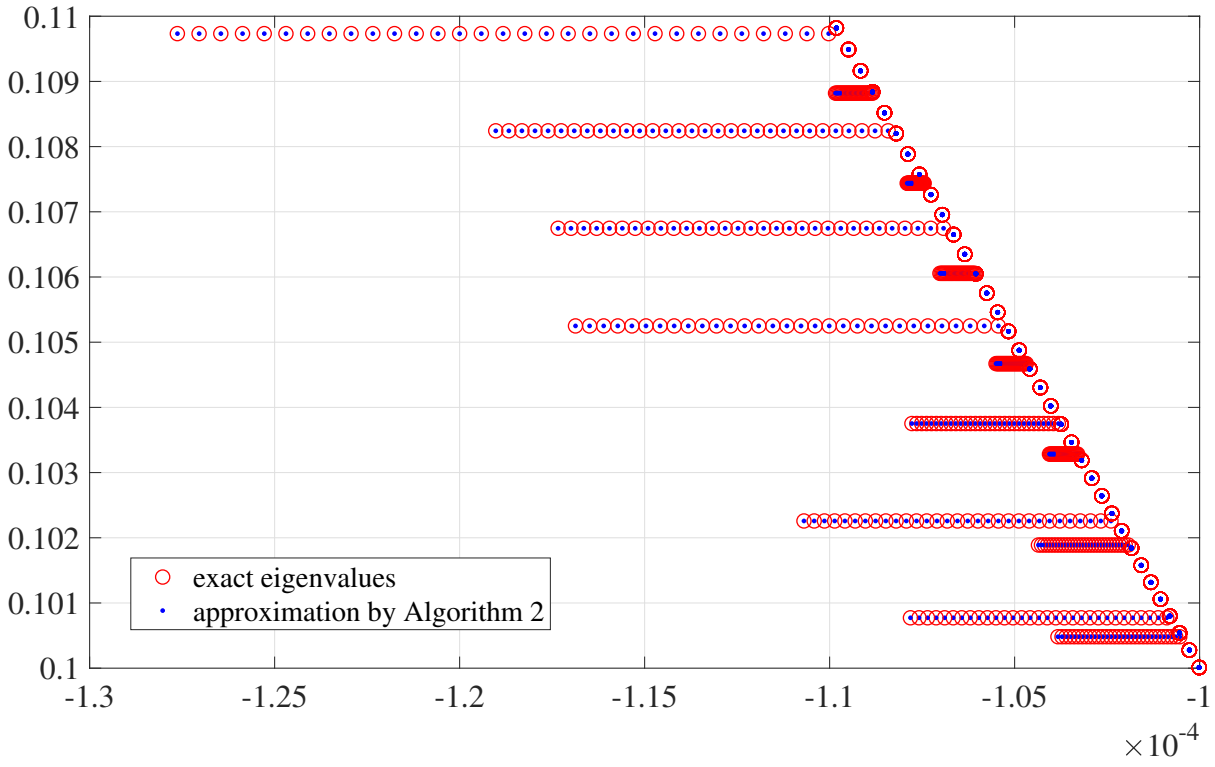

Figure 2.9: Eigenvalue behavior for Example 2.2

The eigenvalue behavior is shown in Figure 2.9, where blue dots denote approximations obtained by Algorithm 2, while red circles denote exact eigenvalues. We would like to note that the reduced dimension was $r = 60$ while $v = 0.05 + 0.1 \cdot i, i = 0, \ldots, 30$. Moreover, since we consider the undamped eigenfrequencies larger than 0.1 and smaller than 0.11, in Figure 2.9 we

plot eigenvalues whose imaginary parts belong between 0.1 and 0.11. From this figure, but also from Figure 2.2, we can note that we have achieved satisfactory eigenvalue tracking even for moderate viscosities v, while we ensure efficient error bounds for the obtained approximation.

In this section we presented how one can compute approximations of eigenvalues (all or just a selected part of them) when the parameter is of the modest magnitude. This is an addition to the work presented in papers [75], [94] where authors present approximations for the cases when, respectively, parameter is very small and very large.

The quality of the error bounds as well as the accuracy of the achieved eigenvalues was illustrated in several numerical experiments.

Furthermore, in the next section we present first order approximation of eigenvalues which can be used to see how eigenvalue will change if we make a small change in the parameter.

2.2 First order approximations of eigenvalues

In this section we present the first order approximation of eigenvalues, based on Taylor's formula for multivariate functions, and the corresponding left and right eigenvectors. This result leads to efficient approximation of parameter dependent eigenvalues and answers the second question from the beginning of this chapter: "How one can efficiently calculate eigenvalues $\lambda_i(\mathbf{p})$, for all $i = 1,\dots,2n$, if we already have eigenvalues $\lambda_i(\mathbf{p}^0)$ calculated, for all $i = 1,\dots,2n$, and the change in parameter is small enough, i.e., $\|\mathbf{p}-\mathbf{p}^0\| \leq \varepsilon_{\mathbf{p}^0}$, for given tolerance $\varepsilon_{\mathbf{p}^0}$?"

For details on Taylor's formula see [83, 54]. In this section, for the sake of generality, we again assume that all matrices $M(\mathbf{p})$, $C(\mathbf{p})$, and $K(\mathbf{p})$ depend on parameter $\mathbf{p}$.

Let $x(\mathbf{p})$ and $y(\mathbf{p})$ be the right and left eigenvector, respectively, that correspond to the eigenvalue $\lambda(\mathbf{p})$ of the PQEP for a given parameter $\mathbf{p} = [p_1, p_2, \dots, p_m]^T \in \mathbb{R}^m$.

The PQEP has $2n$ eigenvalues, so we denote i-th eigenvalue with $\lambda_i(\mathbf{p})$ for all $i \in \{1,\dots,2n\}$. For the i-th eigenvalue and corresponding eigenvector (1.1) can be written as:

$$(\lambda_i^2(\mathbf{p})M(\mathbf{p}) + \lambda_i(\mathbf{p})C(\mathbf{p}) + K(\mathbf{p}))x_i(\mathbf{p}) = 0, \quad x_i(\mathbf{p}) \neq 0. \tag{2.25}$$

We assume that eigenvalues given by (2.25) are all simple. Furthermore, since we are using Taylor's formula for eigenpair approximation, we need to assume or prove that the derivatives exist.

Theorem 2.2 ([2]) If $\lambda(\mathbf{p}^0)$ is a simple eigenvalue of (2.25) for $\mathbf{p} = \mathbf{p}^0$ then there is a neighborhood of $\mathbf{p}^0$ on which there exists an eigenvalue function $\lambda_i(\mathbf{p})$ and right and left eigenvector function that are all analytic functions of $\mathbf{p}$.

In [2], the authors state a general version of Theorem 2.2. However, the detailed proof is not given.

For easier notation in the following lemma and theorem we denote

$$\mathscr{P}(\lambda(\mathbf{p})) \quad = \quad \lambda^2(\mathbf{p})M(\mathbf{p}) + \lambda(\mathbf{p})C(\mathbf{p}) + K(\mathbf{p}). \tag{2.26}$$

Lemma 2.3 ([2]) Let $\lambda(\mathbf{p}^0)$ be a semisimple eigenvalue of $\mathscr{P}(\lambda(\mathbf{p}))$ and $x(\mathbf{p}^0) \in \mathrm{Ker}(\mathscr{P}(\lambda(\mathbf{p}^0)))$, $x(\mathbf{p}^0) \neq 0$. Then $(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0) \notin \mathrm{Im}(\mathscr{P}(\lambda(\mathbf{p}^0)))$.

Proof. We need to use the "local Smith form" for $\mathscr{P}(\lambda(\mathbf{p}))$ [36]. In a neighbourhood of an eigenvalue $\lambda(\mathbf{p}^0)$ we have

$$\mathscr{P}(\lambda(\mathbf{p})) \quad = \quad E(\lambda(\mathbf{p}))D(\lambda(\mathbf{p}))F(\lambda(\mathbf{p})), \tag{2.27}$$

where

$$D(\lambda(\mathbf{p})) = \mathrm{diag}[(\lambda(\mathbf{p}) - \lambda(\mathbf{p}^0))^{\alpha_1}, \ldots, (\lambda(\mathbf{p}) - \lambda(\mathbf{p}^0))^{\alpha_n}], \tag{2.28}$$

$E(\lambda(\mathbf{p}))$ and $F(\lambda(\mathbf{p}))$ are matrix polynomial invertible at $\lambda(\mathbf{p}^0)$ and $\alpha_1 \geq \alpha_2 \geq \ldots \alpha_n \geq 0$ are the "partial multiplicities" of $\lambda(\mathbf{p}^0)$(see [36] for more details). Since properties of $\mathscr{P}(\lambda(\mathbf{p}))$ at a fixed $\mathbf{p}$ are all that are required here, we omit $\mathbf{p}$ in further notations.

Let $E_0 = E(\lambda(\mathbf{p}^0)), F_0 = F(\lambda(\mathbf{p}^0))$ and $\lambda(\mathbf{p}^0)$ be a semisimple eigenvalue. Thus, for some k, $\alpha_1 = \cdots = \alpha_k = 1$ and $\alpha_{k+1} = \cdots = \alpha_n = 0$. Thus, if $D_1 = \frac{\partial D(\lambda(\mathbf{p}^0))}{\partial \lambda} = \mathrm{diag}[I_k, 0]$ and $D_2 = D(\lambda(\mathbf{p}^0)) = \mathrm{diag}[0, I_{n-k}]$, then

$$\mathscr{P}(\lambda(\mathbf{p}^0)) = E_0 D_2 F_0, \tag{2.29}$$

and $\mathscr{P}(\lambda(\mathbf{p}^0))x(\mathbf{p}^0) = 0$ implies $D_2 F_0 x(\mathbf{p}^0) = 0$, since E_0 is regular. Thus

$$\begin{aligned} \frac{\partial \mathscr{P}(\lambda(\mathbf{p}^0))}{\partial \lambda} x(\mathbf{p}^0) =& (2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0) \\ =& \left(\frac{\partial E(\lambda(\mathbf{p}^0))}{\partial \lambda} D_2 F_0 + E_0 D_1 F_0 + E_0 D_2 \frac{\partial F(\lambda(\mathbf{p}^0))}{\partial \lambda} \right) x(\mathbf{p}^0) \\ =& (E_0 D_1 F_0)\, x(\mathbf{p}^0) + \left(E_0 D_2 \frac{\partial F(\lambda(\mathbf{p}^0))}{\partial \lambda} \right) x(\mathbf{p}^0). \end{aligned} \tag{2.30}$$

Let us assume that $(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0) \in \mathrm{Im}(\mathscr{P}(\lambda(\mathbf{p}^0)))$ then there exists u such that $\frac{\partial \mathscr{P}(\lambda(\mathbf{p}^0))}{\partial \lambda} x(\mathbf{p}^0) = \mathscr{P}(\lambda(\mathbf{p}^0))u$. Furthermore, from (2.30), (2.29) and the face that E_0 is regular we obtain

$$\left(D_1 F_0 + D_2 \frac{\partial F(\lambda(\mathbf{p}^0))}{\partial \lambda} \right) x(\mathbf{p}^0) = D_2 F_0 u. \tag{2.31}$$

Now from

$$D_1 F_0 x(\mathbf{p}^0) = D_2 \left(F_0 u - \frac{\partial F(\lambda(\mathbf{p}^0))}{\partial \lambda} \right) \tag{2.32}$$

it follows that $D_1 F_0 x(\mathbf{p}^0) = 0$, since D_1 annihilates $(k+1)$st to nth element in a vector and D_2 annihilates first k elements of a vector, which means that both sides od equality (2.32) have to be 0. As we know $D_2 F_0 x(\mathbf{p}^0) = 0$, we can conclude that $F_0 x(\mathbf{p}^0) = 0$, i.e. since F_0 is regular, $x(\mathbf{p}^0) = 0$ which is contradiction.

Therefore, if $\mathscr{P}(\lambda(\mathbf{p}^0))x(\mathbf{p}^0) = 0$, and $x(\mathbf{p}^0) \neq 0$ then

$$(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0) \notin \mathrm{Im}(\mathscr{P}(\lambda(\mathbf{p}^0))). \tag{2.33}$$

□

Theorem 2.4 ([2]) $\mathscr{P}(\lambda(\mathbf{p}))$ has a simple eigenvalue $\lambda(\mathbf{p}^0)$ if and only if $\mathscr{P}(\lambda(\mathbf{p}^0))$ has rank $n-1$, and $y(\mathbf{p}^0)^*(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0) \neq 0$ where $x(\mathbf{p}^0)$ and $y(\mathbf{p}^0)$ are right and left eigenvector of $\mathscr{P}(\lambda(\mathbf{p}^0))$, respectively.

Proof. [$\Longrightarrow$] Let $\lambda(\mathbf{p}^0)$ be a simple eigenvalue, then it is also semisimple eigenvalue, so in (2.28) we have $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_n = 0$, and $\mathrm{rank}\mathscr{P}(\lambda(\mathbf{p}^0)) = n-1$. If $x(\mathbf{p}^0)$ is right eigenvector that corresponds to $\lambda(\mathbf{p}^0)$, then, from Lemma 2.3 follows that $(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0) \notin \mathrm{Im}(\mathscr{P}(\lambda(\mathbf{p}^0)))$. If $y(\mathbf{p}^0)$ is left eigenvalue that corresponds to $\lambda(\mathbf{p}^0)$, i.e., $y(\mathbf{p}^0)^*\mathscr{P}(\lambda(\mathbf{p}^0)) = 0$, $y(\mathbf{p}^0) \neq 0$, then $y(\mathbf{p}^0)$ is orthogonal to the $(n-1)$ - dimensional subspace $\mathrm{Im}\mathscr{P}(\lambda(\mathbf{p}^0))$, but not to the vector $(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0)$ since (2.33). Therefore, $y(\mathbf{p}^0)^*(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0) \neq 0$.
[$\Longleftarrow$] Let $y(\mathbf{p}^0)^*(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0) \neq 0$ holds. If $\mathrm{rank}\mathscr{P}(\lambda(\mathbf{p}^0)) = n-1$, then from (2.28) follows that $\alpha_2 = \cdots = \alpha_n = 0$. If $\alpha_1 \geq 2$, which means that eigenvalue is semisimple, then from Lemma 2.3

$$\begin{aligned}
\frac{\partial \mathscr{P}(\lambda(\mathbf{p}^0))}{\partial \lambda} x(\mathbf{p}^0) =& (2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0) \\
=& \left(\frac{\partial E(\lambda(\mathbf{p}^0))}{\partial \lambda} D_2 F_0 + E_0 \frac{\partial D(\lambda(\mathbf{p}^0))}{\partial \lambda} F_0 + E_0 D_2 \frac{\partial F(\lambda(\mathbf{p}^0))}{\partial \lambda} \right) x(\mathbf{p}^0) \\
=& \left(\frac{\partial E(\lambda(\mathbf{p}^0))}{\partial \lambda} D_2 F_0 \right) x(\mathbf{p}^0) + \left(E_0 D_2 \frac{\partial F(\lambda(\mathbf{p}^0))}{\partial \lambda} \right) x(\mathbf{p}^0),
\end{aligned} \tag{2.34}$$

since

$$\frac{\partial D(\lambda)}{\partial \lambda} = \mathrm{diag}[\alpha_1(\lambda(\mathbf{p}) - \lambda(\mathbf{p}^0))^{\alpha_1 - 1}, 0, \ldots, 0]$$

and $\alpha_1 \geq 2$ i.e.,

$$D_1 = \frac{\partial D(\lambda(\mathbf{p}^0))}{\partial \lambda} = 0.$$

Furthermore, $\mathscr{P}(\lambda(\mathbf{p}^0))x(\mathbf{p}^0) = 0$ implies $D_2 F_0 x(\mathbf{p}^0) = 0$ and $y(\mathbf{p}^0)^*\mathscr{P}(\lambda(\mathbf{p}^0)) = 0$ implies

$E_0 D_2 x(\mathbf{p}^0) = 0$, since both E_0 and F_0 are regular, and thus $y(\mathbf{p}^0)^*(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0)+C(\mathbf{p}^0))x(\mathbf{p}^0) = 0$, which is contradiction. Hence $y(\mathbf{p}^0)^*(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0)+C(\mathbf{p}^0))x(\mathbf{p}^0) \neq 0$ implies $\alpha_1 = 1$, and $\lambda(\mathbf{p}^0)$ is simple eigenvalue. □

According to Theorem 2.2 and Theorem 2.4 we assume that all eigenvalues are simple, which imply

$$\operatorname{rank}(\lambda_i^2(\mathbf{p})M(v) + \lambda_i(\mathbf{p})C(\mathbf{p}) + K(\mathbf{p})) = n-1\,.$$

Now, using the partial derivative of (2.25) with respect to p_k we get

$$\begin{aligned}\Bigg(2\lambda_i(\mathbf{p})\frac{\partial\lambda_i}{\partial p_k}(\mathbf{p})M(\mathbf{p}) + \lambda_i^2(\mathbf{p})\frac{\partial M}{\partial p_k}(\mathbf{p}) + \\ \frac{\partial\lambda_i}{\partial p_k}(\mathbf{p})C(\mathbf{p}) + \lambda_i(\mathbf{p})\frac{\partial C}{\partial p_k}(\mathbf{p}) + \frac{\partial K}{\partial p_k}(\mathbf{p})\Bigg)x_i(\mathbf{p}) + \\ \left(\lambda_i^2(\mathbf{p})M(\mathbf{p}) + \lambda_i(\mathbf{p})C(\mathbf{p}) + K(\mathbf{p})\right)\frac{\partial x_i}{\partial p_k}(\mathbf{p}) = 0.\end{aligned} \tag{2.35}$$

From this we get the following lemmas.

Lemma 2.5 ([2]) Let $\lambda_i(\mathbf{p})$ denotes function $\lambda_i\colon \mathbb{R}^m \to \mathbb{C}$ that corresponds to the i-th eigenvalue which is simple. Furthermore, let $x_i(\mathbf{p}), y_i(\mathbf{p})$ denote functions $x_i\colon \mathbb{R}^m \to \mathbb{C}^n, y_i\colon \mathbb{R}^m \to \mathbb{C}^n$ that correspond to the i-th right and left eigenvector, respectively and $y_i^*(\mathbf{p})(2\lambda_i(\mathbf{p})M(\mathbf{p}) + C(\mathbf{p}))x_i(\mathbf{p}) \neq 0$, for all $i \in \{1,\ldots,2n\}$, then for $\mathbf{p} \in \Omega_{\mathbf{p}^0} = \{\mathbf{p} \in \mathbb{R}^m\colon ||\mathbf{p}-\mathbf{p}^0|| \leq \varepsilon_{\mathbf{p}^0}\}$, where $\varepsilon_{\mathbf{p}^0}$ is given tolerance, the partial derivative of $\lambda_i(\mathbf{p})$ with respect to p_i is given by

$$\frac{\partial\lambda_i}{\partial p_k}(\mathbf{p}) = -\frac{y_i^*(\mathbf{p})\left(\lambda_i^2(\mathbf{p})\frac{\partial M}{\partial p_k}(\mathbf{p}) + \lambda_i(\mathbf{p})\frac{\partial C}{\partial p_k}(\mathbf{p}) + \frac{\partial K}{\partial p_k}(\mathbf{p})\right)x_i(\mathbf{p})}{y_i^*(\mathbf{p})(2\lambda_i(\mathbf{p})M(\mathbf{p}) + C(\mathbf{p}))x_i(\mathbf{p})}, \quad i = 1,\ldots,2n. \tag{2.36}$$

Proof. By multiplying the equation (2.35) by $y_i^*(\mathbf{p})$ from the left side one obtains

$$\begin{aligned}y_i^*(\mathbf{p})\Bigg(\frac{\partial\lambda_i}{\partial p_k}(\mathbf{p})(2\lambda_i(\mathbf{p})M(\mathbf{p}) + C(\mathbf{p})) + \lambda_i^2(\mathbf{p})\frac{\partial M}{\partial p_k}(\mathbf{p}) + \\ \lambda_i(\mathbf{p})\frac{\partial C}{\partial p_k}(\mathbf{p}) + \frac{\partial K}{\partial p_k}(\mathbf{p})\Bigg)x_i(\mathbf{p}) = 0,\end{aligned} \tag{2.37}$$

since $y_i^*(\mathbf{p})\left(\lambda_i^2(\mathbf{p})M(\mathbf{p}) + \lambda_i(\mathbf{p})C(\mathbf{p}) + K(\mathbf{p})\right) = 0$. Now, (2.35) follows directly from (2.37). □

The similar formula for derivative of eigenvalue for $\mathbf{p} \in \mathbb{R}$ can be found in [112], and in [109] Aa et al. derived formulas for eigenvalue and eigenvector derivatives for a general complex-valued eigensystem. In [65] authors present a numerical way of computing partial derivative of eigenvectors.

Lemma 2.6 ([2]) Let the assumptions from Lemma 2.5 hold, then the partial derivative of $x_i(\mathbf{p})$ over p_k is given by

$$\begin{aligned}\frac{\partial x_i}{\partial p_k}(\mathbf{p}) = &PI_i(\mathbf{p})\left(2\lambda_i(\mathbf{p})\frac{\partial \lambda_i}{\partial p_k}(\mathbf{p})M(\mathbf{p}) + \frac{\partial \lambda_i}{\partial p_k}(\mathbf{p})C(\mathbf{p}) + \right.\\ &\left.\lambda_i^2(\mathbf{p})\frac{\partial M}{\partial p_k}(\mathbf{p}) + \lambda_i(\mathbf{p})\frac{\partial C}{\partial p_k}(\mathbf{p}) + \frac{\partial K}{\partial p_k}(\mathbf{p})\right)x_i(\mathbf{p}), \quad i = 1,\ldots,2n,\end{aligned} \tag{2.38}$$

where

$$PI_i(\mathbf{p}) = -\left(\lambda_i^2(\mathbf{p})M(\mathbf{p}) + \lambda_i(\mathbf{p})C(\mathbf{p}) + K(\mathbf{p})\right)^{\dagger}. \tag{2.39}$$

Proof. The proof follows directly from (2.35). □

Term $PI_i(\mathbf{p})$ includes Moore-Penrose pseudo-inverse of a matrix. For a matrix A, pseudo-inverse is denoted by $A^{\dagger}$, more details on pseudo-inverse of a matrix can be found, e.g., in [29]. From Lemma 2.5 and Lemma 2.6 we can write gradient vectors of functions $\lambda_i(\mathbf{p})$ and $x_i^{(l)}(\mathbf{p})$ in the following way:

$$\mathcal{J}(\lambda_i(\mathbf{p})) = \left[\begin{array}{cccc}\frac{\partial \lambda_i}{\partial p_1}(\mathbf{p}) & \frac{\partial \lambda_i}{\partial p_2}(\mathbf{p}) & \ldots & \frac{\partial \lambda_i}{\partial p_m}(\mathbf{p})\end{array}\right], \tag{2.40}$$

$$\mathcal{J}(x_i^{(l)}(\mathbf{p})) = \left[\begin{array}{cccc}\frac{\partial x_i^{(l)}}{\partial p_1}(\mathbf{p}) & \frac{\partial x_i^{(l)}}{\partial p_2}(\mathbf{p}) & \ldots & \frac{\partial x_i^{(l)}}{\partial p_m}(\mathbf{p})\end{array}\right], \tag{2.41}$$

where $x_i^{(l)}(\mathbf{p})$ is lth component of vector $x_i(\mathbf{p})$, $l = 1,\ldots,n$.

Since we assumed that eigenvalues of PQEP (2.25) are simple, and therefore analytical with corresponding partial derivatives, we can obtain the first order approximation based on Taylor's theorem for multivariate functions. More precisely, the following theorem holds.

Theorem 2.7 ([2]) Let the assumptions from Lemma 2.5 hold. Moreover let $\mathcal{H}(\lambda_i(\mathbf{p}))$ be the Hessian matrix of function $\lambda_i(\mathbf{p})$. For parameters $\mathbf{p} \in \mathbb{R}^m$ and $\mathbf{p}^0 \in \mathbb{R}^m$ there exist scalars $0 \leq t_1, t_2 \leq 1$ such that

$$\begin{aligned}\lambda_i(\mathbf{p}) = &\lambda_i(\mathbf{p}^0) + \mathcal{J}(\lambda_i(\mathbf{p}^0))(\mathbf{p}-\mathbf{p}^0) + \\ &\frac{1}{2}(\mathbf{p}-\mathbf{p}^0)^T\mathcal{H}(\lambda_i(\mathbf{p}^0 + t_1(\mathbf{p}-\mathbf{p}^0)))(\mathbf{p}-\mathbf{p}^0),\end{aligned} \tag{2.42}$$

$$\begin{aligned}x_i^{(l)}(\mathbf{p}) = &x_i^{(l)}(\mathbf{p}^0) + \mathcal{J}(x_i^{(l)}(\mathbf{p}^0))(\mathbf{p}-\mathbf{p}^0) + \\ &\frac{1}{2}(\mathbf{p}-\mathbf{p}^0)^T\mathcal{H}(x_i^{(l)}(\mathbf{p}^0 + t_2(\mathbf{p}-\mathbf{p}^0)))(\mathbf{p}-\mathbf{p}^0) \quad \text{for } l = 1,\ldots,n.\end{aligned} \tag{2.43}$$

Where $\mathcal{J}(\lambda_i(\mathbf{p}^0))$ and $\mathcal{J}(x_i^{(l)}(\mathbf{p}^0))$ are given by (2.40) and (2.41), respectively.

Now, we are able to calculate approximations of the eigenvalues and provide error bounds for obtained approximations.

Corollary 2.8 Let the assumptions from Lemma 2.5 hold, then the approximation of eigenvalue $\widetilde{\lambda}_i(\mathbf{p})$ can be calculated as

$$\widetilde{\lambda}_i(\mathbf{p}) = \lambda_i(\mathbf{p}^0) + \mathcal{J}(\lambda_i(\mathbf{p}^0))(\mathbf{p}-\mathbf{p}^0), \tag{2.44}$$

and the error bound is given by

$$|\lambda_i(\mathbf{p}) - \widetilde{\lambda}_i(\mathbf{p})| \lessapprox \frac{1}{2} M_i \|(\mathbf{p}-\mathbf{p}^0)\|^2, \tag{2.45}$$

where $M_i = \max\limits_{t\in[0,1]} (\|\mathcal{H}(\lambda_i(\mathbf{p}^0 + t(\mathbf{p}-\mathbf{p}^0))\|)$. Approximation of eigenvector $\widetilde{x_i}(\mathbf{p})$ can be calculated using

$$\widetilde{x_i}(\mathbf{p}) = x_i(\mathbf{p}^0) + \begin{bmatrix} \mathcal{J}(x_i^{(1)}(\mathbf{p}^0)) \\ \mathcal{J}(x_i^{(2)}(\mathbf{p}^0)) \\ \vdots \\ \mathcal{J}(x_i^{(n)}(\mathbf{p}^0)) \end{bmatrix} (\mathbf{p}-\mathbf{p}^0). \tag{2.46}$$

Moreover elements of $\mathcal{H}(\lambda_i(\mathbf{u}))$, where $\mathbf{u} = \mathbf{p}^0 + t(\mathbf{p}-\mathbf{p}^0), t \in [0,1]$, are given by the following formula

$$\begin{aligned} \mathcal{H}(\lambda_i(\mathbf{u}))_{k,j} &= \frac{\partial^2 \lambda_i}{\partial p_k \partial p_j}(\mathbf{u}) \\ &= -\frac{y_i^*(\mathbf{u}) \left(\frac{\partial \lambda_i}{\partial p_k}(\mathbf{u}) \frac{\partial \lambda_i}{\partial p_j}(\mathbf{u}) 2M(\mathbf{u}) + Z_{kj}(\mathbf{u}) \right) x_i(\mathbf{u})}{y_i^*(\mathbf{u})(2\lambda_i(\mathbf{u})M(\mathbf{u}) + C(\mathbf{u}))x_i(\mathbf{u})} \\ &\quad - \frac{y_i^*(\mathbf{u}) \left(\lambda_i^2(\mathbf{u}) \frac{\partial^2 M}{\partial p_k \partial p_j}(\mathbf{u}) + \lambda_i(\mathbf{u}) \frac{\partial^2 C}{\partial p_k \partial p_j}(\mathbf{u}) + \frac{\partial^2 K}{\partial p_k \partial p_j}(\mathbf{u}) \right) x_i(\mathbf{u})}{y_i^*(\mathbf{u})(2\lambda_i(\mathbf{u})M(\mathbf{u}) + C(\mathbf{u}))x_i(\mathbf{u})} \\ &\quad - \frac{y_i^*(\mathbf{u})(-N_k(\mathbf{u})PI_i(\mathbf{u})N_j(\mathbf{u}) - N_j(\mathbf{u})PI_i(\mathbf{u})N_k(\mathbf{u}))x_i(\mathbf{u})}{y_i^*(\mathbf{u})(2\lambda_i(\mathbf{u})M(\mathbf{u}) + C(\mathbf{u}))x_i(\mathbf{u})}, \end{aligned} \tag{2.47}$$

where

$$\begin{aligned} Z_{kj}(\mathbf{u}) &= \frac{\partial \lambda_i}{\partial p_k}(\mathbf{u}) \left(2\lambda_i(\mathbf{u}) \frac{\partial M}{\partial p_j}(\mathbf{u}) + \frac{\partial C}{\partial p_j}(\mathbf{u}) \right) + \\ &\quad \frac{\partial \lambda_i}{\partial p_j}(\mathbf{u}) \left(2\lambda_i(\mathbf{u}) \frac{\partial M}{\partial p_k}(\mathbf{u}) + \frac{\partial C}{\partial p_k}(\mathbf{u}) \right), \\ N_j(\mathbf{u}) &= 2\lambda_k(\mathbf{u}) \frac{\partial \lambda_i}{\partial p_j}(\mathbf{u}) M(\mathbf{u}) + \frac{\partial \lambda_i}{\partial p_j}(\mathbf{u}) C(\mathbf{u}) + \\ &\quad \lambda_i^2(u) \frac{\partial M}{\partial p_j}(\mathbf{u}) + \lambda_i(\mathbf{u}) \frac{\partial C}{\partial p_j}(\mathbf{u}) + \frac{\partial K}{\partial p_j}(\mathbf{u}) \; k, j = 1, \ldots, m, \end{aligned}$$

and $PI_i(\mathbf{u})$ for $i = 1,\dots,2n$ is given by (2.39). In general, computing M_i is not easy, since we maximize norm of the Hessian over parameter t and for each computation of Hessian we need eigenvalues and eigenvectors that correspond to PQEP (1.1) with parameter $\mathbf{p}^0 + t(\mathbf{p}-\mathbf{p}^0)$, for given $\mathbf{p},\mathbf{p}^0$, i.e., computation of M_i requires solutions of a sequence of related PQEPs, which is time consuming. As a result instead of computing M_i, we approximate M_i by using $\widetilde{\lambda}_i(\mathbf{p})$ instead of $\lambda_i(\mathbf{p})$, i.e.,

$$\widetilde{M}_i = \max_{t\in[0,1]} (\|\mathcal{H}(\widetilde{\lambda}_i(\mathbf{p}^0 + t(\mathbf{p}-\mathbf{p}^0))\|). \tag{2.48}$$

In Chapter 3 we will need the bound for

$$|\lambda_i(\mathbf{p}) - \widetilde{\lambda}_i(\mathbf{p})| \tag{2.49}$$

for the efficient estimation of the perturbation bounds related to quadratic eigenvalue problem. For this estimation of the perturbation bounds, we could also use the Gershgorian type bounds given by (2.18), where we computed all approximations of eigenvalues by dimension reduction. The bounds given by (2.18) are not efficient for computation if we want high accuracy, i.e., the higher accuracy we want, the higher reduced dimension r is (we need to solve bigger eigenvalue problem in step 4 of Algorithm 3). Thus, we will use the bound (2.45) with the approximation (2.48) instead M_i since it can be efficiently computed. The accuracy of the approximation $\tilde{M}_i$ will be shown in the next subsection.

2.2.1 Comparison with the first order approximations from perturbation theory

In [14] authors consider the perturbation theory for the polynomial eigenvalue problem

$$(\lambda^n(A_n+\Delta A_n)+\cdots+\lambda(A_1+\Delta A_1)+(A_0+\Delta A_0))x = 0, \quad x \neq 0 \tag{2.50}$$

and provide different first order approximation for the perturbed eigenvalue $\lambda+\Delta\lambda$. For the case of quadratic eigenvalue problem, this approximation is given by

$$\widetilde{\lambda} = \lambda - \frac{y^*(\lambda^2\Delta M + \lambda\Delta C + \Delta K)x}{y^*(2M\lambda + C)x}, \tag{2.51}$$

where

$$|\lambda+\Delta\lambda-\widetilde{\lambda}| \lessapprox \mathcal{O}(\|[\Delta M,\Delta C,\Delta K]\|_F^2). \tag{2.52}$$

Since we assumed that the change in the parameter is small enough, we will consider the first order approximation of an eigenvalue

$$\widetilde{\lambda}(\mathbf{p}) = \lambda(\mathbf{p}^0) + \mathcal{J}(\lambda(\mathbf{p}^0))(\mathbf{p}-\mathbf{p}^0),$$

given by (2.44) as the approximation of the perturbed eigenvalue $\lambda(\mathbf{p})$, i.e., we have the perturbation in parameter, $\mathbf{p} = \mathbf{p}^0 + \Delta\mathbf{p}$, where $\mathbf{p^0} = [\mathbf{v}_M^0; \mathbf{v}^0; \mathbf{v}_K^0]$ is parameter of unperturbed system and $\mathbf{p} = [\mathbf{v}_M; \mathbf{v}; \mathbf{v}_K]$ is parameter of perturbed system and $\Delta\mathbf{p}$ is the perturbation of the parameter. Thus, mass, damping and stiffness matrices of the perturbed system are $M(\mathbf{p}) = M(\mathbf{v}_M) = M(\mathbf{v}_M^0) + \Delta M$, $C(\mathbf{p}) = C(\mathbf{v}) = C(\mathbf{v}^0) + \Delta C$, and $K(\mathbf{p}) = K(\mathbf{v}_K) = K(\mathbf{v}_K^0) + \Delta K$, respectively, where $\Delta M, \Delta C, \Delta K$ are corresponding perturbation matrices and $M(\mathbf{p}^0) = M(\mathbf{v}_M^0)$, $C(\mathbf{p}^0) = C(\mathbf{v}^0)$, $K(\mathbf{p}^0) = K(\mathbf{v}_K^0)$ are matrices of unperturbed system.

To distinguish the approximations (2.44) and (2.51) we will denote one from (2.51) with $\widetilde{\mu}$, and empathize the parameter dependence, i.e.,

$$\widetilde{\mu} = \lambda(\mathbf{p}^0) - \frac{y^*(\mathbf{p}^0)(\lambda^2(\mathbf{p}^0)\Delta M + \lambda(\mathbf{p}^0)\Delta C + \Delta K)x(\mathbf{p}^0)}{y^*(\mathbf{p}^0)(2M(\mathbf{p}^0)\lambda(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0)}. \tag{2.53}$$

In the following we show the similarities and differences of these two approximations. Since first order approximation can be written as (2.44), i.e.,

$$\widetilde{\lambda}(\mathbf{p}) = \lambda(\mathbf{p}^0) - \frac{y^*(\mathbf{p}^0)Bx(\mathbf{p}^0)}{y^*(\mathbf{p}^0)(2\lambda(\mathbf{p}^0)M(\mathbf{p}^0) + C(\mathbf{p}^0))x(\mathbf{p}^0)}, \tag{2.54}$$

where $B = \sum_{k=1}^{m} \left(\lambda^2(\mathbf{p}^0)\frac{\partial M}{\partial p_k}(\mathbf{p}^0) + \lambda(\mathbf{p}^0)\frac{\partial C}{\partial p_k}(\mathbf{p}^0) + \frac{\partial K}{\partial p_k}(\mathbf{p}^0)\right)(p_k - p_k^0)$.

Now we see that approximations from (2.54) and (2.53) are equal if and only if

$$B = (\lambda^2(\mathbf{p}^0)\Delta M + \lambda(\mathbf{p}^0)\Delta C + \Delta K)x(\mathbf{p}^0). \tag{2.55}$$

If we write B as

$$\begin{aligned} B =& \lambda^2(\mathbf{p}^0)\sum_{k=1}^{m}\frac{\partial M}{\partial p_k}(\mathbf{p}^0)(p_k - p_k^0) + \lambda(\mathbf{p}^0)\sum_{k=1}^{m}\frac{\partial C}{\partial p_k}(\mathbf{p}^0)(p_k - p_k^0) \\ &+ \sum_{k=1}^{m}\frac{\partial K}{\partial p_k}(\mathbf{p}^0)(p_k - p_k^0), \end{aligned} \tag{2.56}$$

then (2.55) holds if and only if

$$\begin{aligned} \Delta M &= \sum_{k=1}^{m}\frac{\partial M}{\partial p_k}(\mathbf{p}^0)(p_k - p_k^0), \\ \Delta C &= \sum_{k=1}^{m}\frac{\partial C}{\partial p_k}(\mathbf{p}^0)(p_k - p_k^0), \\ \Delta K &= \sum_{k=1}^{m}\frac{\partial K}{\partial p_k}(\mathbf{p}^0)(p_k - p_k^0). \end{aligned} \tag{2.57}$$

In the following example we show the case when equality (2.55) holds and the case when it doesn't.

Example 2.3 In the first case we consider affine parameter-dependance in system matrices, and in the second case we consider quadratic parameter-dependance only in damping matrix.

Case 1: As we said first we have affine parameter-dependance, i.e.,

$$\begin{aligned} M(\mathbf{p}) &= M_0 + \sum_{k=1}^{m} M_k p_k, & C(\mathbf{p}) &= C_0 + \sum_{k=1}^{m} C_k p_k, & K(\mathbf{p}) &= K_0 + \sum_{k=1}^{m} K_k p_k, \\ M(\mathbf{p}^0) &= M_0 + \sum_{k=1}^{m} M_k p_k^0, & C(\mathbf{p}^0) &= C_0 + \sum_{k=1}^{m} C_k p_k^0, & K(\mathbf{p}^0) &= K_0 + \sum_{k=1}^{m} K_k p_k^0. \end{aligned} \tag{2.58}$$

From (2.58) follows

$$\Delta M = \sum_{k=1}^{m} M_k (p_k - p_k^0), \quad \Delta C = \sum_{k=1}^{m} C_k (p_k - p_k^0), \quad \Delta K = \sum_{k=1}^{m} K_k (p_k - p_k^0),$$

and since

$$\frac{\partial M}{\partial p_k}(\mathbf{p}^0) = M_k, \quad \frac{\partial C}{\partial p_k}(\mathbf{p}^0) = C_k, \quad \frac{\partial K}{\partial p_k}(\mathbf{p}^0) = K_k, \quad \text{for} \quad k = 1, \ldots, m,$$

we conclude that (2.57) holds, i.e., approximations are equal.

Case 2: In this case for the sake of simplicity we consider arbitrary parameter-dependance only in damping matrix, i.e.,

$$\begin{aligned} M(\mathbf{p}) &= M(\mathbf{p}^0), & K(\mathbf{p}) &= K(\mathbf{p}^0), \\ C(\mathbf{p}) &= C_0 + \sum_{k=1}^{m} C_k f(p_k), & C(\mathbf{p}^0) &= C_0 + \sum_{k=1}^{m} C_k f(p_k^0), \end{aligned} \tag{2.59}$$

where $f : \mathbb{R} \to \mathbb{R}$ is any function. Then from (2.59) follows

$$\Delta M = 0, \quad \Delta C = \sum_{k=1}^{m} C_k (f(p_k) - f(p_k^0)), \quad \Delta K = 0,$$

and since

$$\frac{\partial M}{\partial p_k}(\mathbf{p}^0) = 0, \quad \frac{\partial C}{\partial p_k}(\mathbf{p}^0) = f'(p_k^0) C_k, \quad \frac{\partial K}{\partial p_k}(\mathbf{p}^0) = 0, \quad \text{for} \quad k = 1, \ldots, m.$$

Since

$$\sum_{k=1}^{m} \frac{\partial C}{\partial p_k}(\mathbf{p}^0)(p_k - p_k^0) = \sum_{k=1}^{m} f'(p_k^0) C_k (p_k - p_k^0)$$

we conclude that (2.57) holds if and only if $f'(p_k^0)(p_k - p_k^0) = f(p_k) - f(p_k^0)$, for $k = 1, \ldots, m$, which is not always satisfied, but when $\Delta \mathbf{p} \to 0$ $|\widetilde{\lambda}(\mathbf{p}) - \widetilde{\mu}| \to 0$.

One can notice that only for affine parameter-dependance, approximations $\widetilde{\mu}$ and $\widetilde{\lambda}(\mathbf{p})$ will be equal. Furthermore, a significant advantage of our approximation $\widetilde{\lambda}(\mathbf{p})$ is that we provided

the upper bound for

$$|\lambda(\mathbf{p}) - \widetilde{\lambda}(\mathbf{p})| \lessapprox \frac{1}{2}\widetilde{M}_i \|(\mathbf{p}-\mathbf{p}^0)\|^2,$$

where $\widetilde{M}_i$ is given by (2.48), which can be efficiently computed.

2.2.2 Numerical experiments

In this subsection, we illustrate the accuracy of the first order approximation of the perturbed eigenvalue problem. We consider the mechanical system, the so-called the *n-mass oscillator* shown in Figure 2.10.

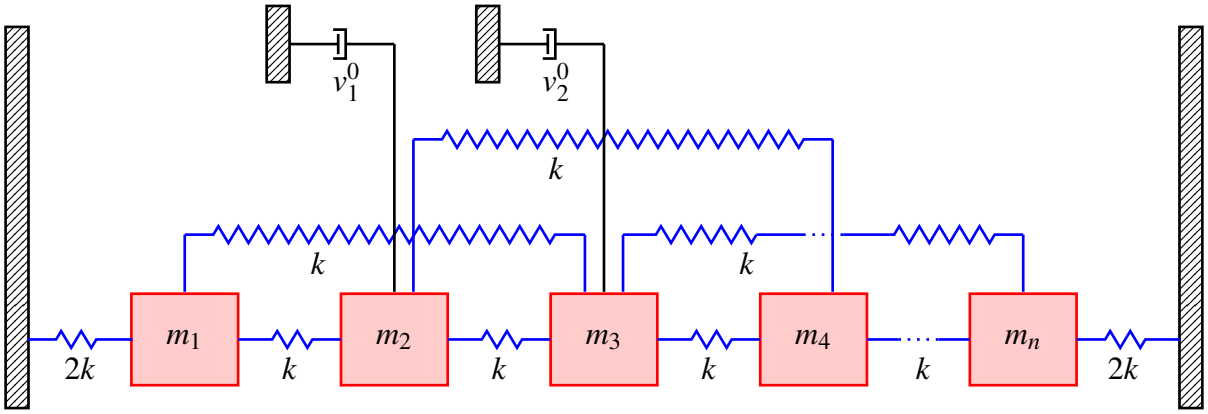

Figure 2.10: n-mass oscillator

Example 2.4 The dimension of the system is $n = 100$. In this example the mass matrix does not depend on the vector of parameters and it is denoted by

$$M = \mathrm{diag}(m_1, \ldots, m_n), \quad \text{where} \quad m_i = 2 \cdot i, i = 1, \ldots, n.$$

The stiffness matrix is given by

$$K_0 = K(\mathbf{p}^0) \quad = \quad K + p_3^0 \cdot \widetilde{K}(\mathbf{p}^0), \tag{2.60}$$

where

$$K = \begin{bmatrix} 4k & -k & -k & & & & \\ -k & 4k & -k & -k & & & \\ -k & -k & 4k & -k & -k & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -k & -k & 4k & -k & -k \\ & & & -k & -k & 4k & -k \\ & & & & -k & -k & 4k \end{bmatrix}, \quad \widetilde{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \widetilde{K}_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{K}_1 = \widetilde{K}(30:55,30:55) = \begin{bmatrix} 2k & 0 & -k & & \\ 0 & 2k & \ddots & \ddots & \\ -k & \ddots & \ddots & \ddots & -k \\ & \ddots & \ddots & \ddots & 0 \\ & & -k & 0 & 2k \end{bmatrix},$$

where $k = 0.1$ and parameter dependant block of the stiffness matrix in our mechanical system is shown in Figure 2.11.

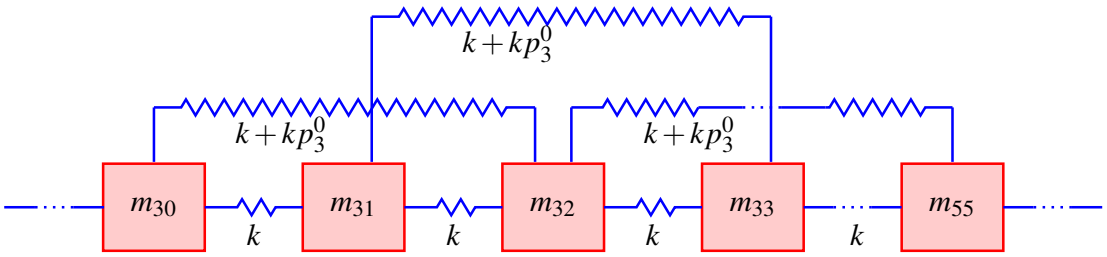

Figure 2.11: Block that contains parameter p_3^0 within n-mass oscillator represented by matrix in (2.60), i.e., $K_0(30:55,30:55)$

Finally, denote the damping matrix $C(\mathbf{p}^0) = C_{int} + C_{\text{ext}}(\mathbf{v}^0)$, where $\mathbf{p}^0 = [v_1^0, v_2^0, p_3^0]^T$, and internal damping $C_{\text{int}} = \alpha_c C_{\text{crit}}$, with $\alpha_c = 0.002$ and C_{crit} in given by (1.21). Matrix $C_{\text{ext}}(\mathbf{v}^0)$ has only two nontrivial elements on positions $(35,35)$ and $(50,50)$. This means that we put two grounded dampers on the 35th and the 50th mass with viscosities v_1^0, v_2^0, respectively, i.e.,

$$C_{\text{ext}}(\mathbf{v}^0) = v_1^0 e_{35} e_{35}^T + v_2^0 e_{50} e_{50}^T.$$

We consider the following (perturbed) QEP:

$$(\lambda^2(\mathbf{p})M + C(\mathbf{p})\lambda(\mathbf{p}) + K(\mathbf{p}))x(\mathbf{p}) = 0, \tag{2.61}$$

with a small change δ in parameter $\mathbf{p}$, i.e., $\mathbf{p} = \mathbf{p}^0 + \delta$. In this example, we consider the case where $\mathbf{p} = (\frac{v}{4} + \delta, v + \delta, \delta)$ and $\mathbf{p}^0 = (\frac{v}{4}, v, 0)$.

Figure 2.12 shows the relative true error between perturbed eigenvalue and its approximation calculated in MATLAB with double precision. It also shows the relative approximation of the error bounds for eigenvalues of the system given in (2.48). Figure 2.12 doesn't show relative errors for all the eigenvalues, but only for the eigenvalues that have relative true error greater than 10^{-13}.

In the first subplot of Figure 2.12 the viscosity parameter is $v = 10$ and perturbation is $\delta = 0.005$. For example, for 193th eigenvalue we obtain the maximal approximated relative error bound, i.e.,

$$\frac{1}{2}\frac{\widetilde{M}_{193}||\mathbf{p} - \mathbf{p}^0||^2}{|\lambda_{193}(\mathbf{p})|} = 5.2836 \cdot 10^{-3},$$

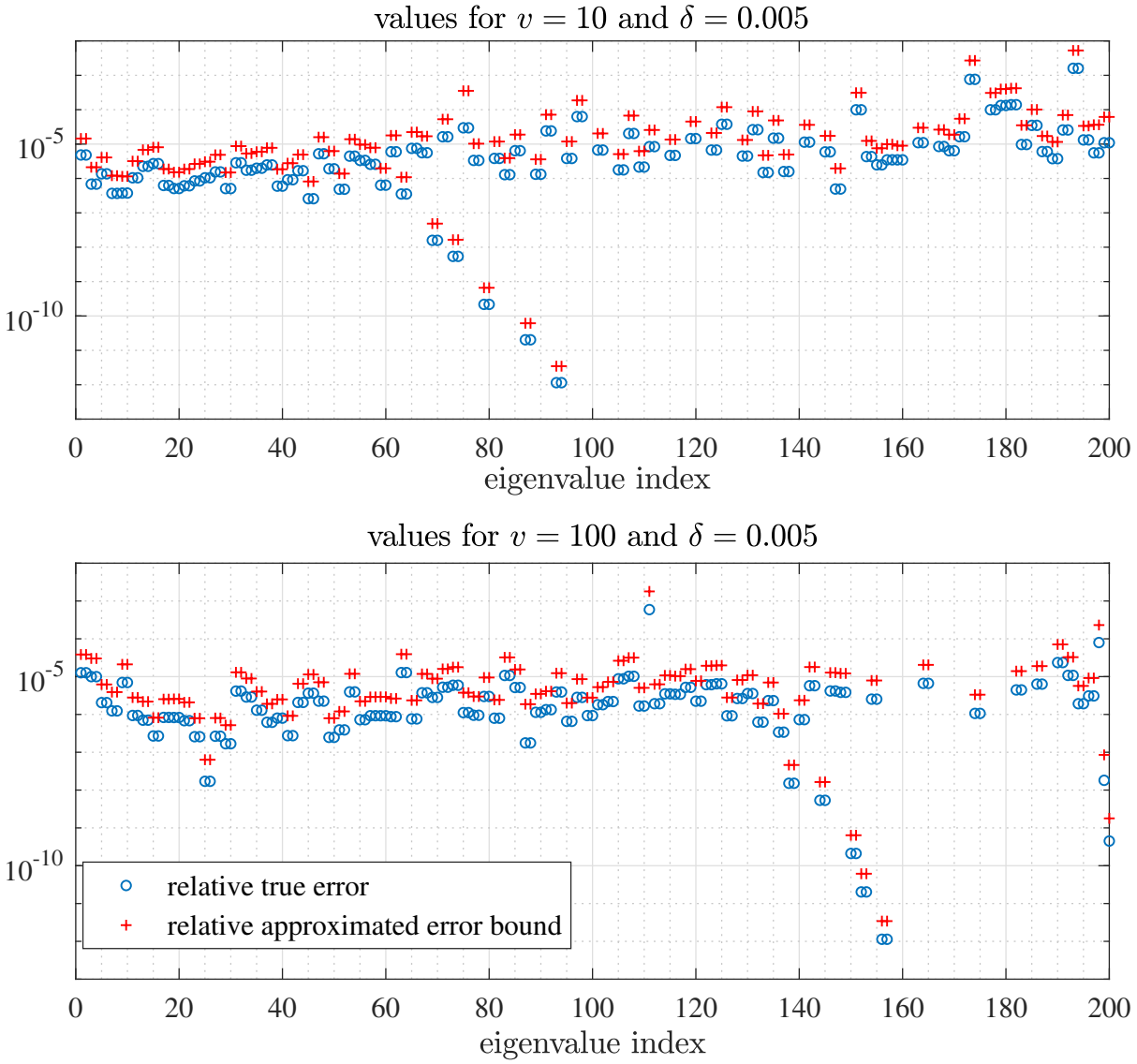

Figure 2.12: Comparison of relative true error $\frac{|\lambda(\mathbf{p})-\widetilde{\lambda}(\mathbf{p})|}{|\lambda(\mathbf{p})|}$ and relative approximated error bound of eigenvalues, where $v = 10$ and $\delta = 0.005$, and $v = 100$ and $\delta = 0.005$

and relative true error for 193th eigenvalue is

$$\frac{|\lambda_{193}(\mathbf{p}) - \widetilde{\lambda}_{193}(\mathbf{p})|}{|\lambda_{193}(\mathbf{p})|} = 1.5986 \cdot 10^{-3}.$$

In the second subplot of Figure 2.12 the viscosity parameter is $v = 100$ and perturbation is $\delta = 0.005$. In this case the maximal approximated relative error bound is obtained for the 111st eigenvalue, i.e.,

$$\frac{1}{2}\frac{\widetilde{M}_{111}||\mathbf{p} - \mathbf{p}^0||^2}{|\lambda_{111}(\mathbf{p})|} = 1.7910 \cdot 10^{-3}.$$

and relative true error for the 111st eigenvalue is

$$\frac{|\lambda_{111}(\mathbf{p}) - \widetilde{\lambda}_{111}(\mathbf{p})|}{|\lambda_{111}(\mathbf{p})|} = 5.9159 \cdot 10^{-4}.$$

From Figure 2.12 we can conclude that approximated error bounds track the true error when we change viscosity.

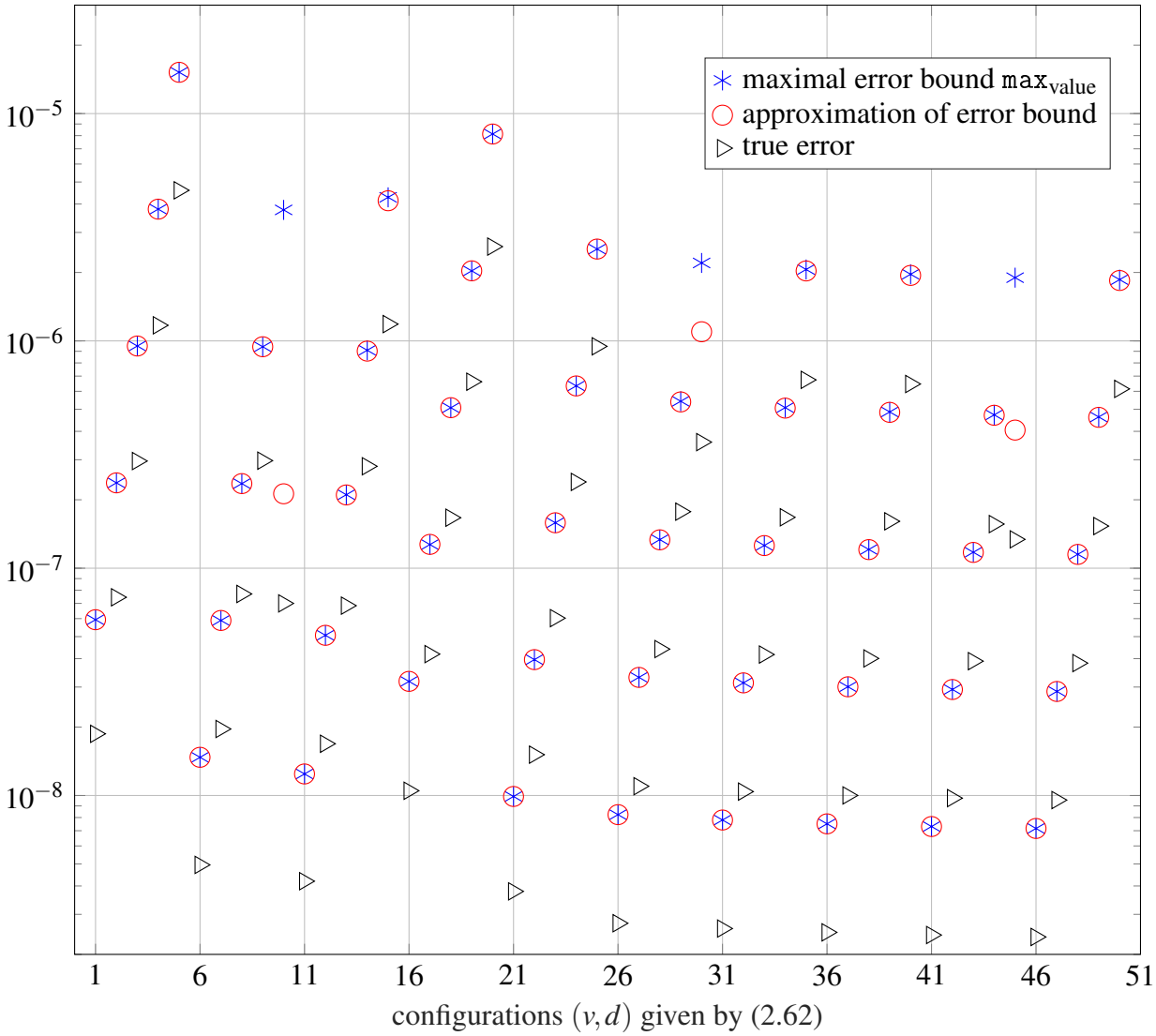

Figure 2.13: Comparison of maximal error bound and corresponding approximation of error bound with corresponding true error for different perturbation and viscosity parameters, e.g., points with axis $1,\ldots,5$ correspond respectively to configurations $(10,5),(10,4),\ldots,(10,1)$.

In Figure 2.13 we consider configurations

$$(v,d), \quad \text{for} \quad v \in \{10,\ldots,100\}, \quad d \in \{5,\ldots,1\} \tag{2.62}$$

where d determines magnitude of perturbation, i.e., perturbation is $\frac{0.01}{2^d}$, and c corresponds to viscosity parameter. For these configurations we calculated maximal error bound

$\texttt{max}_{\text{value}} = \frac{1}{2}M_i||\mathbf{p}-\mathbf{p}^0||^2$, corresponding approximation of error bound $\frac{1}{2}\widetilde{M}_i||\mathbf{p}-\mathbf{p}^0||^2$ and corresponding true error, $|\lambda_i(\mathbf{p})-\widetilde{\lambda}_i(\mathbf{p})|$. Here i is chosen among all 200 eigenvalues, i.e.

$$\texttt{max}_{\text{value}} = \frac{1}{2}M_i||\mathbf{p}-\mathbf{p}^0||^2 = \max_{l=1,\ldots,2n}\frac{1}{2}M_l||\mathbf{p}-\mathbf{p}^0||^2, \tag{2.63}$$

for all configurations (2.62). We should note that if we fix the viscosity parameter and we variate perturbation, i for which we obtain $\texttt{max}_{\text{value}}$ in (2.63) doesn't have to be the same. From Figure 2.13, we can see that the exact error bound can easily be replaced with its approximation.

In this section we have shown the approximation of the eigenvalues of the PQEP (1.1) which can be used when we have a small change in the parameter, and also we have shown approximation of the corresponding error bound which can be used in perturbation theory of PQEP.

In contrast to approximations from this section and from Section 2.1 in the following section we assume that the damping matrix has the structure that can be efficiently exploited, which is shown in Subsection 2.3.1.

2.3 Approximations by modified Rayleigh quotient

In this subsection we show how one can use the structure of the damping matrix to accelerate the computation of the eigenvalues of the PQEP (1.1). We assume that matrices C_i, $i = 1,\ldots,s$ in the external damping (1.3) are matrices of rank one, i.e.,

$$C_i = g_i g_i^T, \quad i = 1,\ldots,s,$$

where $g_i \in \mathbb{R}^n$ is a vector that determines the geometry of the ith damper. In this case we can transform our PQEP (1.1) into multiple connected diagonal plus rank one (DPR1) eigenvalue problems, which is described later in this subsection. Therefore, in this subsection we propose an efficient way of eigenvalue computation based on eigenvalue computation of DPR1 matrices.

Since we have a special structure of the external damping (1.3), again as in Subsection 2.1 our parameter $\mathbf{p} = \mathbf{v}$, i.e., we have only viscosities of the dampers as parameters.

To understand our new method, we first describe the following method of [47] for complex symmetric DPR1 (CSymDPR1) matrices, as an modified version of this will be a major subroutine of our approach.

Before dealing with the transformation of PQEP (1.1) into multiple CSymDPR1 eigenvalue problem, we present an algorithm for efficient eigenvalue computation of CSymDPR1 matrices.

Let $\mathbf{A}$ be an $2n \times 2n$ complex symmetric matrix of the form

$$\mathbf{A} = \mathbf{D} + \zeta uu^T, \tag{2.64}$$

where $\mathbf{D} = \text{diag}(d_1, d_2, \ldots, d_{2n})$ is a diagonal matrix of order $2n$, $u = [u_1, u_2, \cdots, u_{2n}]^T$ is a vector

and $\zeta \neq 0$ is a scalar. Without loss of generality, we make the following assumptions:

- $\zeta > 0$ (otherwise we consider the matrix $\mathbf{A} = -\mathbf{D} - \zeta uu^T$),
- $\mathbf{A}$ is irreducible, i.e., $u_i \neq 0, i = 1, \ldots, 2n$, and $d_i \neq d_j,$ for all $i \neq j,\ i, j = 1, \ldots, 2n$.

Indeed, if $u_i = 0$ for some i, then the diagonal element d_i is an eigenvalue whose corresponding eigenvector is the i-th canonical vector, and if $d_i = d_j$, then d_i is an eigenvalue of the matrix $\mathbf{A}$, see [46].

Let us assume that $\mathbf{A}$ is diagonalizable matrix, i.e.,

$$\mathbf{A} = \mathbf{W}\Lambda\mathbf{W}^T$$

is the eigenvalue decomposition of $\mathbf{A}$, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n})$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$, and $\mathbf{W} = \begin{bmatrix} w_1 & \cdots & w_{2n} \end{bmatrix}$ is a matrix whose columns are the corresponding eigenvectors.

The eigenvalue problem for a CSymDPR1 matrix $\mathbf{A}$ can be solved by any of the standard methods for the symmetric eigenvalue problems (see, e.g., [114]). However, because of the special structure of diagonal-plus-rank-one matrices, we can use the following approach (see, e.g., [24], [37]). The eigenvalues of $\mathbf{A}$ are the zeros of the secular function:

$$f(\lambda) = 1 + \zeta \sum_{i=1}^{2n} \frac{u_i^2}{d_i - \lambda} = 1 + \zeta u^T (\mathbf{D} - \lambda \mathbf{I})^{-1} u, \tag{2.65}$$

and the corresponding eigenvectors are given by

$$w_i = \frac{x_i}{\|x_i\|_2}, \text{ where } x_i = (\mathbf{D} - \lambda_i \mathbf{I})^{-1} u, \quad i = 1, \ldots, 2n. \tag{2.66}$$

The secular function (2.65) can be solved by, for example, `mpsolve` from the package MPSolve ([16], [17]).

Eigenvector matrix of CSymDPR1 matrices has a Cauchy-like structure. More on Cauchy-like matrices can be found in [76]. This structure can be useful in acceleration of matrix multiplication.

If $\mathbf{A}$ is a real matrix, the diagonal elements of the matrix $\mathbf{D}$, are called poles of the function f and for $\zeta > 0$, f is strictly increasing between the poles, implying the strict interlacing property $\lambda_1 > d_1 > \lambda_2 > d_2 > \cdots > \lambda_{2n} > d_{2n}$ and eigenvalues can be computed highly accurately by bisection (for more details see [46]). Orthogonality of eigenvectors depends on accuracy of eigenvalues.

In the complex symmetric case there is no interlacing. In this case for eigenvalues computation, we use Modified Rayleigh Quotient Iteration (MRQI) as follows:

$$\rho = \eta \frac{x^T \mathbf{A} x}{x^T x}, \text{ where } \quad x := (\mathbf{D} - \rho \mathbf{I})^{-1} u, \tag{2.67}$$

where $\eta \in \mathbb{R}$ is a step size that enhances the convergency of ρ. Once ρ has converged, it can be deflated (see formulas in [77]), thus obtaining CSymDPR1 of size $2n-1$. In the method from [47] the parameter η doesn't change, i.e., $\eta = 1$, first few steps use RQI (see [47, Section 3.2]), then MRQI is used until convergence. The change in step size and the fact that we use MRQI in each step are the differences between our method and the existing method.

The deflation formula comes from shifted inverse power method (see, e.g., [15]) where we recompute components of u. For example if we computed the approximation of the eigenvalue λ_1 by using the shift d_1 then the obtained CSymDPR1 of size $2n-1$, i.e., the deflated $\mathbf{A}$ has the following form:

$$\mathbf{A}_{\mathrm{d}} = \mathbf{D}_{\mathrm{d}} + \zeta u_{\mathrm{d}} u_{\mathrm{d}}^T, \tag{2.68}$$

where

$$\mathbf{D}_{\mathrm{d}} = \mathrm{diag}(d_2, \ldots, d_{2n}) \quad \text{and} \quad (u_{\mathrm{d}})_i = u_i \sqrt{\frac{d_i - d_1}{d_i - \lambda_1}}, \quad i = 2, \ldots, 2n. \tag{2.69}$$

By applying Sherman-Morrison-Woodbury formula to compute $\mathbf{A}^{-1}$ one obtains

$$\mathbf{A}^{-1} = \mathbf{D}^{-1} + \gamma \mathbf{D}^{-1} u u^T \mathbf{D}^{-1}, \quad \gamma = -\frac{\zeta}{1 + \zeta u^T \mathbf{D}^{-1} u},$$

which is again a CSymDPR1 matrix and can be computed in $O(2n)$ operations, i.e., it is enough to compute the new scalar and two vectors, one is the diagonal of the diagonal matrix and the other one is the rank 1 update, to know the CSymDPR1 matrix.

We now show how to transform our PQEP (1.1) into multiple connected DPR1 eigenvalue problems. Let Φ be the matrix that simultaneously diagonalizes M and K, as in (1.4).

As already mentioned in Section 1.2 we obtain the following:

$$\mathbf{A}(\mathbf{v})\mathbf{y}(\mathbf{v}) = \lambda(\mathbf{v})\mathbf{y}(\mathbf{v}),$$

where

$$\begin{aligned} \mathbf{A}(\mathbf{v}) &= \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T C_{\mathrm{int}} \Phi - \hat{C}(\mathbf{v}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \Omega \\ -\Omega & -\alpha\Omega \end{bmatrix} - \begin{bmatrix} 0 \\ \Phi^T G \end{bmatrix} \begin{bmatrix} v_1 & & \\ & \ddots & \\ & & v_s \end{bmatrix} \begin{bmatrix} 0 & G^T \Phi \end{bmatrix}, \end{aligned} \tag{2.70}$$

$$\mathbf{y}(\mathbf{v}) = \begin{bmatrix} y_1(\mathbf{v}) \\ y_2(\mathbf{v}) \end{bmatrix} = \begin{bmatrix} \Omega \Phi^{-1} x(\mathbf{v}) \\ \lambda(\mathbf{v}) \Phi^{-1} x(\mathbf{v}) \end{bmatrix}, \tag{2.71}$$

Algorithm 4: Eigensolver for CSymDPR1 matrices

Require: $\mathbf{D} = \operatorname{diag}(d_1, \ldots, d_{2n})$, $u \in \mathbb{C}^{2n}$, $\zeta \in \mathbb{R}_+$ from (2.64),
Ensure: Eigenvalues $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{2n})$ and eigenvectors $\mathbf{W} = \begin{bmatrix} w_1 & \cdots & w_{2n} \end{bmatrix}$ of $\mathbf{A}$.

for $l = 1 : 2n$, (loop for each eigenvalue) **do**
  for $i = 1 : 2n$, (loop for each diagonal element) **do**
    $\mu = \max_{i=1,\ldots,2n} |d_i|$ (shift in a diagonal element)
    $\hat{d}_i = d_i - \mu$
  end for
  set initial values: z_u unit vector, $\delta = 1$, $\gamma = 0$, $\eta = 1$
  while not converged **do**
    if progress is too slow **then**
      $\eta = \frac{\eta}{2}$
      reset initial values: z_u unit vector, $\delta = 1$, $\gamma = 0$,
    end if
    $\delta = \eta z_u^T (\widehat{\mathbf{D}} + \zeta u u^T) z_u / \|z_u\|^2$
    $\gamma = \gamma + \delta$
    $z_u = (\widehat{\mathbf{D}} - \delta I)^{-1} u$
  end while
  $\lambda_l = \mu + \gamma$ (computing λ)
  Compute eigenvector w_l by formula (2.66)
  Deflate λ_l from $\mathbf{D} + \zeta u u^T$ via the procedure given in [77, Section 7.5]
end for

where $G = \begin{bmatrix} g_1 & \cdots & g_s \end{bmatrix}$ is a matrix whose columns are vectors g_i that determine the geometry of the ith damper. Notice that we doubled the dimension of our eigenvalue problem.

Let $\mathbf{P}$ be the perfect shuffle permutation as given in (2.1). Now, instead of PQEP (1.1) we consider the following eigenvalue problem

$$\mathbf{P}^T \mathbf{A}(\mathbf{v}) \mathbf{P} \mathbf{P}^T y(\mathbf{v}) = \widehat{\mathbf{A}}(\mathbf{v}) \mathbf{P}^T y(\mathbf{v}) = \lambda(\mathbf{v}) \mathbf{P}^T y(\mathbf{v}). \tag{2.72}$$

where the structure of $\widehat{\mathbf{A}}(\mathbf{v})$ is given by:

$$\widehat{\mathbf{A}}(\mathbf{v}) = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{bmatrix} - \widehat{G} \begin{bmatrix} v_1 & & \\ & \ddots & \\ & & v_s \end{bmatrix} \widehat{G}^T, \tag{2.73}$$

where

$$D_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & \alpha\omega_i \end{bmatrix}, \qquad i = 1, \ldots, n, \tag{2.74}$$

$$\widehat{G} = \begin{bmatrix} 0 & \dots & 0 \\ (\Phi^T G)_{11} & \dots & (\Phi^T G)_{1s} \\ 0 & \dots & 0 \\ (\Phi^T G_{21} & \dots & (\Phi^T G)_{2s} \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ (\Phi^T G)_{n1} & \dots & (\Phi^T G)_{ns} \end{bmatrix} = \begin{bmatrix} \hat{g}_1 & \dots & \hat{g}_s \end{bmatrix}, \tag{2.75}$$

where $\hat{g}_l$ is l-th column of matrix $\widehat{G}$.

Now the PQEP (1.1) has the following form

$$\left(\begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{bmatrix} - \sum_{l=1}^{s} v_l \hat{g}_l \hat{g}_l^T \right) \mathbf{P}^T y(\mathbf{v}) = \lambda(\mathbf{v}) \mathbf{P}^T y(\mathbf{v}). \tag{2.76}$$

Let $\Psi_i \in \mathbb{R}^{2\times 2}$ be the matrix that diagonalizes matrix D_i, then matrix

$$\Psi = \begin{bmatrix} \Psi_1 & & \\ & \ddots & \\ & & \Psi_n \end{bmatrix} \tag{2.77}$$

diagonalizes block diagonal matrix from (2.76) and transforms that eigenvalue problem into

$$\widetilde{\mathbf{A}}(\mathbf{v}) w(\mathbf{v}) = \lambda(\mathbf{v}) w(\mathbf{v}), \tag{2.78}$$

where

$$\widetilde{\mathbf{A}}(\mathbf{v}) = (\mathbf{D} - v_1 \Psi^{-1} \hat{b}_1 \hat{b}_1^T \Psi - \dots - v_s \Psi^{-1} \hat{b}_s \hat{b}_s^T \Psi), \tag{2.79}$$

$w(\mathbf{v}) = \Psi^{-1} \mathbf{P}^T y(\mathbf{v})$ and

$$\mathbf{D} = \Psi^{-1} \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{bmatrix} \Psi.$$

Since matrices $\Phi, \mathbf{P}, \Psi_i, D_i, i = 1, \dots, n$ are not dependant on parameter $\mathbf{v}$, this previous part can be done off-line in the process of damping optimization, which is shown in Chapter 4.

In the following part we show how one can iteratively apply Algorithm 4 on matrix (2.79) to compute eigenvalues and eigenvectors and this is done on-line in optimization process for each change in parameter $\mathbf{v}$. For the sake of easier notation we won't write dependence on parameter $\mathbf{v}$ in the following description of iterative application of Algorithm 4.

Let ξ_1 and L_1 be eigenvectors and diagonal matrix containing eigenvalues of the DPR1 matrix

Algorithm 5: Approximation of eigenvalues and eigenvectors of PQEP (1.1)

Require: $M,K \in \mathbb{R}^{n\times n}$, $\alpha \in \mathbb{R}$ for C_{int} , $b_l \in \mathbb{R}^n$ and parameter $v_l, l = 1,\dots,s$, $\Phi \in \mathbb{C}^{n\times n}$ such that (1.4) hold, perfect shuffle matrix P from (2.73), Ψ from (2.77) so we can obtain matrix $D - \sum_{l=1}^{s} v_l \Psi^{-1}\hat{g}_l\hat{g}_l^T\Psi$

Ensure: approximation of eigenvalues L and eigenvectors X

1: $L_0 = D$
2: $U := [u_1,\dots,u_n] = \Psi^{-1}[\hat{g}_1,\dots,\hat{g}_s], Z := [z_1,\dots,z_n] = \Psi^T[\hat{g}_1,\dots,\hat{g}_s]$, $v = [v_1,\dots,v_s]^T$
3: **for** $l = 1,\dots,s$ **do**
4: **if** $u_l \neq z_l$ **then**
5: $\hat{z}_l = Su_l$, where S is given by (2.82)
6: **end if**
7: compute L_l and ξ_l by using Algorithm 4 on $(L_{l-1}, \hat{z}_l, v_l)$
8: $\xi_l = S^{-1}\xi_l$
9: $U = \xi_l^{-1}U$, $Z = \xi_l^T Z$
10: **end for**
11: $L = D$, $X = \Phi\Omega^{-1}[P\Psi\xi_1\xi_2\dots\xi_s](1:n,:)$[1]

[1] $[P\Psi\xi_1\xi_2\dots\xi_s](1:n,:)$ takes first n rows of matrix $[P\Psi\xi_1\xi_2\dots\xi_s]$

$D - v_1\Psi^{-1}\hat{g}_1\hat{g}_1^T\Psi$, respectively, then

$$\left(L_1 - v_2\xi_1^{-1}\Psi^{-1}\hat{g}_2\hat{g}_2^T\Psi\xi_1 - \sum_{l=3}^{s} v_l\xi_1^{-1}\Psi^{-1}\hat{g}_l\hat{g}_l^T\Psi\xi_1\right)\xi_1^{-1}w = \lambda\xi_1^{-1}w. \tag{2.80}$$

Next we compute eigenvalues and eigenvectors of $L_1 - v_2\xi_1^{-1}\Psi^{-1}\hat{g}_2\hat{g}_2^T\Psi\xi_1$. If we denote $\xi_0 = I$ and $L_0 = D$ then we can write this iterative application of Algorithm 4 as follows. Compute ξ_i and L_i, which are respectively eigenvectors and diagonal matrix containing eigenvalues of the

$$L_{i-1} - v_i\overline{\xi}_{i-1}^{-1}\Psi^{-1}\hat{g}_i\hat{g}_i^T\Psi\overline{\xi}_{i-1}, \tag{2.81}$$

where $\overline{\xi}_{i-1} = \xi_1\dots\xi_{i-1}$, for $i = 1,\dots,s$. In the last step we obtain L_s, which contains eigenvalues and ξ_s from which one can obtain eigenvectors of QEP (1.1) as shown in Algorithm 5.

This iterative application of Algorithm 4 on matrix with a structure (2.79) is summarised in Algorithm 5.

Remark 2.2. One can notice that every DPR1 matrix can be rewritten as CSymDPR1 with the same eigenvalues, i.e., if we have matrix $D + \rho uz^T$, then we set

$$S = \operatorname{diag}\left(\sqrt{\frac{z_1}{u_1}},\dots,\sqrt{\frac{z_{2n}}{u_{2n}}}\right) \tag{2.82}$$

and obtain CSymDPR1 $D + \rho\hat{z}\hat{z}$, where $\hat{z} = Su$ and also $u^TS^2 = z^T$ holds. Now if (λ, x) is the eigenpair of $D + \rho uz^T$, then (λ, Sx) is the eigenpair of $D + \rho\hat{z}\hat{z}$.

2.3.1 Numerical experiments

In this subsection, we show how accurate and efficient Algorithm 5 is. In this examples we consider the mechanical system, so-called $n-$*mass oscillator* as shown in Figure 2.14.

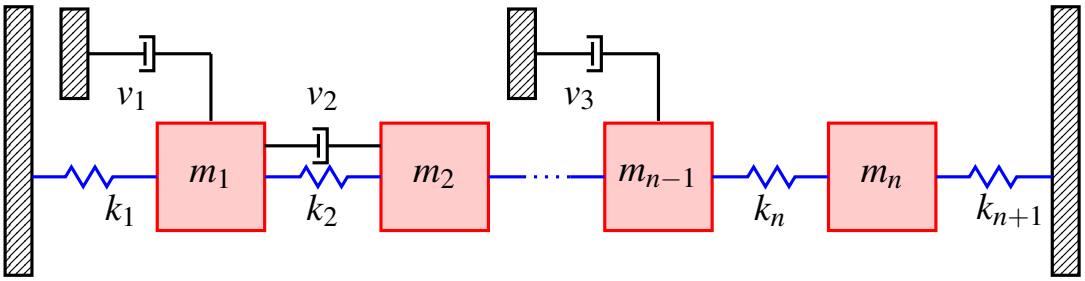

Figure 2.14: n-mass oscillator

Matrices that correspond to this mechanical system are:

$$
\begin{aligned}
M &= \operatorname{diag}(m_1, m_2, \ldots, m_n), \\
K &= \begin{bmatrix}
k_1+k_2 & -k_2 & & & & \\
-k_2 & k_2+k_3 & -k_3 & & & \\
& \ddots & \ddots & \ddots & & \\
& & -k_{n-1} & k_{n-1}+k_n & -k_n & \\
& & & -k_n & k_n+k_{n+1}
\end{bmatrix},
\end{aligned}
$$

$$
C_{\text{ext}} = v_1 e_k e_k^T + v_2 (e_j - e_{j+1})(e_j - e_{j+1})^T + v_3 e_l e_l^T,
$$

where $e_k e_k^T$ and $e_l e_l^T$ means that grounded dampers are on masses m_k and m_l and $(e_j - e_{j+1})(e_j - e_{j+1})^T$ means that one damper is between masses m_j and m_{j+1}. For $k = 1, j = 1, l = n-1$ we have configuration as it is given by Figure 2.14.

All computations have been performed on the machine Intel Core i7-6500 CPU with 8GB of RAM with MATLAB R2018a on Windows 10. We don't show the difference between Algorithm 5 and algorithm from [47], since latter algorithm didn't converge all the time.

Example 2.5 We consider the mechanical system shown in Figure 2.14 for dimensions $n = 50 \cdot i, i = 1, \ldots, 20$, $k_i = 0.01, i = 1, \ldots, n+1$ and $m_i = 10 + \frac{990}{n-1} \cdot (i-1), i = 1, \ldots, n$, and in internal damping given by (1.20), $\alpha_c = 0.002$. To show robustness of our approach we tested it on a random viscosity $\mathbf{v}_r = [v_1^r, v_2^r, v_3^r]^T$, where $v_1^r, v_2^r, v_3^r \in [0.1, 1.1]$ for two different configurations of damper's positions.

$$
\begin{aligned}
\text{Configuration A} : (k, j, l) &= \left(\frac{n}{10}, \frac{3n}{10}, \frac{5n}{10} \right), \\
\text{Configuration B} : (k, j, l) &= \left(\frac{5n}{10}, \frac{7n}{10}, \frac{9n}{10} \right),
\end{aligned}
\tag{2.83}
$$

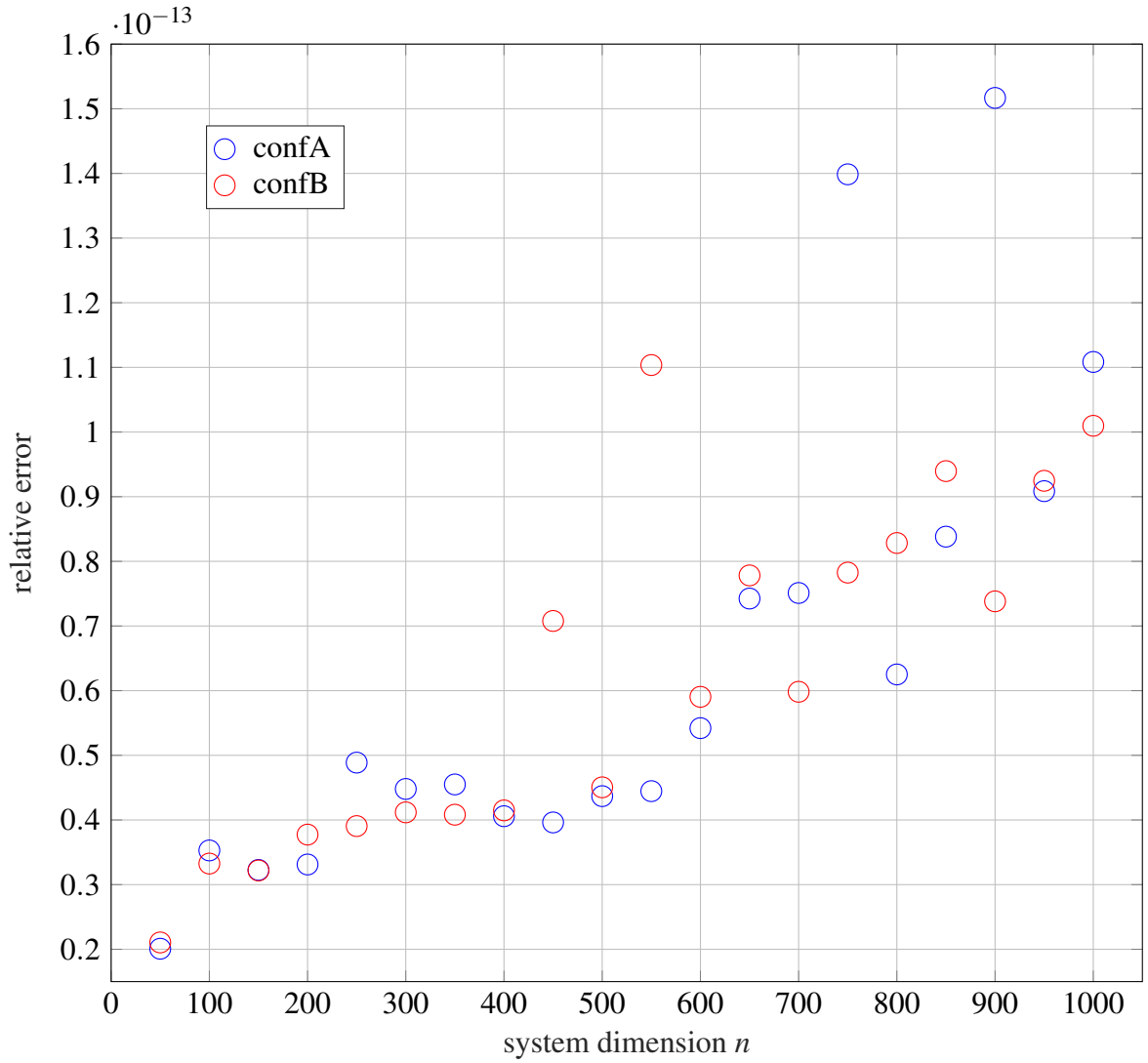

Figure 2.15: Average relative error in eigenvalues for different system dimension

In order to show how accurate Algorithm 5 is, Figure 2.15 shows average relative error in eigenvalues for two different configurations (2.83). Relative errors are computed using

$$\frac{|\lambda_j^p(\mathbf{v}) - \lambda_j(\mathbf{v})|}{|\lambda_j^p(\mathbf{v})|}, \qquad j = 1, \ldots, 2n,$$

where $\lambda_j(\mathbf{v})$ are eigenvalues computed with Algorithm 5 and $\lambda_j^p(\mathbf{v})$ are eigenvalues computed with MATLAB's function `polyeig` (for more see [97], [96], [28]). The average is taken over all $j = 1, \ldots, 2n$. One can notice a slight increase in relative error as dimension increases, but it is still around 10^{-13} which gives good accuracy.

Figure 2.16 shows run times for computation of eigenvalues and eigenvectors with Algorithm 5 and MATLAB's function `polyeig` and `quadeig` (more details on `quadeig` in [41]), for 2 different positions of dampers, i.e., for fixed configuration of positions of dampers, we increased system dimension and recorded run-times needed to run Algorithm 5, and run-times needed to

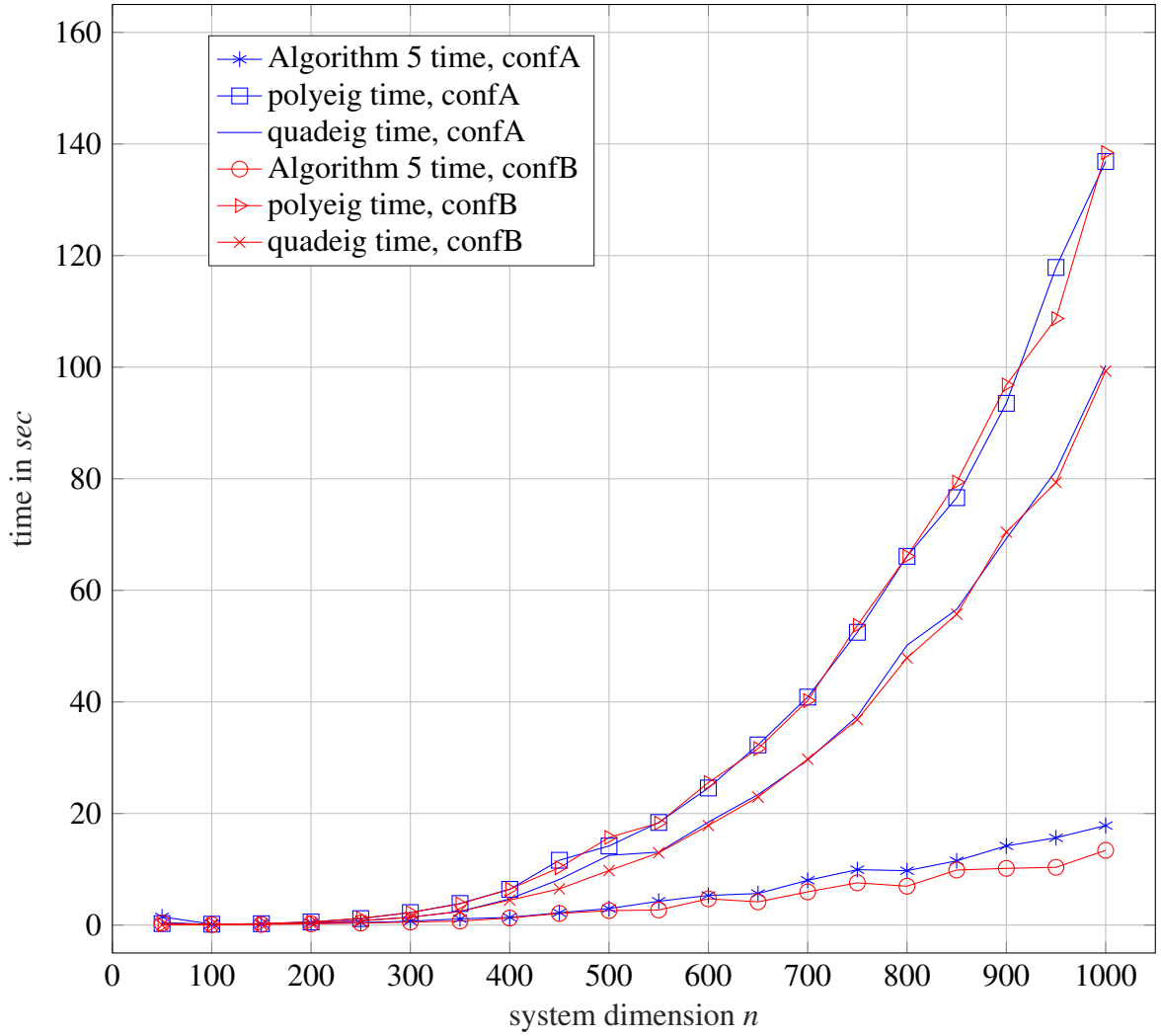

Figure 2.16: Computation time depending on system dimension

run `quadeig` and `polyeig`.

From Figure 2.16 one can see that for lower dimension times are almost the same, but with the increase of dimension the Algorithm 5 is much faster then `polyeig` and `quadeig`. Overall, Figure 2.16 verifies our new eigenvalue computation is much faster than alternative eigensolvers, since it shows a significant gain even when we just solve a single problem. This is very important since we want to apply our approximation to damping optimization where we repetitively compute eigenvalues and eigenvectors to determine which parameter is the optimal one, and the speed up is more significant.

2.4 Conclusion

In this chapter, we have shown how one can efficiently approximate the whole or just one important part of the spectrum of the parameter dependent quadratic eigenvalue problem, for a variety of viscosity parameters $\mathbf{v} = [v_1, \ldots, v_s]^T$, $v_i \in \mathbb{R}_+$. Furthermore, we have shown how one

can efficiently approximate eigenvalues when the change in parameter is small. Also we have shown how one can exploit the structure of the damping matrix for efficient computation of the approximations of the eigenvalues.

Our approach from Section 2.1 generalizes the results from [75] and [112], which are given for $v_i \ll 1$, on the v_i of the modest magnitude. Moreover, even for the case when $v_i \ll 1$, our approximations are more accurate than those in [75] and [112]. At the same time, for all three approximation approaches we have derived the corresponding error bounds. The quality of the error bounds as well as the accuracy of the achieved eigenvalues was illustrated in several numerical experiments.

Approximations from Section 2.2 are appropriate in case when we have a small change in parameter $\mathbf{p}$ (perturbation in parameter $\mathbf{p}$). These approximations and the corresponding error bounds are applied in the next chapter, which is related to perturbation theory.

The approach for computation of eigenvalues from Section 2.3 exploits the structure of damping matrix and it is shown in numerical experiment that this approach is very efficient and applicable in damping optimization, which will be shown in Chapter 4.

CHAPTER 3

Perturbation bounds of PQEP

In this chapter we present results related to perturbation theory of PQEP (1.1). Perturbation theory arises when it comes to the sensitivity of a problem as well as the stability and robustness of numerical methods. Sensitivity of a problem to perturbations in data is measured by the conditioning number. A condition number of a problem quantifies the rate of change of the solution with respect to the input data. On the other hand the stability of a method for solving the problem is characterized by backward error. Backward error measures how far a problem has to be perturbed for an approximate solution to be an exact solution of the perturbed problem. The combination of backward error estimate and a condition number provides an approximate upper bound on the error in a computed solution, i.e.,

$$\text{error in solution} \lessapprox \text{condition number} \times \text{backward error.}$$

For a numerical method we say that it is robust if it does not produce a large perturbation in the solution for a very small perturbation in data.

Since we consider PQEP (1.1) where all matrices depend on parameters, we have the perturbation in parameter, $\mathbf{p} = \mathbf{p}^0 + \Delta\mathbf{p}$, where $\mathbf{p^0} = [\mathbf{v}_M^0; \mathbf{v}^0; \mathbf{v}_K^0]$ is parameter of unperturbed system and $\mathbf{p} = [\mathbf{v}_M; \mathbf{v}; \mathbf{v}_K]$ is parameter of perturbed system and $\Delta\mathbf{p}$ is the perturbation of the parameter.

Thus, mass, damping and stiffness matrices of the perturbed system are $M(\mathbf{p}) = M(\mathbf{v}_M) = M(\mathbf{v}_M^0) + \Delta M$, $C(\mathbf{p}) = C(\mathbf{v}) = C(\mathbf{v}^0) + \Delta C$, and $K(\mathbf{p}) = K(\mathbf{v}_K) = K(\mathbf{v}_K^0) + \Delta K$, respectively, where $\Delta M, \Delta C, \Delta K$ are corresponding perturbation matrices and $M(\mathbf{p}^0) = M(\mathbf{v}_M^0)$, $C(\mathbf{p}^0) = C(\mathbf{v}^0)$, $K(\mathbf{p}^0) = K(\mathbf{v}_K^0)$ are matrices of unperturbed system.

Thus, the PQEP of an unperturbed system is given by:

$$(\lambda^2(\mathbf{p}^0)M(\mathbf{p}^0) + \lambda(\mathbf{p}^0)C(\mathbf{p}^0) + K(\mathbf{p}^0))x(\mathbf{p}^0) = 0, \quad x(\mathbf{p}^0) \neq 0, \tag{3.1}$$

while

$$(\lambda^2(\mathbf{p})M(\mathbf{p}) + \lambda(\mathbf{p})C(\mathbf{p}) + K(\mathbf{p}))x(\mathbf{p}) = 0, \quad x(\mathbf{p}) \neq 0 \tag{3.2}$$

is the PQEP of the perturbed system. One can see that first order approximations of eigenval-

ues, $\widetilde{\lambda}(\mathbf{p})$ given in Section 2.2 can be taken as approximations of the eigenvalues $\lambda(\mathbf{p})$ of the perturbed PQEP.

Usually in perturbation theory of the quadratic eigenvalue problem the following is considered:

$$((\lambda+\Delta\lambda)^2(M+\Delta M)+(\lambda+\Delta\lambda)(C+\Delta C)+(K+\Delta K))(x+\Delta x) \;=\; 0, \tag{3.3}$$

where $(x+\Delta x)\neq 0$ and perturbation matrices are ΔM, ΔC, ΔK, see [97]. The following backward error and eigenvalue condition number are given in their original form as in [97, 43]. Furthermore, let us consider an approximate eigenpair $(\widetilde{\lambda},\widetilde{x})$ of

$$(\lambda^2 M+\lambda C+K)x=0, \quad x\neq 0. \tag{3.4}$$

We can interpret $(\widetilde{\lambda},\widetilde{x})$ as an approximation of an exact eigenpair of perturbed quadratic eigenvalue problem (3.3), where choice of perturbations ΔM, ΔC, ΔK is not unique. Backward error of eigenpair $(\widetilde{\lambda},\widetilde{x})$ is defined as the size of the smallest perturbations given in a following way:

$$\begin{aligned}\eta(\widetilde{\lambda},\widetilde{x}) &= \min\{\varepsilon\colon (\widetilde{\lambda}^2(\Delta M+M)+\lambda(\Delta C+C)+(\Delta K+K))\widetilde{x}=0,\ \widetilde{x}\neq 0,\\ &\qquad \|\Delta M\|\leq\varepsilon\|M\|,\ \|\Delta C\|\leq\varepsilon\|C\|,\ \|\Delta K\|\leq\varepsilon\|K\|\}\\ &= \frac{\|(\widetilde{\lambda}^2M+\widetilde{\lambda}C+K)\widetilde{x}\|}{(|\widetilde{\lambda}|^2\|M\|+|\widetilde{\lambda}|\|C\|+\|K\|)\|\widetilde{x}\|},\end{aligned} \tag{3.5}$$

while eigenvalue condition number is given by:

$$\begin{aligned}\kappa(\lambda) &= \limsup_{\varepsilon\to 0}\left\{\frac{|\Delta\lambda|}{\varepsilon|\lambda|}\colon \text{(3.3) holds }, \|\Delta M\|\leq\varepsilon\|M\|,\ \|\Delta C\|\leq\varepsilon\|C\|,\ \|\Delta K\|\leq\varepsilon\|K\|\right\}\\ &= \frac{(|\lambda|^2\|M\|+\lambda\|C\|+\|K\|)\|y\|\|x\|}{|\lambda||y^*(2\lambda M+C)x|}.\end{aligned} \tag{3.6}$$

Furthermore, proofs of equalities in (3.5) and (3.6) can be found in [96]. An overview on perturbation theory for the quadratic eigenvalue problem is presented in [97], [91] and for hyperbolic quadratic eigenvalue problem is presented in [64]. The perturbation theory for the quadratic eigenvalue problem has been considered in [64], [45], while the relative perturbation theory has been studied in [9], [101], [102]. Moreover, the perturbation theory for the polynomial and nonlinear eigenvalue problems have been considered in [71], [42], [14]. In [14] authors provide a first order approximation of a perturbed eigenvalue $\lambda+\Delta\lambda$ when the eigenvalue is simple which was compared with our first order approximation (2.44) in Subsection 2.2.1.

Further on, we continue with our setting, where perturbed quantities depend on $\mathbf{p}$ and unperturbed quantities depend on $\mathbf{p}^0$. In [9], [101], [102] authors measure the relative error in

eigenvalue, which can be defined as

$$\frac{|\lambda(\mathbf{p})-\lambda(\mathbf{p}^0)|}{|\lambda(\mathbf{p}^0)|} \quad \text{or} \quad \frac{|\lambda(\mathbf{p})-\lambda(\mathbf{p}^0)|}{|\lambda(\mathbf{p})|} \tag{3.7}$$

and also the perturbation of invariant subspaces. For the perturbation of invariant subspaces we need to define the distance between two subspaces (see [93], [37]). Let $\mathcal{X}(\mathbf{p})$ and $\mathcal{Y}(\mathbf{p})$ be subspaces of $\mathbb{C}^n$ and that $\dim(\mathcal{X}(\mathbf{p})) = \dim(\mathcal{Y}(\mathbf{p})) = l$. We define distance between these two spaces by

$$\operatorname{dist}(\mathcal{X}(\mathbf{p}),\mathcal{Y}(\mathbf{p})) = \|P_{\mathcal{X}(\mathbf{p})} - P_{\mathcal{Y}(\mathbf{p})}\|, \tag{3.8}$$

where $P_{\mathcal{X}(\mathbf{p})}, P_{\mathcal{Y}(\mathbf{p})}$ are orthogonal projection onto $\mathcal{X}(\mathbf{p}), \mathcal{Y}(\mathbf{p})$, respectively. If we denote $\Theta(\mathbf{p}) = \operatorname{diag}(\theta_1(\mathbf{p}),\ldots,\theta_l(\mathbf{p}))$, where $\theta_i(\mathbf{p})$, $i = 1,\ldots,l$, are canonical angles between $\mathcal{X}(\mathbf{p})$ and $\mathcal{Y}(\mathbf{p})$, then

$$\|P_{\mathcal{X}(\mathbf{p})} - P_{\mathcal{Y}(\mathbf{p})}\| = \sin\theta_1(\mathbf{p}) = \|\sin\Theta(\mathbf{p})\| = \sin\sphericalangle(X(\mathbf{p}),Y(\mathbf{p})). \tag{3.9}$$

The greatest canonical angle, θ_1, is defined by:

$$\cos\theta_1(\mathbf{p}) = \min_{\substack{x(\mathbf{p})\in\mathcal{X}(\mathbf{p}),\\ x(\mathbf{p})\neq 0}} \max_{\substack{y(\mathbf{p})\in\mathcal{Y}(\mathbf{p}),\\ y(\mathbf{p})\neq 0}} \frac{y(\mathbf{p})^* x(\mathbf{p})}{\|x(\mathbf{p})\|\|y(\mathbf{p})\|}. \tag{3.10}$$

Very often the perturbation bound for eigenvalues (3.7), and also the perturbation bound for invariant subspaces (3.9) depend on perturbed (depending on $\mathbf{p}$) as well as unperturbed (depending on $\mathbf{p}^0$) quantities, making them difficult to use even for

$$\|\Delta\mathbf{p}\| = \|\mathbf{p}-\mathbf{p}^0\| \ll \varepsilon_{\mathbf{p}}, \tag{3.11}$$

where $\varepsilon_{\mathbf{p}}$ is given tolerance, since the quantities of the perturbed system are suppose to be unknown. This dependence on perturbed parameter $\mathbf{p}$ is usually in a gap function. The perturbation bound for invariant eigensubspaces $X(\mathbf{p}), X(\mathbf{p}^0)$ is usually given in a following form

$$\sin\sphericalangle(X(\mathbf{p}),X(\mathbf{p}^0)) \leq \frac{\texttt{residual}}{\texttt{gap}}$$

as in [72], [21], where authors consider residual of certain linearization of the PQEP (1.1) as the `residual` and the measures of the separation between eigenvalues is denoted as `gap`. In a similar way there are gap functions in the denominator of the bounds given in [101], [102] which will be shown later in this chapter.

This chapter is organized as follows. Section 3.1 presents new type of perturbation bounds between individual unperturbed and perturbed eigenvectors of PQEP (1.1).

To overcome the problem of the dependence on perturbed parameter $\mathbf{p}$ in gap functions, in Subsection 3.1.1 we present a set of bounds for different gap functions, where only unperturbed quantities are incorporated in the bounds. Results from Subsection 3.1.1 are based on the triangular inequality and the first order approximation of involved perturbed eigenvalues, from Section 2.2 given by (2.44) and the corresponding error bound given by (2.45).

The same approach can be used for efficient computation of bounds for gap functions that occur in perturbation bounds from [102] and [101], which is shown in Section 3.2.3.

3.1 New eigenvector perturbation bounds

In this section we present a new type of perturbation bounds between individual unperturbed and perturbed eigenvector of PQEP (1.1) and also upper bounds for gap functions that occur in this bound. We then apply our new approach regarding the gap functions to existing bounds given in [102] and [101], in order to improve efficiency of their calculation.

First, we discuss a certain residual type of bound for the unperturbed and perturbed eigenvectors respectively from PQEPs (3.1) and (3.2).

For the purpose of simplifying notation, $x(\mathbf{p}^0)$ is denoted by x and it represents an eigenvector from the unperturbed system given by (3.1) and $x(\mathbf{p})$ is denoted by $\widehat{x}$ and it represents an eigenvector from the perturbed system given by (3.2), while the perturbed eigenvalues of $\lambda(\mathbf{p}^0)$ are denoted by $\lambda(\mathbf{p})$.

Let $(\lambda_i(\mathbf{p}^0), x_i)$ and $(\lambda_j(\mathbf{p}^0), y_j)$ be the i-th right and the j-th left eigenpair of the unperturbed PQEP (3.1), respectively, for $i, j = 1, \ldots, 2n$.

Since, the x_i and y_j are right and left eigenvectors, respectively, the following equalities hold

$$\lambda_i^2(\mathbf{p}^0)M(\mathbf{p}^0)x_i + \lambda_i(\mathbf{p}^0)C(\mathbf{p}^0)x_i + K(\mathbf{p}^0)x_i = 0, \tag{3.12}$$

$$\lambda_j^2(\mathbf{p}^0)y_j^*M(\mathbf{p}^0) + \lambda_j(\mathbf{p}^0)y_j^*C(\mathbf{p}^0) + y_j^*K(\mathbf{p}^0) = 0. \tag{3.13}$$

Now multiplying the first equality from the left with y_j^* and the second equality from the right with x_i and subtracting them, we obtain

$$(\lambda_i^2(\mathbf{p}^0) - \lambda_j^2(\mathbf{p}^0))y_j^*M(\mathbf{p}^0)x_i + (\lambda_i(\mathbf{p}^0) - \lambda_j(\mathbf{p}^0))y_j^*C(\mathbf{p}^0)x_i = 0\,.$$

If

$$\lambda_i(\mathbf{p}^0) \neq \pm\lambda_j(\mathbf{p}^0)\,,$$

then the following equality holds

$$y_j^*\left(M(\mathbf{p}^0) + \frac{C(\mathbf{p}^0)}{\lambda_i(\mathbf{p}^0) + \lambda_j(\mathbf{p}^0)}\right)x_i = 0, \tag{3.14}$$

for all $i \neq j$.

Remark 3.1. Note that in a similar way as we obtained (3.14) from (3.12) and (3.13), instead of annihilating $K(\mathbf{p}^0)$ we can annihilate $M(\mathbf{p}^0)$ or $C(\mathbf{p}^0)$ and obtain

$$\begin{aligned} y_j^* \left(M(\mathbf{p}^0) - \frac{K(\mathbf{p}^0)}{\lambda_i(\mathbf{p}^0)\lambda_j(\mathbf{p}^0)} \right) x_i &= 0, \\ y_j^* \left(C(\mathbf{p}^0) + \frac{K(\mathbf{p}^0)}{\frac{\lambda_i(\mathbf{p}^0)+\lambda_j(\mathbf{p}^0)}{\lambda_i(\mathbf{p}^0)\lambda_j(\mathbf{p}^0)}} \right) x_i &= 0, \end{aligned}$$

respectively, under additional assumption that all $\lambda_i(\mathbf{p}^0) \neq 0, i = 1, \ldots, 2n$.

Let us denote

$$T_{ij} = \frac{C(\mathbf{p}^0)}{\lambda_i(\mathbf{p}) + \lambda_j(\mathbf{p}^0)}, \qquad T_{ij}^0 = \frac{C(\mathbf{p}^0)}{\lambda_i(\mathbf{p}^0) + \lambda_j(\mathbf{p}^0)}.$$

Theorem 3.1 For the PQEP given by (3.1) with simple eigenvalues $\lambda_j(\mathbf{p}^0)$ with corresponding right eigenvectors x_i and left eigenvectors y_j, let (3.2) correspond to the perturbed PQEP with simple eigenvalues $\lambda_i(\mathbf{p})$ with corresponding right eigenvectors $\widehat{x_i}$ such that $\|x_i\| = \|\widehat{x_i}\| = 1$ for all $i, j \in \{1, \ldots, 2n\}$, where $\lambda_i(\mathbf{p}) \neq \pm\lambda_j(\mathbf{p}^0)$ and $\lambda_i(\mathbf{p}^0) \neq \pm\lambda_j(\mathbf{p}^0)$. Denote

$$\gamma_{ij} = \frac{\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta\mathbf{p} + \frac{1}{2}(\mathbf{p}-\mathbf{p}^0)^T \mathcal{H}(\xi)(\mathbf{p}-\mathbf{p}^0)}{\lambda_i(\mathbf{p}^0) + \lambda_j(\mathbf{p}^0)} \tag{3.15}$$

where $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}^0$, $\xi = \lambda_k(\mathbf{p}^0 + t_1\Delta\mathbf{p})$, $t_1 \in [0,1]$ and assume $|\gamma_{ij}| < 1$. Then, for perturbation $\delta = \max\{\|\Delta\mathbf{p}\|, \|\Delta M\|, \|\Delta C\|, \|\Delta K\|\}$, small enough the following inequality holds,

$$\begin{aligned} \left| y_j^* \left(M(\mathbf{p}^0) + T_{ij}^0 \right) \widehat{x_i} \right| \lessapprox & \left| y_j^* \left(T_{ij}^0 \frac{\mathcal{J}_i(\mathbf{p}^0)\Delta\mathbf{p}}{\lambda_i(\mathbf{p}^0) + \lambda_j(\mathbf{p}^0)} \right) x_i \right| + \frac{\|y_j^* \Delta M x_i\|}{\frac{|\lambda_j^2(\mathbf{p}^0) - \lambda_i{}^2(\mathbf{p})|}{|\lambda_i{}^2(\mathbf{p})|}} \\ & + \frac{\|y_j^* \Delta C x_i\|}{\frac{|\lambda_j^2(\mathbf{p}^0) - \lambda_i{}^2(\mathbf{p})|}{|\lambda_i(\mathbf{p})|}} + \frac{\|y_j^* \Delta K x_i\|}{|\lambda_j^2(\mathbf{p}^0) - \lambda_i{}^2(\mathbf{p})|}, \quad \text{for } i \neq j, \end{aligned} \tag{3.16}$$

where $\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta\mathbf{p} = \sum_{k=1}^{m} \frac{\partial\lambda_i}{\partial p_k}(\mathbf{p}^0)(p_k - p_k^0)$ and $\Delta M = M(\mathbf{p}) - M(\mathbf{p}^0)$, $\Delta C = C(\mathbf{p}) - C(\mathbf{p}^0)$, $\Delta K = K(\mathbf{p}) - K(\mathbf{p}^0)$.

Proof. The following equalities hold:

$$\begin{aligned} \lambda_i^2(\mathbf{p}) M(\mathbf{p}) \widehat{x_i} + \lambda_i(\mathbf{p}) C(\mathbf{p}) \widehat{x_i} + K(\mathbf{p}) \widehat{x_i} = 0, \\ \lambda_j^2(\mathbf{p}^0) y_j^* M(\mathbf{p}^0) + \lambda_j(\mathbf{p}^0) y_j^* C(\mathbf{p}^0) + y_j^* K(\mathbf{p}^0) = 0. \end{aligned}$$

Now, by multiplying the first equality from the left with y_j^* and the second from the right with

$\widehat{x_i}$, one gets

$$(\lambda_i^2(\mathbf{p})-\lambda_j^2(\mathbf{p}^0))y_j^*M(\mathbf{p}^0)\widehat{x_i}+(\lambda_i(\mathbf{p})-\lambda_j(\mathbf{p}^0))y_j^*C(\mathbf{p}^0)\widehat{x_i}= \\ -\lambda_i^2(\mathbf{p})y_j^*(M(\mathbf{p})-M(\mathbf{p}^0))\widehat{x_i}-\lambda_i(\mathbf{p})y_j^*(C(\mathbf{p})-C(\mathbf{p}^0))\widehat{x_i}-y_j^*(K(\mathbf{p})-K(\mathbf{p}^0))\widehat{x_i}.$$

Therefore we obtain the following equality

$$y_j^*\left(M(\mathbf{p}^0)+T_{ij}\right)\widehat{x_i}=\frac{y_j^*\Delta M\widehat{x_i}}{\frac{\lambda_j^2(\mathbf{p}^0)-{\lambda_i}^2(\mathbf{p})}{{\lambda_i}^2(\mathbf{p})}}+\frac{y_j^*\Delta C\widehat{x_i}}{\frac{\lambda_j^2(\mathbf{p}^0)-{\lambda_i}^2(\mathbf{p})}{\lambda_i(\mathbf{p})}}+\frac{y_j^*\Delta K\widehat{x_i}}{\lambda_j^2(\mathbf{p}^0)-{\lambda_i}^2(\mathbf{p})}. \tag{3.17}$$

Now, if we use first order approximation for $\lambda_i(\mathbf{p})$ given by (2.42) we obtain

$$\begin{aligned} T_{ij} &= \frac{C(\mathbf{p}^0)}{\lambda_i(\mathbf{p}^0)+\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta\mathbf{p}+\frac{1}{2}(\mathbf{p}-\mathbf{p}^0)^T\mathcal{H}(\xi)(\mathbf{p}-\mathbf{p}^0)+\lambda_j(\mathbf{p}^0)} \\ &= \frac{C(\mathbf{p}^0)}{(\lambda_i(\mathbf{p}^0)+\lambda_j(\mathbf{p}^0))\left(1+\frac{\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta\mathbf{p}+\frac{1}{2}(\mathbf{p}-\mathbf{p}^0)^T\mathcal{H}(\xi)(\mathbf{p}-\mathbf{p}^0)}{\lambda_i(\mathbf{p}^0)+\lambda_j(\mathbf{p}^0)}\right)} \\ &= T_{ij}^0\frac{1}{1+\gamma_{ij}}. \end{aligned} \tag{3.18}$$

Since $|\gamma_{ij}|<1$,

$$\begin{aligned} T_{ij} &= T_{ij}^0+T_{ij}^0\sum_{k=1}^{\infty}(-\gamma_{ij})^k \\ &= T_{ij}^0-T_{ij}^0\frac{\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta\mathbf{p}}{\lambda_i(\mathbf{p}^0)+\lambda_j(\mathbf{p}^0)}+\mathcal{O}(\|\Delta\mathbf{p}\|^2). \end{aligned} \tag{3.19}$$

Now, by applying (3.19) in (3.17), we obtain

$$\begin{aligned} y_j^*\left(M(\mathbf{p}^0)+T_{ij}^0\right)\widehat{x_i}=y_j^*\left(T_{ij}^0\frac{\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta\mathbf{p}}{\lambda_i(\mathbf{p}^0)+\lambda_j(\mathbf{p}^0)}\right)\widehat{x_i}+\frac{y_j^*\Delta M\widehat{x_i}}{\frac{\lambda_j^2(\mathbf{p}^0)-{\lambda_i}^2(\mathbf{p})}{{\lambda_i}^2(\mathbf{p})}} \\ +\frac{y_j^*\Delta C\widehat{x_i}}{\frac{\lambda_j^2(\mathbf{p}^0)-{\lambda_i}^2(\mathbf{p})}{\lambda_i(\mathbf{p})}}+\frac{y_j^*\Delta K\widehat{x_i}}{\lambda_j^2(\mathbf{p}^0)-{\lambda_i}^2(\mathbf{p})}+\mathcal{O}(\|\Delta\mathbf{p}\|^2). \end{aligned} \tag{3.20}$$

Considering (2.43) let

$$\widehat{x_i}=x_i+\Delta x_i \tag{3.21}$$

where $\|\Delta x_i\| = \mathcal{O}(\|\Delta \mathbf{p}\|^2)$. If we substitute (3.21) in (3.20) we obtain

$$\begin{aligned} y_j^* \left(M(\mathbf{p}^0) + T_{ij}^0\right) \widehat{x}_i = y_j^* \left(T_{ij}^0 \frac{\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta \mathbf{p}}{\lambda_i(\mathbf{p}^0) + \lambda_j(\mathbf{p}^0)} \right) x_i + \frac{y_j^* \Delta M x_i}{\frac{\lambda_j^2(\mathbf{p}^0) - {\lambda_i}^2(\mathbf{p})}{\lambda_i(\mathbf{p})}} \\ + \frac{y_j^* \Delta C x_i}{\frac{\lambda_j^2(\mathbf{p}^0) - {\lambda_i}^2(\mathbf{p})}{\lambda_i(\mathbf{p})}} + \frac{y_j^* \Delta K x_i}{\lambda_j^2(\mathbf{p}^0) - {\lambda_i}^2(\mathbf{p})} + \mathcal{O}(\delta^2). \end{aligned} \tag{3.22}$$

Now, for δ small enough by taking the norm we obtain (3.16). □

Furthermore, let the matrix $X = [x_{i_1}, \ldots, x_{i_q}]$ ($Y = [y_{j_1}, \ldots, y_{j_p}]$) contain linearly independent right (left) eigenvectors of PQEP (3.2) which correspond to eigenvalues $\lambda_{i_1}(\mathbf{p}^0), \ldots, \lambda_{i_q}(\mathbf{p}^0)$, ($\lambda_{j_1}(\mathbf{p}^0), \ldots, \lambda_{j_p}(\mathbf{p}^0)$). We assume that $\operatorname{rank} X = p$ ($\operatorname{rank} Y = q$), i.e., matrix X has the full column rank. Suppose $i_k \neq j_l$ for $k = 1, \ldots, q$, $l = 1, \ldots, p$.

By bounding the denominators on the right-hand side of (3.16) we have the following corollary.

Corollary 3.2 For chosen $i \in \{i_1, \ldots, i_p\}$, $j \in \{j_1, \ldots, j_q\}$, let assumptions from Lemma 3.1 hold. The following inequality holds

$$\begin{aligned} \left| y_j^* \left(M(\mathbf{p}^0) + T_{ij}^0\right) \widehat{x}_i \right| \lessapprox \left\| y_j^* \left(T_{ij}^0 \frac{\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta \mathbf{p}}{\lambda_i(\mathbf{p}^0) + \lambda_j(\mathbf{p}^0)} \right) x_i \right\| \\ + \frac{\|y_j^* \Delta M x_i\|}{\mathrm{g}_1} + \frac{\|y_j^* \Delta C x_i\|}{\mathrm{g}_2} + \frac{\|y_j^* \Delta K x_i\|}{\mathrm{g}_3}, \end{aligned} \tag{3.23}$$

where

$$\mathrm{g}_1 = \min_{\substack{i=i_1,\ldots,i_p \\ j=j_1,\ldots,j_q \\ i \neq j}} \frac{|\lambda_j^2(\mathbf{p}^0) - {\lambda_i}^2(\mathbf{p})|}{|\lambda_i^2(\mathbf{p})|}, \tag{3.24}$$

$$\mathrm{g}_2 = \min_{\substack{i=i_1,\ldots,i_p \\ j=j_1,\ldots,j_q \\ i \neq j}} \frac{|\lambda_j^2(\mathbf{p}^0) - {\lambda_i}^2(\mathbf{p})|}{|\lambda_i(\mathbf{p})|}, \tag{3.25}$$

$$\mathrm{g}_3 = \min_{\substack{i=i_1,\ldots,i_p \\ j=j_1,\ldots,j_q \\ i \neq j}} |\lambda_j^2(\mathbf{p}^0) - {\lambda_i}^2(\mathbf{p})|. \tag{3.26}$$

Proof. The proof follows directly from Theorem 3.1. □

Note that in bounds (3.16) we apply a first order approximation on perturbed eigenvector, while the right hand side still possesses perturbed and unperturbed eigenvalues. Also, the right hand side in bound (3.23) possesses perturbed and unperturbed eigenvalues in the gap functions (3.24), (3.25), (3.26). Our result in the next subsection demonstrates how one can avoid using perturbed eigenvalues in bounds, i.e., we derive a new bound that depends only on unperturbed eigenvalues for small enough perturbation.

3.1.1 Application of first order approximation bounds

In this subsection we present a lower bound for gap functions, that contain unperturbed eigenvalues and approximation of perturbed eigenvalues given in Corollary 2.8 by (2.44) and corresponding error bound given by (2.45).

As we already mentioned $\lambda(\mathbf{p})$ represents an eigenvalue of the perturbed PQEP, then if we denote the approximation of an eigenvalue $\lambda(\mathbf{p})$ with $\widetilde{\lambda}(\mathbf{p})$, then the following corollary holds:

Corollary 3.3 Let all assumptions from Theorem 3.1 and Lemma 2.5 hold and let $\widetilde{\lambda}_i(\mathbf{p})$ be first order approximation of eigenvalue $\lambda_i(\mathbf{p})$ given by (2.44) with corresponding error bound (2.45). Furthermore, let the perturbation δ be small enough such that $|\lambda_j^2(\mathbf{p}^0) - \widetilde{\lambda}_i^2(\mathbf{p})| - M_i^{(2)} \geq 0$ for all $i = i_1, \ldots, i_p$, $j = j_1, \ldots, j_q, i \neq j$.

Then the following inequalities hold

$$\mathrm{g}_1 \gtrapprox \min_{\substack{i=i_1,\ldots,i_p \\ j=j_1,\ldots,j_q \\ i\neq j}} \frac{|\lambda_j^2(\mathbf{p}^0) - \widetilde{\lambda}_i^2(\mathbf{p})| - M_i^{(2)}}{|\widetilde{\lambda}_i^2(\mathbf{p})| + M_i^{(2)}}, \tag{3.27}$$

$$\mathrm{g}_2 \gtrapprox \min_{\substack{i=i_1,\ldots,i_p \\ j=j_1,\ldots,j_q \\ i\neq j}} \frac{|\lambda_j^2(\mathbf{p}^0) - \widetilde{\lambda}_i^2(\mathbf{p})| - M_i^{(2)}}{|\widetilde{\lambda}_i(\mathbf{p})| + M_i^{(1)}}, \tag{3.28}$$

$$\mathrm{g}_3 \gtrapprox \min_{\substack{i=i_1,\ldots,i_p \\ j=j_1,\ldots,j_q \\ i\neq j}} (|\lambda_j^2(\mathbf{p}^0) - \widetilde{\lambda}_i^2(\mathbf{p})| - M_i^{(2)}). \tag{3.29}$$

where $M_i^{(1)} = \frac{1}{2} M_i \|\mathbf{p} - \mathbf{p}^0\|^2$ and $M_i^{(2)} = M_i(M_i + 2|\widetilde{\lambda}_i(\mathbf{p})|)$.

Proof. By applying the triangle inequality we get the following inequalities:

$$\begin{aligned}
|\widetilde{\lambda}_i^2(\mathbf{p}) - \lambda_i{}^2(\mathbf{p})| &= |\widetilde{\lambda}_i(\mathbf{p}) - \lambda_i(\mathbf{p})||\widetilde{\lambda}_i(\mathbf{p}) + \lambda_i(\mathbf{p})| \\
&\lessapprox M_i^{(1)}(M_i^{(1)} + 2|\widetilde{\lambda}_i(\mathbf{p})|) = M_i^{(2)}, \\
|\lambda_i{}^2(\mathbf{p})| &\lessapprox M_i^{(2)} + |\widetilde{\lambda}_i^2(\mathbf{p})|, \\
|\lambda_i(\mathbf{p})| &\lessapprox M_i^{(1)} + |\widetilde{\lambda}_i(\mathbf{p})|, \\
|\lambda_j^2(\mathbf{p}^0) - \widetilde{\lambda}_i^2(\mathbf{p})| &\lessapprox |\lambda_j^2(\mathbf{p}^0) - \lambda_i{}^2(\mathbf{p})| + M_i^{(2)}.
\end{aligned}$$

Here, by applying the above inequalities we get the following:

$$\begin{aligned}
\frac{|\lambda_j^2(\mathbf{p}^0)-\lambda_i{}^2(\mathbf{p})|}{|\lambda_i{}^2(\mathbf{p})|} &\gtrapprox \frac{|\lambda_j^2(\mathbf{p}^0)-\widetilde{\lambda}_i^2(\mathbf{p})|-M_i^{(2)}}{|\widetilde{\lambda}_i^2(\mathbf{p})|+M_i^{(2)}},\\
\frac{|\lambda_j^2(\mathbf{p}^0)-\lambda_i{}^2(\mathbf{p})|}{|\lambda_i(\mathbf{p})|} &\gtrapprox \frac{|\lambda_j^2(\mathbf{p}^0)-\widetilde{\lambda}_i^2(\mathbf{p})|-M_i^{(2)}}{|\widetilde{\lambda}_i(\mathbf{p})|+M_i^{(1)}},\\
|\lambda_j^2(\mathbf{p}^0)-\lambda_i{}^2(\mathbf{p})| &\gtrapprox |\lambda_j^2(\mathbf{p}^0)-\widetilde{\lambda}_i^2(\mathbf{p})|-M_i^{(2)}.
\end{aligned} \tag{3.30}$$

By taking the minimum of both sides we obtain inequalities (3.27)-(3.29). □

Now we can present two upper bounds for the expression

$$\left|y_j^*\left(M(\mathbf{p}^0)+T_{ij}^0\right)\widehat{x}_i\right|$$

such that we can efficiently calculate them for different parameters $\mathbf{p}$. Moreover, in order to have an upper bound that can be efficiently implemented, in derived upper bounds (3.27)-(3.29) instead of using M_i we can use approximation $\widetilde{M}_i$ from (2.48). In the next corollaries we state an estimation of the upper bound.

Corollary 3.4 For chosen $i,j\in\{1,\ldots,2n\}$, $i\neq j$, let assumptions from Corollary 3.3 hold. Then the following inequality holds

$$\begin{aligned}
\left|y_j^*\left(M(\mathbf{p}^0)+T_{ij}^0\right)\widehat{x}_i\right| \lessapprox \left|\left|y_j^*\left(T_{ij}^0\frac{\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta\mathbf{p}}{\lambda_i(\mathbf{p}^0)+\lambda_j(\mathbf{p}^0)}\right)x_i\right|\right| + \frac{\|y_j^*\Delta M x_i\|}{\frac{|\widetilde{\lambda}_i^2(\mathbf{p})-\lambda_j^2(\mathbf{p}^0)|-M_i^{(2)}}{|\widetilde{\lambda}_i^2(\mathbf{p})|+M_i^{(2)}}}\\
+\frac{\|y_j^*\Delta C x_i\|}{\frac{|\widetilde{\lambda}_i^2(\mathbf{p})-\lambda_j^2(\mathbf{p}^0)|-M_i^{(2)}}{|\widetilde{\lambda}_i(\mathbf{p})|+M_i^{(1)}}}+\frac{\|y_j^*\Delta K x_i\|}{|\widetilde{\lambda}_i^2(\mathbf{p})-\lambda_j^2(\mathbf{p}^0)|-M_i^{(2)}},
\end{aligned} \tag{3.31}$$

where $M_i^{(1)}=\frac{1}{2}M_i\|(\mathbf{p}-\mathbf{p}^0)\|^2$ and $M_i^{(2)}=M_i^{(1)}(M_i^{(1)}+2|\widetilde{\lambda}_i(\mathbf{p})|)$.

Proof. Proof follows directly from application of inequalities (3.30) in Theorem 3.1. □

Corollary 3.5 For chosen $i,j\in\{1,\ldots,2n\}$, $i\neq j$, let assumptions from Corollary 3.3 hold. Then the following inequality holds

$$\begin{aligned}
\left|y_j^*\left(M(\mathbf{p}^0)+T_{ij}^0\right)\widehat{x}_i\right| \lessapprox \left|\left|y_j^*\left(T_{ij}^0\frac{\mathcal{J}(\lambda_i(\mathbf{p}^0))\Delta\mathbf{p}}{\lambda_i(\mathbf{p}^0)+\lambda_j(\mathbf{p}^0)}\right)x_i\right|\right|\\
+\frac{\|y_j^*\Delta M x_i\|}{\min\limits_{\substack{i=i_1,\ldots,i_p\\ j=j_1,\ldots,j_q\\ i\neq j}}\frac{|\widetilde{\lambda}_i^2(\mathbf{p})-\lambda_j^2(\mathbf{p}^0)|-M_i^{(2)}}{|\widetilde{\lambda}_i^2(\mathbf{p})|+M_i^{(2)}}}
\end{aligned}$$

$$+\frac{\|y_j^*\Delta C x_i\|}{\min\limits_{\substack{i=i_1,\ldots,i_p\\ j=j_1,\ldots,j_q\\ i\neq j}} \frac{|\widetilde{\lambda}_i^2(\mathbf{p})-\lambda_j^2(\mathbf{p}^0)|-M_i^{(2)}}{|\widetilde{\lambda}_i(\mathbf{p})|+M_i^{(1)}}} + \frac{\|y_j^*\Delta K x_i\|}{\min\limits_{\substack{i=i_1,\ldots,i_p\\ j=j_1,\ldots,j_q\\ i\neq j}} (|\widetilde{\lambda}_i^2(\mathbf{p})-\lambda_j^2(\mathbf{p}^0)|-M_i^{(2)})}, \tag{3.32}$$

where $M_i^{(1)} = \frac{1}{2}M_i\|(\mathbf{p}-\mathbf{p}^0)\|^2$ and $M_i^{(2)} = M_i^{(1)}(M_i^{(1)}+2|\widetilde{\lambda}_i(\mathbf{p})|)$.

Proof. Proof follows directly from application of the lower bounds from Corollary 3.3 and Corollary 3.2. □

The quality of these upper bounds will be illustrated with a the next section in numerical example.

3.1.2 Numerical experiments

In this example we show how accurate the bounds from (3.23) and (3.16) are. Since in bounds (3.32) and (3.31) we have $M_i, i = 1,\ldots,2n$, which is difficult to calculate because it implies multiple computation of new (perturbed) eigenvalues, as we have already mentioned in Section 2.2 we can use its approximation, $\widetilde{M}_i$. Moreover, we already shown in Example 2.4 that we can use the approximation $\widetilde{M}_i$ instead of M_i, when the perturbation is small enough.

Example 3.1 Consider the mechanical system shown in Figure 3.1 as in Example 2.4.

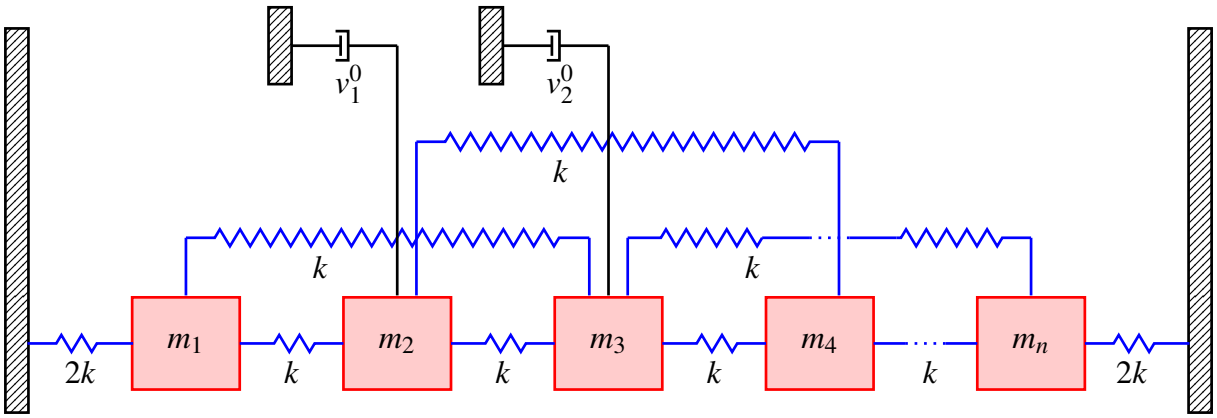

Figure 3.1: n-mass oscillator

Once again the dimension of the system is $n = 100$. In this example the mass matrix does not depend on the vector of parameters and it is given by

$$M = \operatorname{diag}(m_1,\ldots,m_n), \quad \text{where} \quad m_i = 2\cdot i, i = 1,\ldots,n.$$

The stiffness matrix is given by

$$K_0 = K(\mathbf{p}^0) \;\; = \;\; K + p_3^0 \cdot \widetilde{K}(\mathbf{p}^0), \tag{3.33}$$

where

$$K = \begin{bmatrix} 4k & -k & -k & & & & \\ -k & 4k & -k & -k & & & \\ -k & -k & 4k & -k & -k & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -k & -k & 4k & -k & -k \\ & & & -k & -k & 4k & -k \\ & & & & -k & -k & 4k \end{bmatrix}, \quad \widetilde{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \widetilde{K}_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{K}_1 = \widetilde{K}(30:55, 30:55) = \begin{bmatrix} 2k & 0 & -k & & \\ 0 & 2k & \ddots & \ddots & \\ -k & \ddots & \ddots & \ddots & -k \\ & \ddots & \ddots & \ddots & 0 \\ & & -k & 0 & 2k \end{bmatrix},$$

where $k = 0.1$ parameter dependant block of the stiffness matrix in our mechanical system is shown in Figure 2.11.

Finally, denote the damping matrix $C(\mathbf{p}^0) = C_{\text{int}} + C_{\text{ext}}(\mathbf{v}^0)$ where $\mathbf{p}^0 = [v_1^0, v_2^0, p_3^0]^T$, $\mathbf{v}^0 = [v_1^0, v_2^0]^T$, while internal damping $C_{\text{int}} = \alpha_c C_{\text{crit}}$, with $\alpha_c = 0.002$ and C_{crit} is given by (1.21). We put two grounded dampers on the 35th and the 50th mass with viscosities v_1^0, v_2^0, respectively, i.e.,

$$C_{\text{ext}}(\mathbf{p}^0) = v_1^0 e_{35} e_{35}^T + v_2^0 e_{50} e_{50}^T.$$

We will consider the following perturbed QEP:

$$(\lambda^2(\mathbf{p})M + C(\mathbf{p})\lambda(\mathbf{p}) + K(\mathbf{p}))x(\mathbf{p}) = 0. \tag{3.34}$$

In this example, we consider the case where $\mathbf{p} = (\frac{v}{4} + \delta, v + \delta, \delta)$ and $\mathbf{p}^0 = (\frac{v}{4}, v, 0)$.

Figure 3.2 shows the left-hand side (LHS) of inequality (3.16) for different combinations of y_j and $\widehat{x}_i$, together with corresponding right-hand sides (RHS) given in (3.23) and (3.16) as well as corresponding approximation bound of RHS, given in (3.32) and (3.31), for $v = 10$ and $\delta = \frac{0.01}{2^8}$. Here we have used approximated $\widetilde{M}_k$ instead of real M_k, from Corollary 2.8, for calculation of RHS in (3.32) and (3.31).

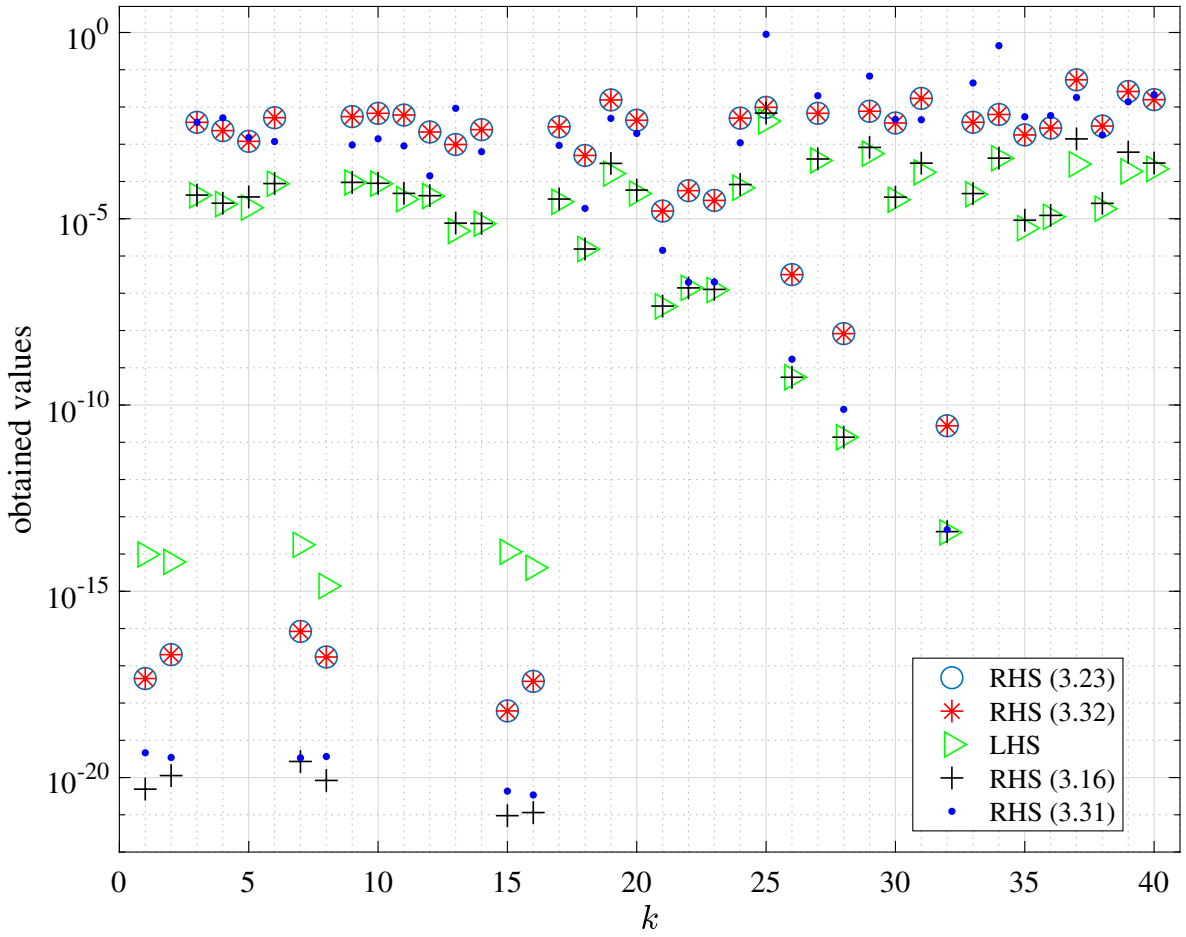

Figure 3.2: Left-hand side (LHS) and right-hand sides (RHS) of inequalities (3.23) and (3.16) and approximation bound of right-hand sides of inequalities (3.32) and (3.31) for different combinations of $(y_j, \widehat{x}_i)$, where $j = 4 \cdot k, i = k + 160$ and $k = 1, \ldots, 40$

As one can see from Figure 3.2, for all but six combinations of eigenvectors y_j and $\widehat{x}_i$, we obtained the satisfactory behaviour of the approximation bounds given by (3.32), (3.31) since those bounds follow the theoretical bounds given by (3.23), (3.16). For the other six combinations LHS is greater than RHS. The reasons are following:

1) we assumed that equalities (3.2) and (3.1) hold, but numerically the RHS of these equalities is equal to $\mathcal{O}(10^{-14})$, instead of 0,

2) in our bounds we neglect everything i.e., $\mathcal{O}(\delta^2)$, where $\delta = \|\Delta \mathbf{p}\|$.

Furthermore, in next section we apply the approach from Section 3.1.1 to bound gap functions that occur in already existing perturbation bounds from [101] and [102].

3.2 Estimation of existing bounds

In this section we show some applications of derived approximations. As we already mentioned, very often in perturbation analysis one can obtain perturbation bounds that depend on gap functions.

Thus, with derived bounds for gap functions one can improve computation efficiency of existing perturbation bounds by following our approach.

Typically, calculation of the perturbation bounds can be inefficient since the gap functions contain both eigenvalues of the unperturbed and perturbed problem. Therefore, we propose an approach for the calculation of the bounds for the gap functions using the unperturbed eigenvalues and the first order eigenvalue approximations of perturbed eigenvalues. This can significantly improve the calculation efficiency of perturbation bounds.

For the illustration of our approach we have chosen the bounds from [101] and [102], which have a more theoretical aspect (both unperturbed and perturbed eigenvalues exist in them).

In this section we present the perturbation bounds from [102, Theorem 2.1] and [101, Theorem 7]. The perturbation bound from [102] measures the difference between norms of two M-scalar products of unperturbed and perturbed eigenvectors (usually non M-orthogonal) of the PQEP (1.1). This bound is related to $\sin\Theta$ type theorems for the eigensubspaces for the considered quadratic eigenvalue problem, for more on this relation see [102].

The bound from [101] corresponds to the $\sin\Theta$ type theorems for the eigensubspaces for the considered quadratic hyperbolic eigenvalue problem.

3.2.1 Efficient estimation of the bounds for Hermitian QEP

Let $x_1, x_2, \ldots, x_n$ be n linearly independent right eigenvectors with the corresponding n eigenvalues $\lambda_1(\mathbf{p}^0), \lambda_2(\mathbf{p}^0), \ldots, \lambda_n(\mathbf{p}^0)$ of PQEP (1.1). In what follows we use the similar notation as in [102] Thus, let $x_1, \ldots, x_k$, be eigenvectors, which correspond to the eigenvalues $\lambda_1(\mathbf{p}^0), \ldots, \lambda_k(\mathbf{p}^0)$, i.e.,

$$\begin{aligned} X &= [X_1, X_2], \quad \text{with } X_1 = [x_1, \ldots, x_k], X_2 = [x_{k+1}, \ldots, x_n], \\ \Lambda(\mathbf{p}^0) &= \operatorname{diag}(\Lambda_1(\mathbf{p}^0), \Lambda_2(\mathbf{p}^0)), \quad \text{with} \\ &\quad \Lambda_1(\mathbf{p}^0) = \operatorname{diag}(\lambda_1(\mathbf{p}^0), \ldots, \lambda_k(\mathbf{p}^0)), \\ &\quad \Lambda_2(\mathbf{p}^0) = \operatorname{diag}(\lambda_{k+1}(\mathbf{p}^0), \ldots, \lambda_n(\mathbf{p}^0)). \end{aligned}$$

Corresponding perturbed eigenvectors are columns of $\widehat{X} = [\widehat{X}_1, \widehat{X}_2]$, and corresponding perturbed eigenvalues are diagonal entries of $\Lambda(\mathbf{p}) = \operatorname{diag}(\Lambda_1(\mathbf{p}), \Lambda_2(\mathbf{p}))$.

Under assumptions that $\overline{\lambda}_i^2(\mathbf{p}^0) - \lambda_j^2(\mathbf{p}) \neq 0$, $\lambda_j(\mathbf{p}^0) \neq 0$ and $\lambda_j(\mathbf{p}) \neq 0$, for all $j = 1, \ldots, k$,

$i = k+1, \ldots, n$, the following inequality holds:

$$\left| \|X_2^* M(\mathbf{p}^0)\widehat{X}_1\|_F^2 - \|X_2^* M(\mathbf{p}^0)X_1\|_F^2 \right| \leq \left| \frac{\|X_2^* C(\mathbf{p}^0)\widehat{X}_1\|_F^2}{\widehat{\mathrm{g}}_0^2} - \frac{\|X_2^* C(\mathbf{p}^0)X_1\|_F^2}{\widehat{\mathrm{g}}_M^2} \right| + \sum_{j=1}^{k}\sum_{i=k+1}^{n} \left(2\frac{|(X_2^* C(\mathbf{p}^0)\widehat{X}_1)_{ij}|}{\widehat{\mathrm{g}}_0} \left(\frac{|(X_2^* \Delta M\widehat{X}_1)_{ij}|}{\widehat{\mathrm{g}}_1} + \frac{|(X_2^* \Delta C\widehat{X}_1)_{ij}|}{\widehat{\mathrm{g}}_2} + \frac{|(X_2^* \Delta K\widehat{X}_1)_{ij}|}{\widehat{\mathrm{g}}_3} \right) + \left(\frac{|(X_2^* \Delta M\widehat{X}_1)_{ij}|}{\widehat{\mathrm{g}}_1} + \frac{|(X_2^* \Delta C\widehat{X}_1)_{ij}|}{\widehat{\mathrm{g}}_2} + \frac{|(X_2^* \delta K\widehat{X}_1)_{ij}|}{\widehat{\mathrm{g}}_3} \right)^2 \right) \tag{3.35}$$

where $\Delta M = M(\mathbf{p}) - M(\mathbf{p}^0)$, $\Delta C = C(\mathbf{p}) - C(\mathbf{p}^0)$, $\Delta K = K(\mathbf{p}) - K(\mathbf{p}^0)$ and

$$\widehat{\mathrm{g}}_0 = \min_{\substack{j=1,\ldots,k \\ i=k+1,\ldots,n}} |\overline{\lambda}_i(\mathbf{p}^0) + \lambda_j(\mathbf{p})|, \tag{3.36}$$

$$\widehat{\mathrm{g}}_M = \max_{\substack{j=1,\ldots,k \\ i=k+1,\ldots,n}} |\overline{\lambda}_i(\mathbf{p}^0) + \lambda_j(\mathbf{p}^0)|, \tag{3.37}$$

$$\widehat{\mathrm{g}}_1 = \min_{\substack{j=1,\ldots,k \\ i=k+1,\ldots,n}} \frac{|\overline{\lambda}_i^2(\mathbf{p}^0) - \lambda_j^2(\mathbf{p})|}{|\lambda_j^2(\mathbf{p})|}, \tag{3.38}$$

$$\widehat{\mathrm{g}}_2 = \min_{\substack{j=1,\ldots,k \\ i=k+1,\ldots,n}} \frac{|\overline{\lambda}_i^2(\mathbf{p}^0) - \lambda_j^2(\mathbf{p})|}{|\lambda_j(\mathbf{p})|}, \tag{3.39}$$

$$\widehat{\mathrm{g}}_3 = \min_{\substack{j=1,\ldots,k \\ i=k+1,\ldots,n}} |\overline{\lambda}_i^2(\mathbf{p}^0) - \lambda_j^2(\mathbf{p})|, \tag{3.40}$$

As one can notice, the gap functions (3.36), (3.38), (3.39), (3.40), contain unperturbed and perturbed eigenvalues, which makes them hard to compute. Since we want to avoid perturbed eigenvalues in bound (3.35) we can bound gap functions (3.36), (3.38), (3.39), (3.40) only by using unperturbed eigenvalues $\lambda(\mathbf{p}^0)$ and approximations of perturbed eigenvalues $\widetilde{\lambda}(\mathbf{p})$.

These bounds for (3.38), (3.39), (3.40) are obtained directly from Corollary 3.3, and following hold:

$$\widehat{\mathrm{g}}_1 \gtrapprox \min_{\substack{j=1,\ldots,k \\ i=k+1,\ldots,n}} \frac{|\overline{\lambda}_i^2(\mathbf{p}^0) - \widetilde{\lambda}_j^2(\mathbf{p})| - M_j^{(2)}}{|\widetilde{\lambda}_j^2(\mathbf{p})| + M_j^{(2)}} =: \mathrm{Bnd}_{\widehat{\mathrm{g}}_1}, \tag{3.41}$$

$$\widehat{\mathrm{g}}_2 \gtrapprox \min_{\substack{j=1,\ldots,k \\ i=k+1,\ldots,n}} \frac{|\overline{\lambda}_i^2(\mathbf{p}^0) - \widetilde{\lambda}_j^2(\mathbf{p})| - M_j^{(2)}}{|\widetilde{\lambda}_j(\mathbf{p})| + M_j^{(1)}} =: \mathrm{Bnd}_{\widehat{\mathrm{g}}_2}, \tag{3.42}$$

$$\widehat{\mathrm{g}}_3 \gtrapprox \min_{\substack{j=1,\ldots,k \\ i=k+1,\ldots,n}} (|\overline{\lambda}_i^2(\mathbf{p}^0) - \widetilde{\lambda}_j^2(\mathbf{p})| - M_j^{(2)}) =: \mathrm{Bnd}_{\widehat{\mathrm{g}}_3}. \tag{3.43}$$

Furthermore, following corollary gives the bound for (3.36).

Corollary 3.6 Let the assumptions from Theorem 3.1 hold and let the perturbation δ be small enough such that $|\overline{\lambda}_i(\mathbf{p}^0)+\widetilde{\lambda}_j(\mathbf{p})|-M_j^{(1)}\geq 0$, for all $j=1,\ldots,k$, $i=k+1,\ldots,n$. Then the following bound holds

$$\widehat{\mathrm{g}}_0 \;\gtrapprox \min_{\substack{j=1,\ldots,k\\ i=k+1,\ldots,n}} |\overline{\lambda}_i(\mathbf{p}^0)+\widetilde{\lambda}_j(\mathbf{p})|-M_j^{(1)} =: \mathrm{Bnd}_{\widehat{\mathrm{g}}_0}, \tag{3.44}$$

where $M_i^{(1)}=\frac{1}{2}M_i\|(\mathbf{p}-\mathbf{p}^0)\|^2$.

Proof. By applying the triangle inequality

$$|\overline{\lambda}_i(\mathbf{p}^0)+\widetilde{\lambda}_j(\mathbf{p})| \;\lessapprox\; |\overline{\lambda}_i(\mathbf{p}^0)+\lambda_j(\mathbf{p})|+M_j^{(1)} \tag{3.45}$$

we obtain

$$|\overline{\lambda}_i(\mathbf{p}^0)+\lambda_j(\mathbf{p})| \;\gtrapprox\; |\overline{\lambda}_i(\mathbf{p}^0)+\widetilde{\lambda}_j(\mathbf{p})|-M_j^{(1)}. \tag{3.46}$$

Now, by taking the minimum of both sides we obtain inequality (3.44). □

Now, by applying bounds (3.44), (3.41),(3.42),(3.43) we obtain the following bound

$$\begin{aligned}\left|\|X_2^*M(\mathbf{p}^0)\widehat{X}_1\|_F^2-\|X_2^*M(\mathbf{p}^0)X_1\|_F^2\right| &\leq \left|\frac{\|X_2^*C(\mathbf{p}^0)\widehat{X}_1\|_F^2}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_0}^2}-\frac{\|X_2^*C(\mathbf{p}^0)X_1\|_F^2}{\widehat{\mathrm{g}}_M^2}\right| \\ +\sum_{j=1}^{k}\sum_{i=k+1}^{n}\left(2\frac{|(X_2^*C(\mathbf{p}^0)\widehat{X}_1)_{ij}|}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_0}}\left(\frac{|(X_2^*\Delta M\widehat{X}_1)_{ij}|}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_1}}+\frac{|(X_2^*\Delta C\widehat{X}_1)_{ij}|}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_2}}+\frac{|(X_2^*\Delta K\widehat{X}_1)_{ij}|}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_3}}\right)\right. & \\ \left.+\left(\frac{|(X_2^*\Delta M\widehat{X}_1)_{ij}|}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_1}}+\frac{|(X_2^*\Delta C\widehat{X}_1)_{ij}|}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_2}}+\frac{|(X_2^*\Delta K\widehat{X}_1)_{ij}|}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_3}}\right)^2\right) &:= \mathrm{Bnd}_{\mathrm{RHS}}.\end{aligned} \tag{3.47}$$

Remark 3.2. Since the calculation of M_j is not easy, as we already mentioned, we will denote with $\widetilde{\mathrm{Bnd}}_{\widehat{\mathrm{g}}_i}$, for $i=0,1,2,3$, lower bounds from (3.44), (3.41), (3.42), (3.43) in which we use $\widetilde{M}_j$ from (2.48) instead of M_j. Also, the corresponding $\mathrm{Bnd}_{\mathrm{RHS}}$ will be denoted as $\widetilde{\mathrm{Bnd}}_{\mathrm{RHS}}$.

3.2.2 Numerical experiment

In the following example we illustrate the bound given by (3.47), which contains the bounds for gap functions $\widehat{g}_i$, $i=0,1,2,3$, given by (3.44), (3.41), (3.42), (3.43), respectively.

Example 3.2 We again consider the mechanical system as in Example 2.4 and Example 3.1, shown in Figure 2.10. Also again, we consider the case where $\mathbf{p}=(\frac{v}{4}+\delta, v+\delta, \delta)$ and $\mathbf{p}^0=(\frac{v}{4}, v, 0)$.
As we already mentioned, in this case we examine bounds given by (3.44), (3.41), (3.42), (3.43). Figure 3.3 shows magnitude of relative error in the bound of the RHS given in (3.47), if we

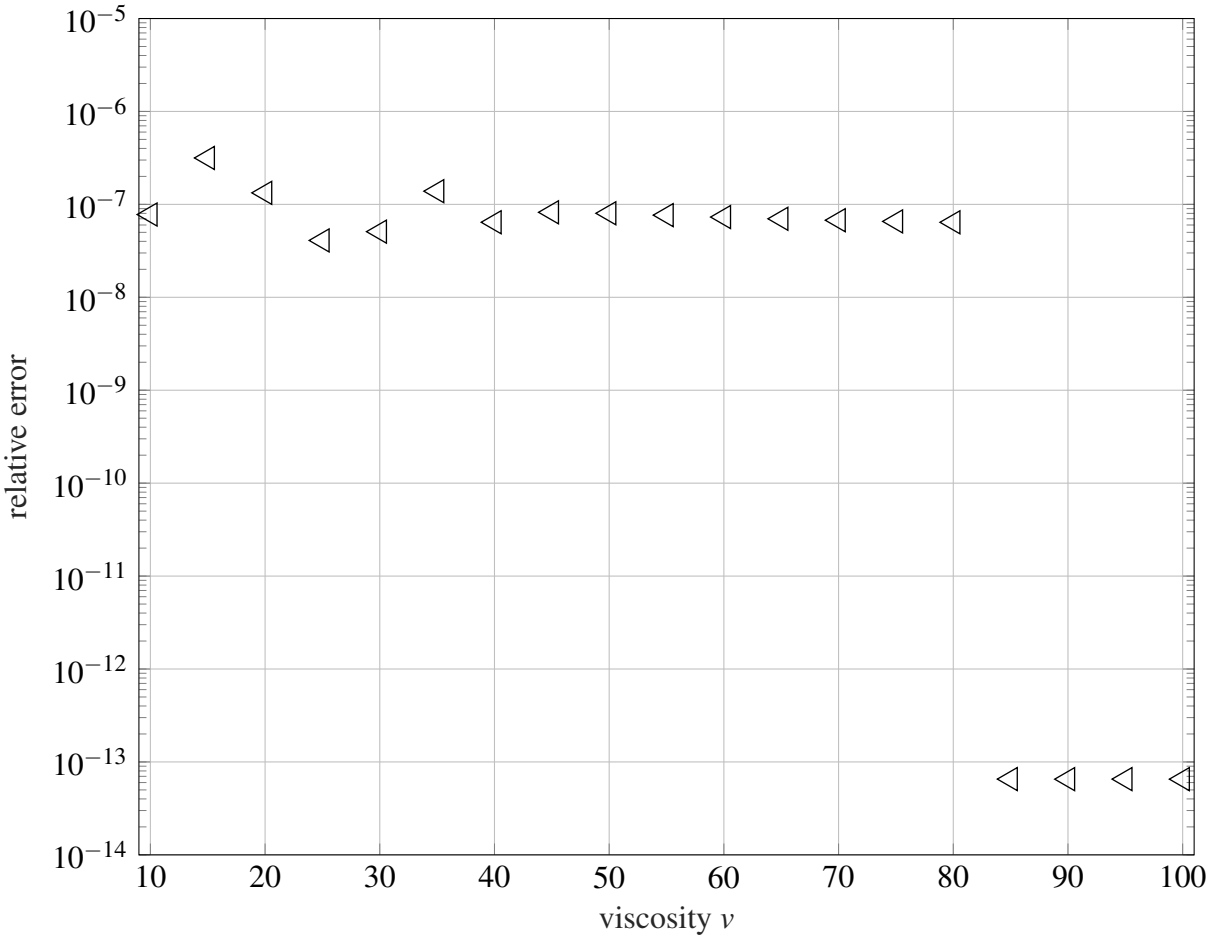

Figure 3.3: Relative error given by (3.48) for perturbation $\delta = 0.0001$ and viscosities $v = 10, 15, 20, \ldots, 100$

replace M_k with $\widetilde{M}_k$ in (3.44), (3.41), (3.42), (3.43), i.e.,

$$\frac{|\mathrm{Bnd}_{\mathrm{RHS}} - \widetilde{\mathrm{Bnd}}_{\mathrm{RHS}}|}{|\mathrm{Bnd}_{\mathrm{RHS}}|}. \tag{3.48}$$

From Figure 3.3 one can see that we can replace $\mathrm{Bnd}_{\mathrm{RHS}}$ with $\widetilde{\mathrm{Bnd}}_{\mathrm{RHS}}$ in (3.35).

Figure 3.4 shows the LHS and RHS of inequality (3.35) denoted as red circles and blue $\times$s, respectively, for viscosity $v = 5 \cdot i$, $i = 2, 3, \ldots, 20$. Furthermore, black triangle represents an approximation of $\mathrm{Bnd}_{\mathrm{RHS}}$ as we mentioned in Remark 3.2. As one can see, our approximation of the bound behaves the same way as the bound (RHS) from article [102] and it does not contain both perturbed and unperturbed eigenvalues.

3.2.3 Efficient estimation of the bounds for hyperbolic QEP

For the problem in this subsection, we need additional assumptions. Namely, we assume that for vector of parameters **p**, the PQEP given by (1.1) is determined with Hermitian positive

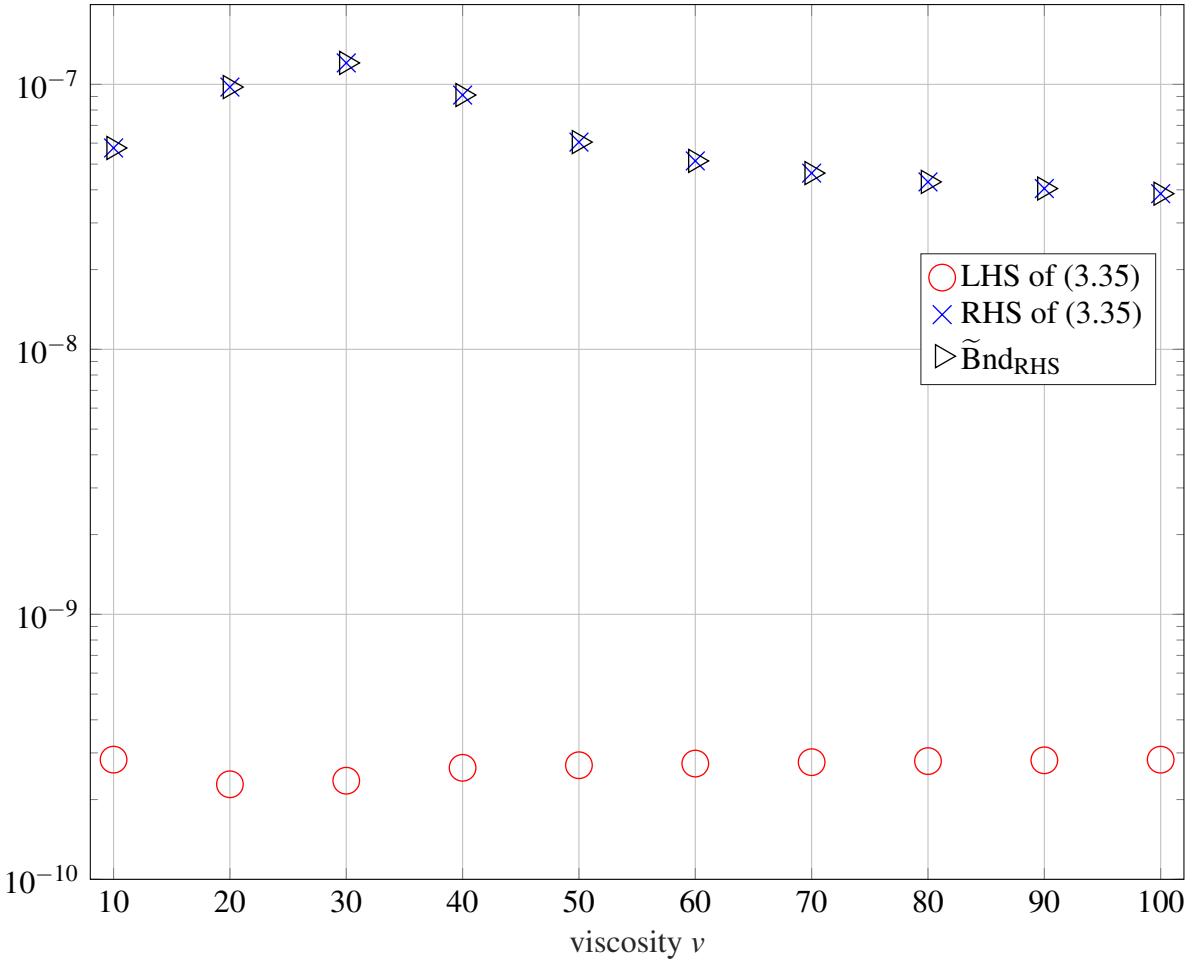

Figure 3.4: LHS and RHS of (3.35) with corresponding bound $\widetilde{\text{Bnd}}_{\text{RHS}}$ for perturbation $\delta = 0.01$ and viscosities $v = 10, 20, \ldots, 100$

definite matrices $M(\mathbf{p})$, $C(\mathbf{p})$, and $K(\mathbf{p})$ such that

$$(\widehat{x}^* C(\mathbf{p})\widehat{x})^2 - 4(\widehat{x}^* M(\mathbf{p})\widehat{x})(\widehat{x}^* K(\mathbf{p})\widehat{x}) > 0, \tag{3.49}$$

holds for all eigenvectors x of PQEP. Now, [101, Theorem 7] gives the $\sin\Theta$ type upper bound for the eigensubspaces for the considered PQEP as follows. Let matrices $M(\mathbf{p})$, $C(\mathbf{p})$, and $K(\mathbf{p})$ determine the perturbed PQEP.

Let $M(\mathbf{p}^0) = L_M L_M^*$ and $K(\mathbf{p}^0) = L_K L_K^*$ be Cholesky (or any other) decompositions. In [101] the authors consider the pair (A, J), where

$$A = \begin{bmatrix} L_K^{-1} C(\mathbf{p}^0) L_K^{-*} & L_K^{-1} L_M \\ L_M^* L_K^{-*} & 0 \end{bmatrix}, \quad J = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}. \tag{3.50}$$

Where the pair $(\widehat{A},\widehat{J})$ that corresponds to perturbed PQEP is given by

$$\widehat{A} = \begin{bmatrix} L_K^{-1}C(\mathbf{p})L_K^{-*} & L_K^{-1}L_M + L_K^{-1}(M(\mathbf{p}) - M(\mathbf{p}^0))L_M^{-*} \\ L_M^{*}L_K^{-*} + L_M^{-1}(M(\mathbf{p}) - M(\mathbf{p}^0))L_K^{-*} & 0 \end{bmatrix}, \tag{3.51}$$
$$\widehat{J} = \begin{bmatrix} -I + L_K^{-1}(K(\mathbf{p}) - K(\mathbf{p}^0))L_K^{-*} & 0 \\ 0 & I + L_M^{-1}(M(\mathbf{p}) - M(\mathbf{p}^0))L_M^{-*} \end{bmatrix}.$$

Let X and $\widehat{X}$ be nonsingular matrices which simultaneously diagonalize the pairs (A,J) and $(\widehat{A},\widehat{J})$, respectively. Then the following bound holds [101, Theorem 7] for $j = 1,\ldots,2n$:

$$|\sin\Theta(X(:,j),\widehat{X}(:,j))| \leq \kappa_2(X)\kappa_2(\widehat{X})\left(\frac{\delta a_F + \delta c_F}{\mathrm{g}_{2,j}} + \frac{\delta b_F}{\mathrm{g}_{1,j}}\right), \tag{3.52}$$

where

$$\mathrm{g}_{1,j} = \min_{i\neq j}\frac{|\lambda_j(\mathbf{p}^0) - \lambda_i(\mathbf{p})|}{|\lambda_j(\mathbf{p}^0)|}, \quad \mathrm{g}_{2,j} = \min_{i\neq j}\frac{|\lambda_j(\mathbf{p}^0) - \lambda_i(\mathbf{p})|}{|\lambda_i(\mathbf{p})|}, \tag{3.53}$$

and

$$\delta a_F = \sqrt{\|L_K^{*}M(\mathbf{p}^0)^{-1}(M(\mathbf{p}) - M(\mathbf{p}^0))L_M^{-*}\|_F^2 + \|L_M^{-1}(M(\mathbf{p}) - M(\mathbf{p}^0))L_M^{-*}\|_F^2},$$
$$\delta b_F = \sqrt{\|L_M^{-1}(M(\mathbf{p}) - M(\mathbf{p}^0))L_M^{-*}\|_F^2 + \|L_K^{-1}(K(\mathbf{p}) - K(\mathbf{p}^0))L_K^{-*}\|_F^2},$$
$$\delta c_F = \sqrt{\|L_M^{-1}(C(\mathbf{p}) - C(\mathbf{p}^0))L_K^{-*} + L_M^{-1}C(\mathbf{p}^0)M(\mathbf{p}^0)^{-1}(M(\mathbf{p}) - M(\mathbf{p}^0))L_M^{-*}\|_F^2}.$$

In order to have an upper bound for $|\sin\Theta(X(:,j),\widehat{X}(:,j))|$ that can be efficiently calculated we will additionally estimate gaps given by (3.53).

Corollary 3.7 Let $M(\mathbf{p}^0), C(\mathbf{p}^0), K(\mathbf{p}^0) \in \mathbb{C}^{n\times n}$ be matrices given as in (3.49). Let $M(\mathbf{p}), C(\mathbf{p})$ and $K(\mathbf{p})$ be corresponding perturbed matrices where (3.49) holds for all $\mathbf{p}$ such that $\|\mathbf{p}-\mathbf{p}^0\| \ll \varepsilon_{\mathbf{p}}$ and $|\lambda_j(\mathbf{p}^0) - \widetilde{\lambda}_i(\mathbf{p})| - M_i^{(1)} \geq 0$ for $i,j = 1,\ldots,2n, i \neq j$, and X and $\widehat{X}$ nonsingular matrices which simultaneously diagonalize the pairs (A,J) and $(\widehat{A},\widehat{J})$, respectively. Then the following bounds hold

$$\mathrm{g}_{1,j} \gtrapprox \min_{i\neq j}\frac{|\lambda_j(\mathbf{p}^0) - \widetilde{\lambda}_i(\mathbf{p})| - M_i^{(1)}}{\lambda_j(\mathbf{p}^0)} =: \mathrm{Bnd}_{\widehat{\mathrm{g}}_{1,j}}, \tag{3.54}$$
$$\mathrm{g}_{2,j} \gtrapprox \min_{i\neq j}\frac{|\lambda_j(\mathbf{p}^0) - \widetilde{\lambda}_i(\mathbf{p})| - M_i^{(1)}}{|\widetilde{\lambda}_i(\mathbf{p})| + M_i^{(1)}} =: \mathrm{Bnd}_{\widehat{\mathrm{g}}_{2,j}}, \tag{3.55}$$

where $M_i^{(1)} = \frac{1}{2}M_i\|(\mathbf{p}-\mathbf{p}^0)\|^2$.

Proof. By applying the triangle inequality on (2.42) and the fact that $\|\mathcal{H}(\lambda_i(\mathbf{p}^0 + t(\mathbf{p}-\mathbf{p}^0))\| \leq$

M_i we get:

$$|\lambda_i(\mathbf{p})| \lessapprox |\widetilde{\lambda}_i(\mathbf{p})| + M_i^{(1)}. \tag{3.56}$$

Furthermore, by adding and subtracting $\lambda_j(\mathbf{p}^0)$ from the left hand side of (2.45) and applying the triangle inequality we obtain

$$|\lambda_j(\mathbf{p}^0) - \lambda_i(\mathbf{p})| \gtrapprox |\lambda_j(\mathbf{p}^0) - \widetilde{\lambda}_i(\mathbf{p})| - M_i^{(1)}. \tag{3.57}$$

By applying the above inequalities we get the following:

$$\frac{|\lambda_j(\mathbf{p}^0) - \lambda_i(\mathbf{p})|}{|\lambda_i(\mathbf{p})|} \gtrapprox \frac{|\lambda_j(\mathbf{p}^0) - \widetilde{\lambda}_i(\mathbf{p})| - M_i^{(1)}}{|\widetilde{\lambda}_i(\mathbf{p})| + M_i^{(1)}}, \tag{3.58}$$

$$\frac{|\lambda_j(\mathbf{p}^0) - \lambda_i(\mathbf{p})|}{|\lambda_j(\mathbf{p}^0)|} \gtrapprox \frac{|\lambda_j(\mathbf{p}^0) - \widetilde{\lambda}_i(\mathbf{p})| - M_i^{(1)}}{|\lambda_j(\mathbf{p}^0)|}. \tag{3.59}$$

Now, by taking the minimum of both sides we obtain inequalities (3.54) - (3.55). □

Now, we can apply our bounds (3.54) and (3.55) from Corollary 3.7 and obtain the upper bound for $|\sin\Theta(X(:,j),\widehat{X}(:,j))|$, $j = 1,\ldots,2n$,

$$|\sin\Theta(X(:,j),\widehat{X}(:,j))| \leq \kappa_2(X)\kappa_2(\widehat{X}) \left(\frac{\delta a_F + \delta c_F}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_{2,j}}} + \frac{\delta b_F}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_{1,j}}} \right), \tag{3.60}$$

3.2.4 Numerical experiments

In this example we illustrate the bounds for $\mathrm{g}_{1,i}$ and $\mathrm{g}_{2,i}$ given in (3.53). We also illustrate how accurate our bound is if we use approximation of M_i, instead of M_i.

Example 3.3 In this example we consider a hyperbolic mechanical system. The dimension of the system is $n = 100$, where the mass and stiffness matrices and internal damping are defined in the same way as in the Example 3.2. Furthermore, the external damping matrix is defined as $C_{\mathrm{ext}}(\mathbf{p}^0) = \mathrm{diag}(v_1^0 I_1, v_2^0 I_2, v_4^0 I_3)$, where I_s are identity matrices with dimensions $40, 50$, and 10, respectively. This means that we have put a grounded damper on each mass, but each block of dampers corresponds to viscosities v_1, v_2 and v_3, respectively.

In this example, parameters $\mathbf{p}$ and $\mathbf{p}^0$ are as follows $\mathbf{p} = (v, \frac{v}{2} + \delta, \delta, 4v + \delta)$ and $\mathbf{p}^0 = (v, \frac{v}{2}, 0, 4v)$.

Let $\mathrm{RHS}_{\mathrm{g}}$ denote the right-hand side of inequality (3.52), and

$$\mathrm{RHS}_{\mathrm{Bnd}_{\widehat{\mathrm{g}}}} = \kappa_2(X)\kappa_2(\widehat{X}) \left(\frac{\delta a_F + \delta c_F}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_{2,j}}} + \frac{\delta b_F}{\mathrm{Bnd}_{\widehat{\mathrm{g}}_{1,j}}} \right).$$

Since, Corollary 3.7 holds,

$$\mathrm{RHS}_{\mathrm{g}} \leq \mathrm{RHS}_{\mathrm{Bnd}_{\widehat{\mathrm{g}}}}.$$

We will illustrate how accurate $\mathrm{RHS}_{\mathrm{Bnd}_{\widehat{\mathrm{g}}}}$ bounds $\mathrm{RHS}_{\mathrm{g}}$ for both when we use M_i and its approximation $\widetilde{M}_i$. Relative error is as defined:

$$\mathrm{Rel}_{\mathrm{err}} = \frac{|\mathrm{RHS}_{\mathrm{g}} - \mathrm{RHS}_{\mathrm{Bnd}_{\widehat{\mathrm{g}}}}|}{\mathrm{RHS}_{\mathrm{g}}}. \tag{3.61}$$

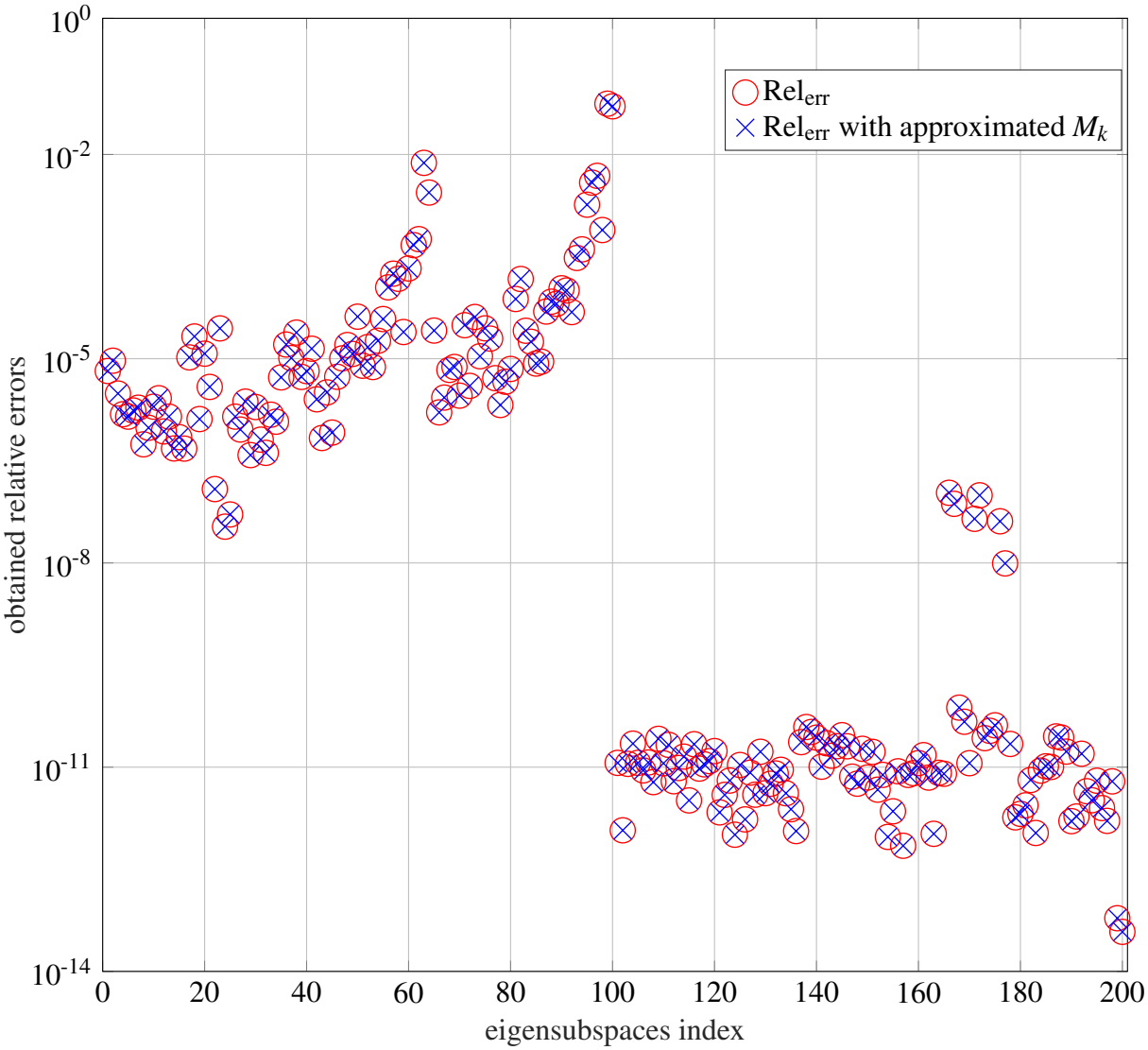

Figure 3.5: Relative error defined by (3.61)

Figure 3.5 shows relative error defined in (3.61) and corresponding relative error when we use approximation of M_i (2.48) instead of M_i, where $v = 1000$ and $\delta = 0.00025$.

From Figure 3.5 we can conclude that we can use error bounds (2.45) calculated with (2.48) instead of the real error bounds, since in calculation of relative errors (3.61) it didn't make many

difference.

3.3 Conclusion

In this chapter we have presented a new perturbation bound between individual unperturbed and perturbed eigenvector of quadratic eigenvalue problems. Usually, these kind of perturbation bounds depend on unperturbed and perturbed eigenvalues and eigenvectors, which reduces their possible applications. Therefore, in order to estimate certain gap functions, we have used the first order approximations for eigenvalues and eigenvectors based on Taylor's formula, and their corresponding bounds from Section 2.2.

Using these approximations we have derived upper bounds for different gap functions that occur in various perturbation bounds, such as the bounds from [101] and [102]. This approach significantly improves calculation efficiency and possible application of corresponding perturbation bounds. The quality of the obtained bounds have been illustrated with numerical examples.

CHAPTER 4

Damping optimization in mechanical systems

As previously mentioned we consider the mechanical system described by mass, stiffness and damping matrices as in (1.2). In this chapter we are mainly devoted to damping optimization, therefore, our parameter contains only dampers' viscosity, i.e., $\mathbf{p} = \mathbf{v}$ and $M(\mathbf{p}) = M, C(\mathbf{p}) = C(\mathbf{v}) = C_{\text{ext}}(\mathbf{v}) + C_{\text{int}}, K(\mathbf{p}) = K$. This means that PQEP (1.1) has the following form:

$$(\lambda(\mathbf{v})^2 M + \lambda(\mathbf{v})C(\mathbf{v}) + K)x(\mathbf{v}) \quad = \quad 0, \qquad x(\mathbf{v}) \neq 0, \tag{4.1}$$

In this chapter we focus on two damping optimization criteria: the minimization of total average energy of the system and the frequency isolation which are introduced in Section 1.2. Also we show difference between eigenvalue based criteria (the minimization of spectral abscissa and the frequency isolation) and minimization of total average energy. Main difference between the minimization of the spectral abscissa and the frequency isolation is the face that we push all eigenvalues further left in complex half plane, while the frequency isolation removes eigenvalues from the specific areas that are dangerous. First one is more appropriate for the stationary case given by (1.2), i.e. the case where there is no external excitation, while the second one is more suitable for the non-stationary case

$$\begin{aligned} M\ddot{q}(\mathbf{v};t) + C(\mathbf{v})\dot{q}(\mathbf{v};t) + Kq(\mathbf{v};t) &= f(t), \\ q(\mathbf{v};0) = q_0(\mathbf{v}) \quad \text{and} \quad \dot{q}(\mathbf{v};0) &= \dot{q}_0(\mathbf{v}), \end{aligned} \tag{4.2}$$

since the external force $f(t)$ can be approximated with a Fourier series and we assume that it is given as the finite sum of sine and cosine functions, i.e.,

$$f(t) = \sum_{j=1}^{N} f_a^j \cos\left(\omega_j t\right) + f_b^j \sin\left(\omega_j t\right), \tag{4.3}$$

where ω_j are the frequencies of the external force, i.e. the dangerous frequencies. For the details on Fourier series one can see [83].

Furthermore, in the following example we show the difference between the minimization of

the total average energy and the minimization of the spectral abscissa and later in Section 4.3 we compare the minimization of total average energy and the frequency isolation criteria given in Section 4.2. It is important to emphasize that the results from these examples are illustrative and do not represent results in general for all mechanical systems.

Example 4.1 Consider the mechanical system shown in Figure 4.1 and it differs from mechanical system shown in Figure 2.10 in Example 2.4 only in the structure of the damping. There is a grounded damper on each mass.

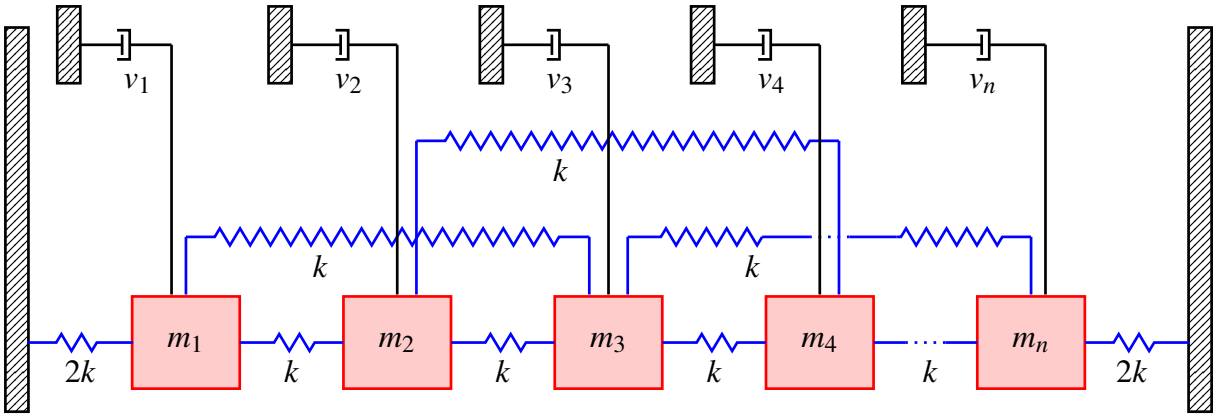

Figure 4.1: n-mass oscillator

This time the dimension of the system is $n = 20$. In this example the mass matrix and the stiffness matrix do not depend on the vector of parameters and they are given by

$$M = \operatorname{diag}(m_1, \ldots, m_n), \quad \text{where}$$
$$m_i = \begin{cases} 200 - 20(i-1) & , i = 1, \ldots, 10 \\ 201 + 20(i-11) & , i = 11, \ldots, 20 \end{cases}, \tag{4.4}$$

and

$$K = \begin{bmatrix} 4k & -k & -k & & & & \\ -k & 4k & -k & -k & & & \\ -k & -k & 4k & -k & -k & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -k & -k & 4k & -k & -k \\ & & & -k & -k & 4k & -k \\ & & & & -k & -k & 4k \end{bmatrix}, \quad \text{where} \quad k = 1. \tag{4.5}$$

The damping matrix is given by $C(\mathbf{v}) = C_{\text{int}} + C_{\text{ext}}(\mathbf{v})$ where internal damping is given by

$C_{\text{int}} = \alpha_c C_{\text{crit}}$, with $\alpha_c = 0.002$ and C_{crit} is given by (1.21). The external damping is given by

$$C_{\text{ext}}(\mathbf{v}) = \text{diag}(v_1, \ldots, v_n). \tag{4.6}$$

We minimized spectral abscissa and total average energy for this system, and obtained the optimal viscosities $\mathbf{v}_{\text{sa}}$ and $\mathbf{v}_{\text{tae}}$ for each criteria respectively. The obtained value for the minimized spectral abscissa is

$$\alpha_{\text{MCK}}(\mathbf{v}_{\text{sa}}) = \max_{\lambda(\mathbf{v}_{\text{sa}}) \in \Lambda(\mathbf{v}_{\text{sa}})} \text{Re}(\lambda(\mathbf{v}_{\text{sa}})) = -0.0244,$$

while the spectral abscissa for the system with no external damping and for the viscosity $\mathbf{v}_{\text{tae}}$ are respectively

$$\alpha_{\text{MCK}}(\mathbf{0}) = -0.002, \quad \alpha_{\text{MCK}}(\mathbf{v}_{\text{tae}}) = -0.0239.$$

The obtained value for minimized total average energy is

$$\text{tr}(\mathbf{Z}\mathbf{X}_{\Phi}(\mathbf{v}_{\text{tae}})) = 356.4998,$$

where $\mathbf{X}_{\Phi}(\mathbf{v}_{\text{tae}})$ is the solution of corresponding Lyapunov equation (1.33).

$$\mathbf{Z} = \begin{bmatrix} I_{10} & 0 & 0 & 0 \\ 0 & 0_{10} & 0 & 0 \\ 0 & 0 & I_{10} & 0 \\ 0 & 0 & 0 & 0_{10} \end{bmatrix}, \tag{4.7}$$

where I_{10} and 0_{10} are respectively 10-dimensional identity and zero matrix. Initial condition in considered system are

$$q(\mathbf{v};0) = e_1 \quad \text{and} \quad \dot{q}(\mathbf{v};0) = 0,$$

where e_1 is first canonical vector of dimension n.

The Figures 4.2 and 4.3 show respectively the behaviour of the displacement and the velocity of masses m_2 and m_{17} over time for the stationary system. From these figures it is hard to conclude whether it is better to choose the minimization of the spectral abscissa or the minimization of the total average energy as a criterion. In both figures the mass reaches the equilibrium point faster with the viscosities obtained by minimization of the spectral abscissa, but we can also see from Figure 4.3 that for the spectral abscissa criterion (blue line) mass m_{17} has larger oscillation at the beginning than for the total average energy criterion (red, dashed line), so the choice of the criterion will depend on the application. We also illustrate the total energy of the system over time which is given in (1.25).

The Figure 4.4 shows the energy of the stationary system (1.2) over time for two different

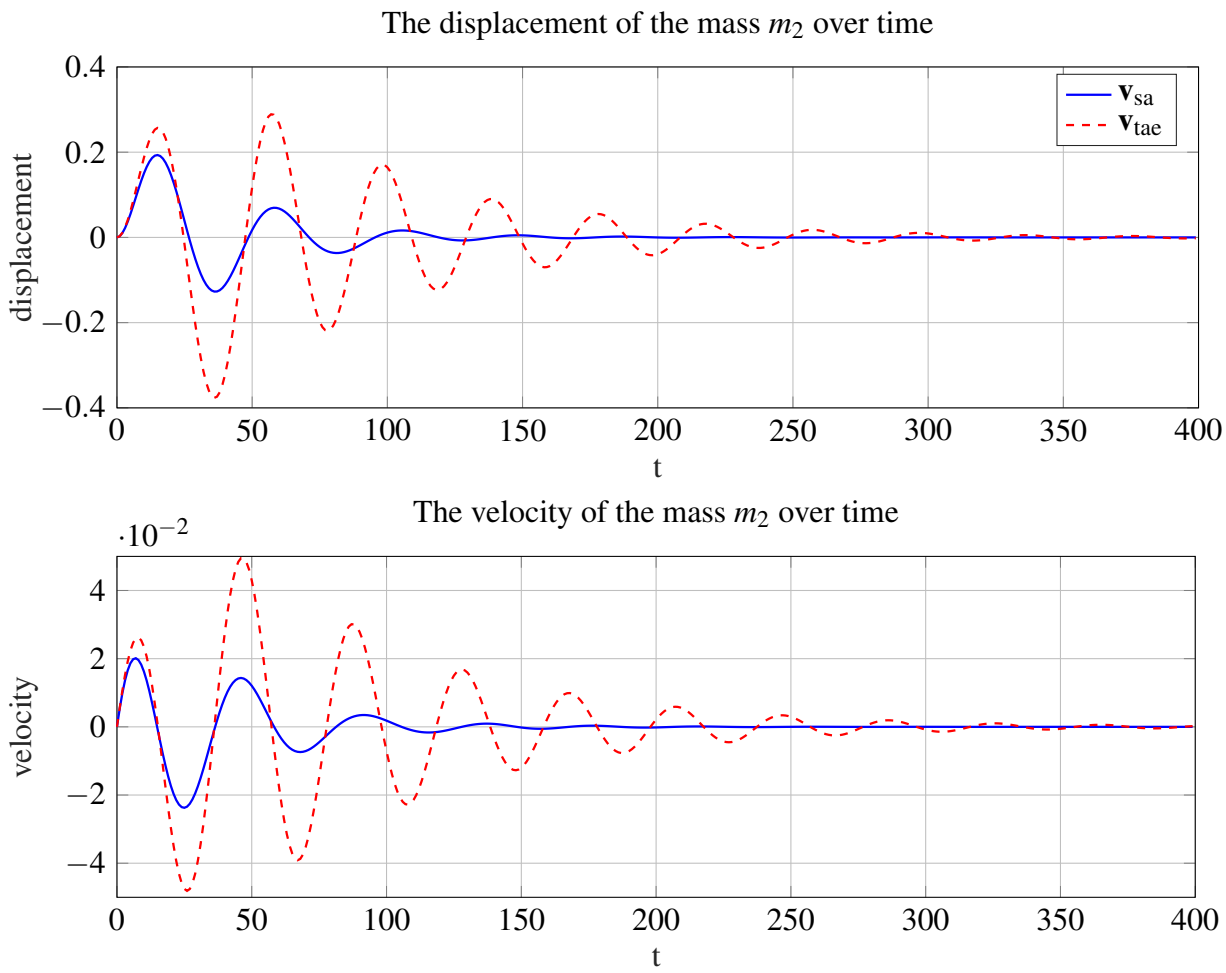

Figure 4.2: The behaviour of the displacement and velocity of the mass m_2 over time for two different sets of viscosities in stationary case

sets of viscosities. The blue line show the total energy for the case when the set of viscosities correspond to the optimal viscosities for the minimization of the spectral abscissa, i.e., $\mathbf{v}_{\text{sa}}$, while the red dashed line show the total energy for the case when the set of viscosities correspond to the optimal viscosities for the minimization of the total average energy, i.e., $\mathbf{v}_{\text{tae}}$. One can see that the spectral abscissa criterion resulted in faster decay of the total energy of the system.

On the other hand, Figure 4.5 shows the same, but for the non-stationary case (4.2), where the external force is given by

$$f(t) = \sum_{j=2}^{5} f_a^j \cos\left(\omega_j t\right) + f_b^j \sin\left(\omega_j t\right), \tag{4.8}$$

where $f_a^j = f_b^j = [5,5,\ldots,5]^T$ are n-dimensional vectors, and ω_j are eigenfrequencies that correspond to the undamped system given by (1.5), more precisely

$$\omega_2 = 0.0490, \quad \omega_3 = 0.0651, \quad \omega_4 = 0.0885, \quad \omega_5 = 0.1071.$$

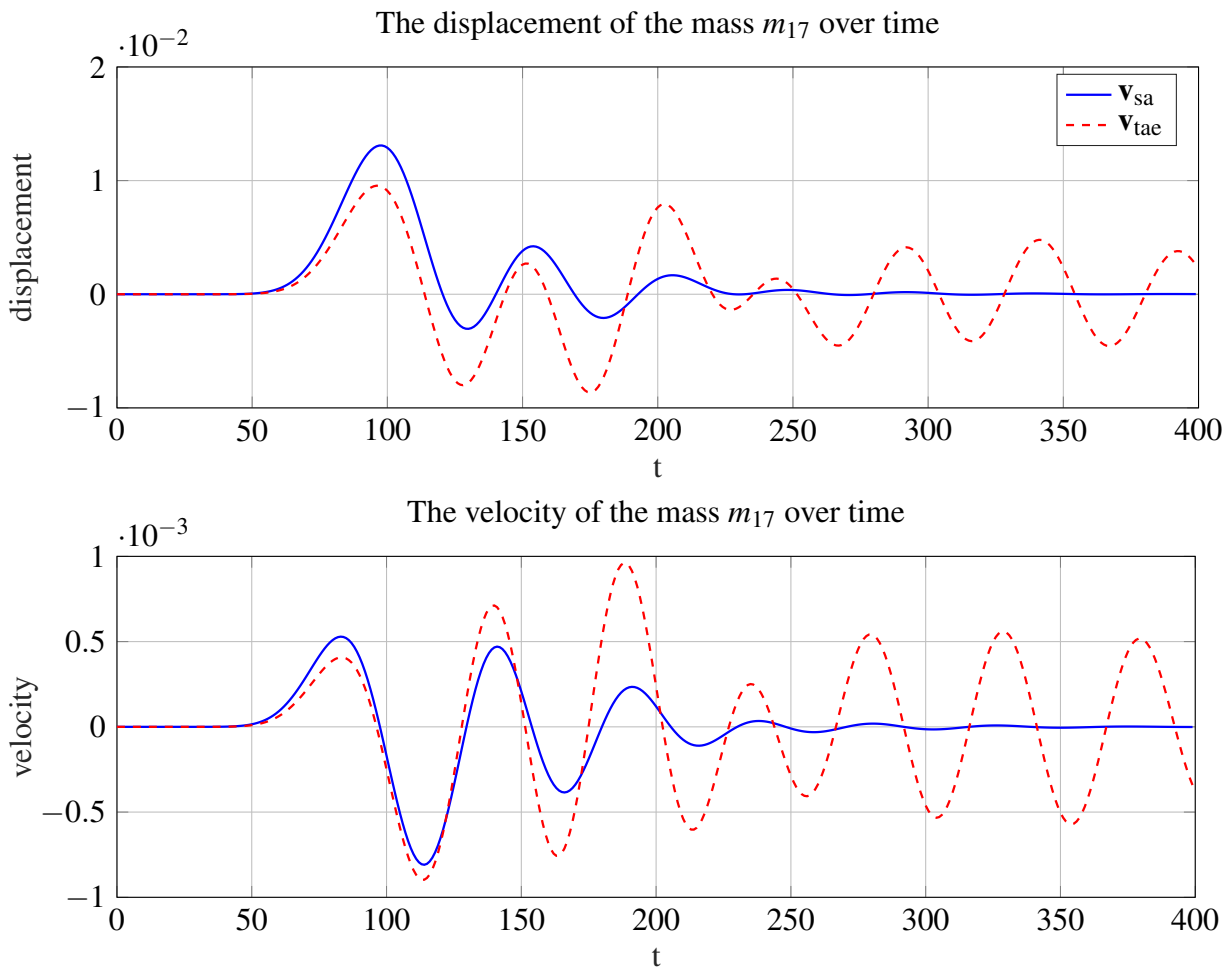

Figure 4.3: The behaviour of the displacement and velocity of the mass m_{17} over time for two different sets of viscosities in stationary case

As one can see in the non-stationary case the minimization of the total average energy criterion produces less energy. The main reason is that the minimization of the spectral abscissa criterion doesn't take into account the dangerous frequencies, while the left side of the Lyapunov equation that corresponds to the minimization of the total average energy criterion takes into account the dangerous frequencies.

Since there are cases when the minimization of the total average energy is more appropriate than the spectral abscissa criterion, in Section 4.1, we precisely define our damping optimization approach while using minimization of total average energy as an optimization criterion. Section 4.1.1 provides an approximation of the solution of the structured Lyapunov equation, while Subsection 4.1.2 gives the approximation of the trace of the solution of the Lyapunov equation when mass and damping matrices have the same eigenvector structure, i.e., we assume that $v_i, i = 1, \ldots, n$, are eigenvectors of both M and $C(\mathbf{v})$. In Subsubsections 4.1.2.1 we provide the approximation of the trace of the solution of the Lyapunov equation, where the damping matrix is structured, while in Subsubsection 4.1.2.2 we have a general damping matrix. The efficiency

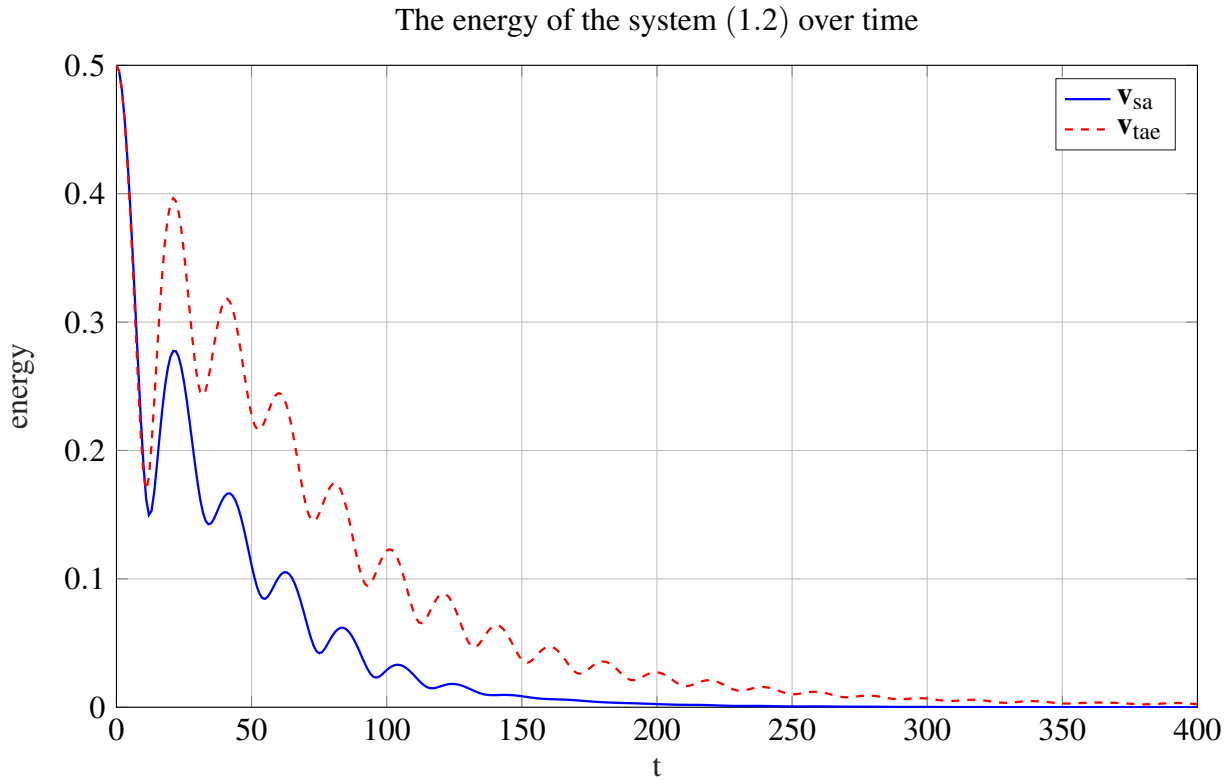

Figure 4.4: The energy for the stationary system (1.2)

and performance of the proposed approach are illustrated in Subsection 4.1.3.

Furthermore in Section 4.2 we present an eigenvalue based approaches which take into account the dangerous frequencies. These approaches are a frequency isolation in damping optimization. First one is based on maximization of the major axis of the ellipses while there are no eigenvalues in their interior, while the second one minimizes the spectral abscissa while taking into account that there are no eigenvalues in the previously determined fixed ellipses. Furthermore, in Subsection 4.2.2 we apply our efficient algorithm for eigenvalue computation from Section 2.3 given by Algorithm 5. The efficiency of this approach is given in Subsection 4.2.3. In Section 4.3 we use both frequency isolation approaches from Section 4.2 in Example 4.1 to illustrate how the system behaves when we push its eigenvalues further away from the dangerous frequencies.

4.1 Total average energy damping optimization

The purpose of this section is to present new results of approximation algorithms for estimating optimal damping, by using minimization of total average energy as a criterion. As we show in some cases determination of optimal damping can be given by an explicit formula, while in more general cases we present a numerical approach to determination of optimal damping which can be efficiently implemented.

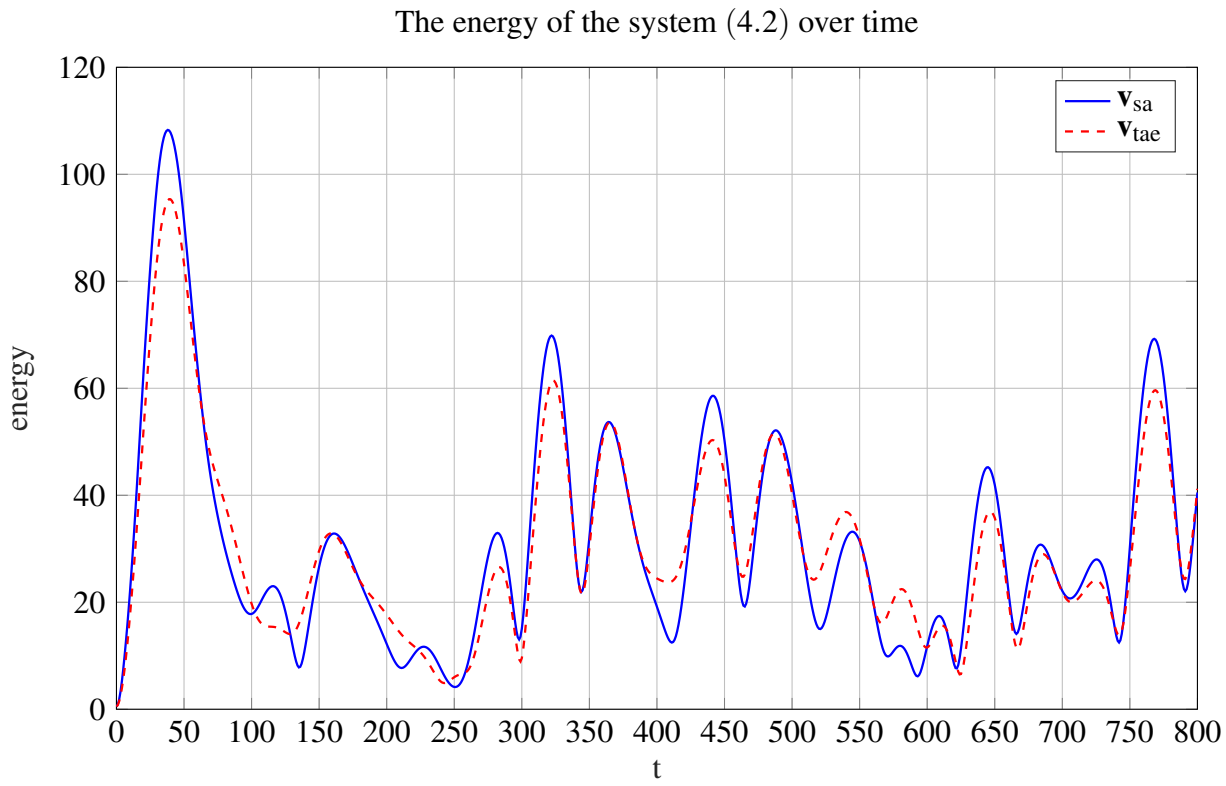

Figure 4.5: The energy over time for the non-stationary system (4.2)

Our approach is based on the fact that for a modally damped mechanical system all three matrices M, $C(\mathbf{v})$ and K can be simultaneously diagonalized. Thus, the main assumption here is that matrices M, $C(\mathbf{v})$ and K are simultaneously diagonalizable or that they are close to the case when all three matrices can be simultaneously diagonalized. Although this approach has been widely used by different scientific communities, especially in engineering, in this section we propose a slightly different perspective, which allows us to determine optimal damping very efficiently for a certain structure of mechanical systems, as we demonstrate later.

Moreover, since only the damping matrix $C(\mathbf{v})$ depends on parameters, usual approaches to viscosity optimization assume preprocessing based on diagonalization of the mass and stiffness matrices, M and K, as in (1.4). On the other hand, in this section we propose a new approach, which is based on diagonalization of the damping matrix $C(\mathbf{v})$, and then calculation of optimal viscosities. As we show in this section, this approach can be very efficient for structured systems which allow us to determine optimal viscosities, explicitly or numerically considerably faster.

First, in the next subsection we present a new algorithm which approximates the solution (as well as its trace) of the corresponding Lyapunov equation, and after that in Subsection 4.1.2 we present a new algorithm for finding optimal total average energy damping.

4.1.1 Approximation of the solution of the structured Lyapunov equation

It is well known that linearization from (2.64) is not unique. Thus for our purpose we rewrite (1.2) as

$$M^{\frac{1}{2}}M^{\frac{1}{2}}\ddot{q}(\mathbf{v};t)+C(\mathbf{v})\dot{q}(\mathbf{v};t)+K^{\frac{1}{2}}K^{\frac{1}{2}}q(\mathbf{v};t)=0, \tag{4.9}$$

since both M and K are positive definite, and t represents the time variable. Thus, since $M^{\frac{1}{2}}$ is positive definite, we obtain

$$I_n\ddot{q}(\mathbf{v};t)+M^{-\frac{1}{2}}C(\mathbf{v})M^{-\frac{1}{2}}\dot{q}(\mathbf{v};t)+M^{-\frac{1}{2}}K^{\frac{1}{2}}K^{\frac{1}{2}}M^{-\frac{1}{2}}q(\mathbf{v};t)=0. \tag{4.10}$$

Now, simply by substituting $y_1(\mathbf{v};t)=K^{\frac{1}{2}}M^{-\frac{1}{2}}q(\mathbf{v};t)$ and $y_2(\mathbf{v};t)=\dot{q}(\mathbf{v};t)$ we obtain the following:

$$\begin{aligned}
\dot{y}_1(\mathbf{v};t) &= K^{\frac{1}{2}}M^{-\frac{1}{2}}\dot{q}(\mathbf{v};t)\\
&= K^{\frac{1}{2}}M^{-\frac{1}{2}}y_2(\mathbf{v};t),\\
\dot{y}_2(\mathbf{v};t) &= \ddot{q}(\mathbf{v};t)\\
&= -M^{-\frac{1}{2}}C(\mathbf{v})M^{-\frac{1}{2}}\dot{q}(\mathbf{v};t)-M^{-\frac{1}{2}}K^{\frac{1}{2}}K^{\frac{1}{2}}M^{-\frac{1}{2}}q(\mathbf{v};t),\\
&= -M^{-\frac{1}{2}}C(\mathbf{v})M^{-\frac{1}{2}}y_2(\mathbf{v};t)-M^{-\frac{1}{2}}K^{\frac{1}{2}}y_1(\mathbf{v};t).
\end{aligned} \tag{4.11}$$

From (4.11) obtain the following linearization:

$$\dot{\mathbf{y}}(\mathbf{v};t)=\mathbf{A}_*(\mathbf{v})\mathbf{y}(\mathbf{v};t), \tag{4.12}$$

where

$$\mathbf{y}(\mathbf{v};t)=\begin{bmatrix}y_1(\mathbf{v};t)\\ y_2(\mathbf{v};t)\end{bmatrix},\quad \mathbf{A}_*(\mathbf{v})=\begin{bmatrix}0 & K^{\frac{1}{2}}M^{-\frac{1}{2}}\\ -M^{-\frac{1}{2}}K^{\frac{1}{2}} & -M^{-\frac{1}{2}}C(\mathbf{v})M^{-\frac{1}{2}}\end{bmatrix}. \tag{4.13}$$

Since we are interested in minimization of total average energy as a criterion, as we have shown in Section 1.2, it follows that we are interested in minimization of the trace

$$\mathrm{tr}(\mathbf{Z}\mathbf{Y}(\mathbf{v})),$$

where $\mathbf{Y}(\mathbf{v})$ is a solution of the Lyapunov equation

$${\mathbf{A}_*}^T(\mathbf{v})\mathbf{Y}(\mathbf{v})+\mathbf{Y}(\mathbf{v})\mathbf{A}_*(\mathbf{v})=-\mathbf{I}, \tag{4.14}$$

and $\mathbf{Z}$ is defined in (1.35).

Next we present a new approach to approximation of the solution of Lyapunov equation

(4.14), which is different from the standard ones, that are based on modal approximation of mechanical systems. Our approach combines two aspects, one is a modal approximation approach and the other is an approach based on the improved error estimates, see e.g. [12], [13] and [8].

For that purpose let

$$M^{-\frac{1}{2}}C(\mathbf{v})M^{-\frac{1}{2}} = U_0(\mathbf{v})\Delta(\mathbf{v})U_0^T(\mathbf{v}), \qquad \Delta(\mathbf{v}) = \operatorname{diag}(\delta_1(\mathbf{v}),\ldots,\delta_n(\mathbf{v})), \tag{4.15}$$

be the eigenvalue decomposition of the "damping matrix" $M^{-\frac{1}{2}}C(\mathbf{v})M^{-\frac{1}{2}}$.

Let

$$\mathbf{T}(\mathbf{v}) = \begin{bmatrix} I & 0 \\ 0 & U_0(\mathbf{v}) \end{bmatrix} \tag{4.16}$$

be the orthogonal matrix, where $U_0(\mathbf{v})$ is defined in (4.15). If one multiplies Lyapunov equation (4.14) from the left and from right by $\mathbf{T}^T(\mathbf{v})$ and $\mathbf{T}(\mathbf{v})$, respectively, then one gets

$$\mathbf{A}^T(\mathbf{v})\mathbf{X}(\mathbf{v}) + \mathbf{X}(\mathbf{v})\mathbf{A}(\mathbf{v}) = -\mathbf{I}, \tag{4.17}$$

where

$$\mathbf{A}(\mathbf{v}) = \mathbf{T}^T(\mathbf{v})\mathbf{A}_*(\mathbf{v})\mathbf{T}(\mathbf{v}) = \begin{bmatrix} 0 & N^T(\mathbf{v}) \\ -N(\mathbf{v}) & -\Delta(\mathbf{v}) \end{bmatrix}, \tag{4.18}$$

where $N(\mathbf{v}) = U_0^T(\mathbf{v})M^{-\frac{1}{2}}K^{\frac{1}{2}}$ and

$$\mathbf{X}(\mathbf{v}) = \begin{bmatrix} X_{11}(\mathbf{v}) & X_{12}(\mathbf{v}) \\ X_{12}^T(\mathbf{v}) & X_{22}(\mathbf{v}) \end{bmatrix}. \tag{4.19}$$

For the sake of easier notation in this section, from now on we omit writing of the parameter $\mathbf{v}$, if not necessary.

Now equation (4.17) can be written as

$$\begin{bmatrix} 0 & -N^T \\ N & -\Delta \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} 0 & N^T \\ -N & -\Delta \end{bmatrix} = -\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{4.20}$$

where

$$\mathbf{Y} = \mathbf{T} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \mathbf{T}^T. \tag{4.21}$$

It obviously holds that $\operatorname{tr}(\mathbf{ZY}) = \operatorname{tr}(\mathbf{Z_U X})$, where

$$\mathbf{Z_U} = \mathbf{T}^T\mathbf{Z}\mathbf{T}. \tag{4.22}$$

Now, from (4.20) one gets

$$N^T X_{12}^T + X_{12}N = I, \tag{4.23}$$

$$-N^T X_{22} + X_{11} N^T - X_{12}\Delta = 0, \tag{4.24}$$

$$N X_{11} - X_{22} N - \Delta X_{12}^T = 0, \tag{4.25}$$

$$N X_{12} + X_{12}^T N^T - \Delta X_{22} - X_{22}\Delta = -I. \tag{4.26}$$

Since $X_{12}N \in \mathbb{R}^n$, we can write it as

$$X_{12}N = \frac{X_{12}N + N^T X_{12}^T}{2} + S, \tag{4.27}$$

where $\frac{X_{12}N+N^T X_{12}^T}{2}$ is symmetric part of the matrix $X_{12}N$ and $S = \frac{X_{12}N-N^T X_{12}^T}{2}$ is skew-symmetric part of the matrix $X_{12}N$, i.e., $S^T = -S$.

Now, from (4.23) and (4.27) it follows

$$N^T X_{12}^T + X_{12} N \;\; = I \iff 2X_{12}N = I + 2S \tag{4.28}$$

Therefore, from (4.28) it follows

$$X_{12} = \frac{1}{2} N^{-1} + S N^{-1}, \quad \text{where} \quad S = -S^T. \tag{4.29}$$

Thus, if one knows the skew-symmetric matrix S from (4.29), then the solution $\mathbf{X}$ is known.

A new approach: as described at the beginning of the section, our approach is based on some interesting properties of a modally damped system. As is well known (see e.g. [112, Theorem 2.3]), the modally damped system satisfies the so-called commuting condition

$$CK^{-1}M = MK^{-1}C.$$

It can also be shown, provided that inverses exist, that the above equality is equivalent to

$$KC^{-1}M = MC^{-1}K \text{ and also to } CM^{-1}K = KM^{-1}C. \tag{4.30}$$

From (4.30) and (1.18) one can see that the mechanical system can be modally damped, i.e., that all three matrices M, C and K can be simultaneously diagonalized, even if some of them are singular.

Next we show that this assumption is equivalent to the assumption on commuting NN^T and Δ^{-1}, i.e., we state the following lemma.

Lemma 4.1 Let $N = U_0^T M^{-\frac{1}{2}} K^{\frac{1}{2}}$ and $\Delta = U_0^T M^{-\frac{1}{2}} C M^{-\frac{1}{2}} U_0$. Then

$$NN^T\Delta^{-1} = \Delta^{-1}NN^T \tag{4.31}$$

if and only if the mechanical system from (1.2) is modally damped, i.e., the left equality from (4.30) holds.

Proof. Since $N = U_0^T M^{-\frac{1}{2}} K^{\frac{1}{2}}$ and $\Delta = U_0^T M^{-\frac{1}{2}} C M^{-\frac{1}{2}} U_0$, it follows that

$$NN^T = U_0^T M^{-\frac{1}{2}} K M^{-\frac{1}{2}} U_0 \quad \text{and} \quad \Delta^{-1} = U_0^T M^{\frac{1}{2}} C^{-1} M^{\frac{1}{2}} U_0.$$

Now, simple multiplication gives

$$\begin{aligned} NN^T \Delta^{-1} &= U_0^T M^{-\frac{1}{2}} K C^{-1} M^{\frac{1}{2}} U_0, \\ \Delta^{-1} NN^T &= U_0^T M^{\frac{1}{2}} C^{-1} K M^{-\frac{1}{2}} U_0, \end{aligned}$$

which together gives the following equivalence

$$\begin{aligned} NN^T\Delta^{-1} = \Delta^{-1}NN^T &\iff && M^{-\frac{1}{2}} K C^{-1} M^{\frac{1}{2}} = M^{\frac{1}{2}} C^{-1} K M^{-\frac{1}{2}} \\ &\iff M^{\frac{1}{2}} \cdot \backslash && M^{-\frac{1}{2}} K C^{-1} M^{\frac{1}{2}} = M^{\frac{1}{2}} C^{-1} K M^{-\frac{1}{2}} \quad / \cdot M^{\frac{1}{2}} \\ &\iff && KC^{-1}M = MC^{-1}K. \end{aligned}$$

□

The assumption from Lemma 4.1 is equivalent to the assumption on commuting X_{12} and N, i.e., if and only if

$$X_{12}N = NX_{12}, \tag{4.32}$$

then (4.30) holds, and the mechanical system is modally damped.

If X_{12} and N commute, i.e., if (4.32) holds, then also $X_{12}^T N^T = N^T X_{12}^T$ and $NX_{12} + X_{12}^T N^T = I$. Therefore, if we subtract (4.26) from (4.23) we obtain

$$\Delta X_{22} + X_{22}\Delta = 2I. \tag{4.33}$$

Now from (4.25) we obtain

$$X_{11} = N^{-1} X_{22} N + N^{-1} \Delta X_{12}^T. \tag{4.34}$$

Furthermore, the assumption that (4.32) holds if and only if S from (4.29) is a zero matrix, i.e., if $S = 0$ then (4.32) trivially holds. On the other hand if (4.32) holds, then

$$\begin{aligned} \left(\frac{1}{2}N^{-1} + SN^{-1}\right) N &= N \left(\frac{1}{2}N^{-1} + SN^{-1}\right), && (4.35) \\ \frac{1}{2}I + S &= \frac{1}{2}I + NSN^{-1}, && (4.36) \\ S &= NSN^{-1}, && (4.37) \end{aligned}$$

which means that S is similar to itself for every matrix N, but since N is not fixed (it depends

on the choice of matrices $M, C(\mathbf{v}), K$ the only two matrices for which (4.37) is satisfied are the identity and zero matrix. S is skew-symmetric, thus $S = 0$.

As it is shown in [23], when the system is a modally damped Lyapunov equation (4.17) has a unique solution and the explicit formula for the solution can be obtained.

Lemma 4.2 The mechanical system is modally damped if and only if the solution of Lyapunov equation (4.17) is

$$\mathbf{X} = \begin{bmatrix} N^{-1}\Delta^{-1}N + \frac{1}{2}N^{-1}\Delta N^{-T} & \frac{1}{2}N^{-1} \\ \frac{1}{2}N^{-T} & \Delta^{-1} \end{bmatrix}. \tag{4.38}$$

Proof. Uniqueness of the solution of the Lyapunov equation follows directly from the fact that **A** given by (4.18) is a Hurwitz matrix and the right hand side of the Lyapunov equation is a negative definite matrix, see [27]. Now remains to check whether X given by (4.38) is really the solution of Lyapunov equation (4.17).

$$\begin{aligned} \begin{bmatrix} 0 & -N^T \\ N & -\Delta \end{bmatrix} \mathbf{X} + \mathbf{X} \begin{bmatrix} 0 & N^T \\ -N & -\Delta \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2}I - \frac{1}{2}I & -N^T\Delta^{-1} + N^{-1}\Delta^{-1}NN^T \\ \Delta^{-1}N - \Delta^{-1}N & -\frac{1}{2}I - \frac{1}{2}I \end{bmatrix} \\ &= \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}, \end{aligned}$$

since Δ^{-1} and NN^T commute, $N^{-1}\Delta^{-1}NN^T = N^{-1}NN^T\Delta^{-1} = N^T\Delta^{-1}$. □

The main idea: Now we do not assume that (4.32) holds, i.e., our mechanical system is no longer modally damped.

But if it is still "good in some sense", or "close" to a modally damped system, we can use the above conclusions to approximate the solution **X** of Lyapunov equation (4.17).

For that purpose, we approximate X_{12} from (4.29) with

$$\widetilde{X}_{12} = \frac{1}{2}N^{-1}. \tag{4.39}$$

Further, from (4.33) it follows that

$$\widetilde{X}_{22} = \Delta^{-1}. \tag{4.40}$$

Once we have derived $\widetilde{X}_{22}$, it is easy to derive the last unknown approximation $\widetilde{X}_{11}$. Indeed, from (4.34) it follows

$$\widetilde{X}_{11} = N^{-1}\widetilde{X}_{22}N + \frac{1}{2}N^{-1}\Delta N^{-T} = N^{-1}\Delta^{-1}N + \frac{1}{2}N^{-1}\Delta N^{-T}. \tag{4.41}$$

In the next theorem, we present the residual error

$$\mathrm{R_{er}} = \|\mathbf{A}^T\widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}\mathbf{A} + \mathbf{I}\| \tag{4.42}$$

obtained by the approximation

$$\widetilde{\mathbf{X}} = \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix}, \tag{4.43}$$

which is equivalent to inserting approximations $\widetilde{X}_{11}$, $\widetilde{X}_{22}$ and $\widetilde{X}_{12}$ into (4.24).

The following theoretical results are used for error estimation made by the above approximation of the solution of the Lyapunov equation. They are also used for the a priori estimation, whether the considered mechanical system can be approximated with a modally damped one.

Theorem 4.3 Let $\widetilde{\mathbf{X}}$ be the approximation of solution (4.19) of Lyapunov equation (4.20) obtained by (4.39), (4.40) and (4.41). Then the residual error $\mathrm{R_{er}}$ is given by

$$\mathrm{R_{er}} = \|N^T\Delta^{-1} - N^{-1}\Delta^{-1}NN^T\|. \tag{4.44}$$

Proof. The proof simply follows by inserting $\widetilde{\mathbf{X}}$ into (4.20). Indeed, from (4.43) and (4.20) one gets

$$\begin{bmatrix} 0 & -N^T \\ N & -\Delta \end{bmatrix} \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix} + \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix} \begin{bmatrix} 0 & N^T \\ -N & -\Delta \end{bmatrix} = - \begin{bmatrix} I & \mathrm{Err} \\ 0 & I \end{bmatrix}, \tag{4.45}$$

where

$$-\mathrm{Err} = -N^T\widetilde{X}_{22} + N^{-1}\widetilde{X}_{22}NN^T + \frac{1}{2}\left(-N^{-1}\Delta + N^{-1}\Delta\right).$$

Now, since

$$\begin{aligned} \mathrm{R_{er}} &= \|\mathbf{A}^T\widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}\mathbf{A} + \mathbf{I}\| \\ &= \| \begin{bmatrix} 0 & \mathrm{Err} \\ 0 & 0 \end{bmatrix} \| \\ &= \|\mathrm{Err}\|, \end{aligned}$$

(4.44) holds, which completes the proof. □

As a consequence of the above theorem and Lemma 4.2, we have that $\mathrm{R_{er}} = 0$, which implies that $\widetilde{\mathbf{X}} = \mathbf{X}$ if and only if

$$NN^T\Delta^{-1} = \Delta^{-1}NN^T,$$

i.e., the system is modally damped.

4.1.2 Total average energy optimization

Using the approximation from (4.43), here we present a new approach to damping optimization. Thus, we assume that the considered mechanical system is close to the perturbed modally damped system, i.e., further on we assume that the residual error $\mathrm{R_{er}}$ from (4.44) is small enough, which means that $KC^{-1}M \approx MC^{-1}K$ in some sense.

For that purpose, let

$$M = U_M \Lambda_M U_M^T, \quad U_M = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}, \quad \Lambda_M = \begin{bmatrix} \mu_1 & \dots & \mu_n \end{bmatrix},$$

be the eigenvalue decomposition of the mass matrix M.

We distinguish two different cases. In the first case, we assume that the damping matrix C has the same eigenvector structure as the mass matrix M, i.e., we assume that

$$C_I = \nu_1 u_1 u_1^T + \nu_2 u_2 u_2^T + \ldots + \nu_n u_n u_n^T, \tag{4.46}$$

where $\nu_i = v_i + \alpha_c$, $i = 1, \ldots, n$.

For the damping matrix close to C_I, and in the case when the number of dampers is equal to the dimension, i.e., when $s = n$, we are able to derive the explicit formula for optimal damping viscosities v_i, $i = 1, \ldots, n$. On the other hand, for the case when the number of dampers is less than the dimension or some viscosities are the same, we present a formula that covers these cases in a more general setting.

Thus, back to the first case, we assume that the damping matrix C is close to C_I from (4.46), i.e.,

$$C \approx (v_1 + \alpha_c) u_1 u_1^T + (v_2 + \alpha_c) u_2 u_2^T + \ldots + (v_n + \alpha_c) u_n u_n^T. \tag{4.47}$$

Below we derive a simple formula for calculation of optimal viscosities $\mathbf{v}$, for which the trace of the approximation $\widetilde{\mathbf{X}}$ from (4.43) is minimal.

If we are interested in damping the s undamped frequencies, then using the matrix Z from (1.35) we obtain that for the matrix Z_U from (4.22) it can be written as

$$\mathbf{Z_U} \doteq \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix},$$

where

$$Z_1 = \mathrm{diag}(0_{t_1}, I_{t_2}, 0_{t_3}), \tag{4.48}$$

$$Z_2 = U_0^T \mathrm{diag}(0_{t_1}, I_{t_2}, 0_{t_3}) U_0. \tag{4.49}$$

Since our penalty function is a trace of the solution of the corresponding Lyapunov equation, note that for the approximation of the trace it holds that

$$\begin{aligned}\operatorname{tr}(\mathbf{Z_U X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) &= \operatorname{tr}(Z_1\widetilde{X}_{11}) + \operatorname{tr}(Z_2\widetilde{X}_{22}) \\ &= \operatorname{tr}(Z_1\widetilde{X}_{22}) + \operatorname{tr}(Z_2\widetilde{X}_{22}) + \frac{1}{2}\operatorname{tr}(N^{-1}Z_1\Delta N^{-T}),\end{aligned}$$

i.e.,

$$\operatorname{tr}(\mathbf{Z_U X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) = \operatorname{tr}(Z_1\Delta^{-1}) + \operatorname{tr}(Z_2\Delta^{-1}) + \frac{1}{2}\operatorname{tr}(Z_1\Delta N^{-T}N^{-1}). \tag{4.50}$$

Approximation (4.50) is our starting point, which allows us to derive an approximation for optimal $\mathbf{v}^* \in \mathbb{R}^n_+$.

Note that from (4.47) and (4.15) it follows that

$$M^{-\frac{1}{2}}CM^{-\frac{1}{2}} = U_0\Delta U_0^T, \quad \text{where } \Delta = \operatorname{diag}(\mathbf{v}+\alpha_c). \tag{4.51}$$

Now from (4.50) and (4.51) one gets

$$\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(\mathbf{v})) = \sum_{i=1}^{n}\frac{(Z_1)_{ii}+(Z_2)_{ii}}{v_i+\alpha_c} + \frac{1}{2}\sum_{i=1}^{n}(v_i+\alpha_c)(Z_1)_{ii}w_i, \tag{4.52}$$

where $w_i = \|W(:,i)\|^2, \quad W = N^{-1}$ for $i = 1,\ldots,n$.

Using the fact that all quantities in (4.52) are nonnegative, simply by using partial derivatives

$$\frac{\partial \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(\mathbf{v}))}{\partial v_i} = -\frac{(Z_1)_{ii}+(Z_2)_{ii}}{(v_i+\alpha_c)^2} + \frac{1}{2}(Z_1)_{ii}w_i, \quad i = 1,\ldots,n, \tag{4.53}$$

and for $i = 1,\ldots,n$ from

$$\frac{\partial \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(\mathbf{v}))}{\partial v_i} = 0 \tag{4.54}$$

$$-\frac{(Z_1)_{ii}+(Z_2)_{ii}}{(v_i+\alpha_c)^2} + \frac{1}{2}(Z_1)_{ii}w_i = 0, \tag{4.55}$$

$$\frac{(v_i+\alpha_c)^2}{(Z_1)_{ii}+(Z_2)_{ii}} = \frac{2}{(Z_1)_{ii}w_i} \tag{4.56}$$

one gets that

$$v_i^* = \sqrt{\frac{2(Z_1)_{ii}+2(Z_2)_{ii}}{(Z_1)_{ii}w_i}} - \alpha_c, \qquad i = 1,\ldots,n.$$

are minimum for the trace (since $\mathcal{H}(\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(\mathbf{v}))) > 0$), i.e.,

$$\mathbf{v}^* = \arg\min \mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(\mathbf{v})).$$

The setting given by (4.46) was just a motivation for a more general case that is considered in the next subsubsection.

4.1.2.1 Damping optimization for the structured case

Throughout this subsubsection we assume that the eigenvalue decomposition of the matrix $M^{-\frac{1}{2}}CM^{-\frac{1}{2}}$ is given by

$$M^{-\frac{1}{2}}CM^{-\frac{1}{2}} = U_0\Delta U_0^T, \quad \Delta = v_1C_1 \oplus v_2C_2 \oplus \cdots \oplus v_sC_s, \tag{4.57}$$

where each matrix C_i, $i = 1,\ldots,s$ is a diagonal matrix and it has a dimension d_i, $i = 1,\ldots,s$, respectively, with $\sum_{i=1}^{s} d_i = n$.

The above assumption means that the matrix Δ is a direct sum of smaller matrices that correspond to the same viscosities and it arises from the fact that very often the damping matrix C can have blocks of dampers which have the same viscosities. Moreover, in the assumed setting, damping blocks with different viscosities do not interlace with each other.

Note that the setting included in (4.46) is also covered by (4.57) since we can also use this approach in the case when $d_i = 1$, $i = 1,\ldots,n$, considering that all viscosities are different. On the other hand, we would like to emphasize that damping of the form (4.57) generalizes damping studied in the previous part, (4.46). Moreover, it also includes more general cases in which the damping matrix C is permutation similar to the block diagonal matrix where each block corresponds to damping parts with its own viscosity parameter.

If we are interested in damping the first s most important eigenfrequencies, then the matrix $\mathbf{Z_U}$ from (4.22) can be written as

$$\mathbf{Z_U} = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix},$$

where Z_1 and Z_2 are given by (4.48) - (4.49).

Also, for the approximation of the trace it holds

$$\mathrm{tr}(\mathbf{Z_U}\mathbf{X}) \approx \mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) = \mathrm{tr}(Z_1\widetilde{X}_{22}) + \mathrm{tr}(Z_2\widetilde{X}_{22}) + \frac{1}{2}\mathrm{tr}(N^{-1}Z_1\Delta N^{-T}),$$

i.e.,

$$\mathrm{tr}(\mathbf{Z_U}\mathbf{X}) \approx \mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) = \mathrm{tr}(Z_1\Delta^{-1}) + \mathrm{tr}(Z_2\Delta^{-1}) + \frac{1}{2}\mathrm{tr}(Z_1\Delta N^{-T}N^{-1}). \tag{4.58}$$

Now using approximation (4.58) together with (4.57) we can derive approximate optimal parameters $\mathbf{v}^* \in \mathbb{R}^s_+$.

In particular, from (4.57) and (4.58) one gets

$$\begin{aligned}\operatorname{tr}(\mathbf{Z}_\mathbf{U}\widetilde{\mathbf{X}}(\mathbf{v})) = &\sum_{i=1}^{s}\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}\frac{(Z_1)_{jj}+(Z_2)_{jj}}{v_i(C_i)_{k_jk_j}+\alpha_c}\\ &+\frac{1}{2}\sum_{i=1}^{s}\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}(Z_1)_{jj}w_j(v_i(C_i)_{k_jk_j}+\alpha_c),\end{aligned} \tag{4.59}$$

where k_j determines the index that depends on j and it holds that $k_j = j-(d_1+\cdots+d_{j-1})$. Moreover, w_i is the square of the 2-norm of the column of the matrix $W = N^{-1}$, i.e., $w_i = \|W(:,i)\|^2$.

In general, for this function we are not able to derive an explicit formula for optimal viscosities. But since in this case, where the matrix which diagonalizes the matrix $M^{-\frac{1}{2}}CM^{-\frac{1}{2}}$ is the same for all viscosities, we can determine optimal viscosities efficiently by a numerical optimization procedure which is described later in Remark 4.1.

Additionally, we are also able to derive an explicit formula for a global minimum if $\sum_{i=1}^{s}\operatorname{rank}(C_i) = n$ and $\alpha_c = 0$. In that case, our objective function has the following form:

$$\begin{aligned}\operatorname{tr}(\mathbf{Z}_\mathbf{U}\widetilde{\mathbf{X}}(\mathbf{v})) = &\sum_{i=1}^{s}\frac{1}{v_i}\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}\frac{(Z_1)_{jj}+(Z_2)_{jj}}{(C_i)_{k_jk_j}}\\ &+\frac{1}{2}\sum_{i=1}^{s}v_i\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}(Z_1)_{jj}w_j(C_i)_{k_jk_j}.\end{aligned} \tag{4.60}$$

Using the fact that all quantities in (4.59) are nonnegative, one easily obtains partial derivatives

$$\frac{\partial\operatorname{tr}(\mathbf{Z}_\mathbf{U}\widetilde{\mathbf{X}}(\mathbf{v}))}{\partial v_i} = -\frac{\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}\frac{(Z_1)_{jj}+(Z_2)_{jj}}{(C_i)_{k_jk_j}}}{v_i^2}+\frac{1}{2}\sum_{j=d_1+\cdots+d_{i-1}}^{d_1+\cdots+d_i}(Z_1)_{jj}w_j(C_i)_{k_jk_j}, \tag{4.61}$$

for $i = 1,\ldots,s$. Now, from

$$\frac{\partial\operatorname{tr}(\mathbf{Z}_\mathbf{U}\widetilde{\mathbf{X}}(\mathbf{v}))}{\partial v_i} = 0, \quad \forall i = 1,\ldots,s, \tag{4.62}$$

one gets that

$$v_i^* = \sqrt{\frac{2\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i}\frac{(Z_1)_{jj}+(Z_2)_{jj}}{(C_i)_{k_jk_j}}}{\sum_{j=d_1+\cdots+d_{i-1}}^{d_1+\cdots+d_i}(Z_1)_{jj}w_j(C_i)_{k_jk_j}}}, \qquad i = 1,\ldots,s, \tag{4.63}$$

are optimal viscosities, i.e.,

$$\mathbf{v}^* = \arg\min \operatorname{tr}(\mathbf{Z}_{\mathbf{U}}\widetilde{\mathbf{X}}(\mathbf{v})).$$

Remark 4.1. The objective function given by (4.59) for the parameter $\alpha_c \neq 0$ can be efficiently optimized using a numerical optimization procedure. In particular, in this case we deal with the minimization of s functions where the ith function f_i is given by

$$f_i(v_i) = \sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} \frac{(Z_1)_{jj}+(Z_2)_{jj}}{v_i(C_i)_{k_jk_j}+\alpha_c} + \frac{1}{2}\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} (Z_1)_{jj}w_j(v_i(C_i)_{k_jk_j}+\alpha_c), \quad (4.64)$$

for $i = 1,\ldots,s$. Here the function f_i is a strictly convex function with global minima v_i^*, for $i = 1,\ldots,s$ respectively, where minima v_i^*, for $i = 1,\ldots,s$ can be efficiently determined using iterative solvers. By this approach we are able to determine optimal parameters v_i^*, for $i = 1,\ldots,s$ that minimize $\operatorname{tr}(\mathbf{Z}_{\mathbf{U}}\widetilde{\mathbf{X}}(\mathbf{v}))$.

4.1.2.2 Damping optimization for the more general structure

In this subsubsection, we consider a more general case, than the two cases already presented in Subsection 4.1.2 , but still we assume that our system corresponds to the configuration where (4.44) is small enough, or that approximation $KC^{-1}M \approx MC^{-1}K$ holds in a certain sense.

Since M is a positive definite and C is a positive semidefinite matrix, there exists an orthogonal matrix U such that

$$M^{-\frac{1}{2}}CM^{-\frac{1}{2}} = U\Delta U^T, \quad \Delta = \operatorname{diag}(\delta_1,\ldots,\delta_n). \quad (4.65)$$

Apart from previous cases where we are able to derive an explicit formula for global minima, in this subsubsection we present a numerical approach to calculation of an approximation of optimal viscosities. The main problem within this *general case* is that the matrix U, which diagonalizes the matrix $M^{-\frac{1}{2}}CM^{-\frac{1}{2}}$, depends on viscosities that determine the damping matrix C, contrary to the cases that are already presented in Subsection 4.1.2 .

Thus, let us assume that s dampers with corresponding viscosities v_i, $i = 1,\ldots,s$ are given, which determine our external damping matrix $C_{\text{ext}}(\mathbf{v})$, i.e., the damping matrix is given by $C(\mathbf{v}) = C_{\text{int}} + C_{\text{ext}}(\mathbf{v})$. Since in general the matrix U from (4.65) depends on viscosities, let U_0 be a unitary matrix which diagonalizes $C(\mathbf{v}^0)$ for initial viscosities $\mathbf{v}^0$.

Now, similar to the beginning of this subsection, we can calculate approximation of the trace of the solution of the corresponding Lyapunov equation for the given viscosities $(\mathbf{v}^0)$. As above, we can show that

$$\operatorname{tr}(\mathbf{Z}_{\mathbf{U}}\widetilde{\mathbf{X}}(\mathbf{v}^0)) = \sum_{i=1}^{n} \frac{(Z_2)_{ii}+(Z_1)_{ii}}{\delta_i} + \frac{1}{2}\sum_{i=1}^{n} \delta_i(Z_1)_{ii}w_i, \quad (4.66)$$

where Z_1 and Z_2 are given by (4.48-4.49) and

$$M^{-\frac{1}{2}}(C(\mathbf{v}^0))M^{-\frac{1}{2}} = U_0 \Delta U_0^T, \qquad \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n), \tag{4.67}$$

$$w_i = \|W_0(:,i)\|^2, \qquad W_0 = N^{-1}, \quad i = 1, \ldots, n. \tag{4.68}$$

Now, we do not have an explicit formula for optimal viscosities, thus we propose the following numerical approach to viscosity optimization.

During the optimization process, the next iteration (for viscosities) ($\mathbf{v}^1$) can be calculated using corresponding matrix U_1, given as

$$M^{-\frac{1}{2}}(C(\mathbf{v}^1))M^{-\frac{1}{2}} = U_1 \Delta U_1^T, \quad \Delta = \mathrm{diag}(\delta_1^1, \ldots, \delta_n^1),$$

which ensures the corresponding trace approximation. It is important to notice here that very often, during the optimization process, the same subspace U is also good for several iteration steps (i.e., for several viscosity updates). Thus, during the optimization process, we first check if the same subspace is good enough, meaning that the residual error

$$\mathrm{err}_{\mathrm{U}} = \|M^U - \mathrm{diag}(M_{11}^U, M_{22}^U, \ldots, M_{nn}^U)\| < \mathtt{tol}_{\mathrm{U}}, \tag{4.69}$$

where $M^U = U_0^T M^{-\frac{1}{2}}(C_{\mathrm{int}} + C(\mathbf{v}^1))M^{-\frac{1}{2}}U_0$ and $\mathtt{tol}_{\mathrm{U}}$ is some given tolerance.

This means that in the optimization process, if $\mathrm{err}_{\mathrm{U}}$ defined in (4.69) for viscosities $\mathbf{v}^1$ is smaller than the tolerance $\mathtt{tol}_{\mathrm{U}}$, we use the unitary matrix U_0 instead of computing U_1 for approximation of the trace $\mathrm{tr}(\mathbf{Z}_{\mathbf{U}}\widetilde{\mathbf{X}}(\mathbf{v}^1))$.

The algorithm for the optimization of viscosities is summarized in Algorithm 6. We would like to emphasize that the main cost in trace approximation (4.66) belongs to the calculation of the matrix U; thus in Algorithm 6, by using the residual tolerance $\mathtt{tol}_{\mathrm{U}}$ we can avoid calculation of the matrix U for some viscosities $\mathbf{v}^i$ (for some i's), which significantly accelerates the optimization process. Moreover, as one could expect, the matrix U does not need to be calculated (up to tolerance $\mathtt{tol}_{\mathrm{U}}$) in each step of iterations (for various viscosities $\mathbf{v}^i$) if a mechanical system has some special structure or if the changes in viscosities are small (which often appears during the optimization process).

Remark 4.2. Note that by using Algorithm 6, we can also minimize objective functions given by (4.59). In that case, the $\mathrm{err}_{\mathrm{U}}$ is zero (up to machine tolerance) and we are able to calculate approximation of the objective function without calculating the matrix U in each step. On the other hand, since in this case the objective function consists of s independent functions, it is even more efficient to use an approach described in Remark 4.1.

Algorithm 6: Computation of optimal viscosities

Require: System matrices; tolerance $\texttt{tol}_\text{U}$ for updating eigensubspace U; starting viscosities $\mathbf{v}^0$.

Ensure: Approximation of optimal viscosities.

1: Calculate approximation of the trace given in (4.66) and U_0 given in (4.67). Set $U = U_0$.

2: Find optimal viscosities by using an appropriate optimization procedure (e.g. the Nelder-Mead algorithm). Evaluate the function value using trace approximation at the given viscosities $\mathbf{v}^i$ as in Steps 3 to 8:

3: Calculate the error for the subspace U from

$$\text{err}_\text{U} = \|M^U - \text{diag}(M^U_{11}, M^U_{22}, \ldots, M^U_{nn})\|,$$

where $M^U = U^T M^{-\frac{1}{2}}(C_\text{int} + C_\text{ext}(\mathbf{v}^i))M^{-\frac{1}{2}}U$.

4: **if** $\text{err}_\text{U} < \texttt{tol}_\text{U}$, **then**

5: Compute the function value at viscosities $(\mathbf{v}^i)$ using

$$\text{tr}(\mathbf{Z}_\mathbf{U}\widetilde{\mathbf{X}}(\mathbf{v}^i)) = \sum_{i=1}^{n} \frac{(Z_2)_{ii} + (Z_1)_{ii}}{\delta_i} + \frac{1}{2}\sum_{i=1}^{n} \delta_i (Z_1)_{ii} w_i,$$

where Z_1 and Z_2 are given by (4.48) - (4.49) and

$$\Delta = \text{diag}(\delta_1, \ldots, \delta_n), \quad \delta_i = (U^T M^{-\frac{1}{2}}(C_\text{int} + C_\text{ext}(\mathbf{v}^i))M^{-\frac{1}{2}}U)_{ii}\,, \; i = 1, \ldots, n.$$

6: **else**

7: Compute new U and Δ, such that

$$M^{-\frac{1}{2}}(C_\text{int} + C_\text{ext}(\mathbf{v}^i))M^{\frac{1}{2}} = U\Delta U^T, \quad \Delta = \text{diag}(\delta_1, \ldots, \delta_n).$$

Compute the function value at viscosities $\mathbf{v}^i$ using formula

$$\text{tr}(\mathbf{Z}_\mathbf{U}\widetilde{\mathbf{X}}(\mathbf{v}^i)) = \sum_{i=1}^{n} \frac{(Z_2)_{ii} + (Z_1)_{ii}}{\delta_i} + \frac{1}{2}\sum_{i=1}^{n} \delta_i (Z_1)_{ii} w_i,$$

where $w_i = \|W_0(:,i)\|^2, \quad W_0 = N^{-1}, \quad i = 1, \ldots, n$ and Z_1, Z_2 are given by (4.48) and (4.49).

8: **end if**

4.1.3 Numerical experiments

In this subsection, we illustrate the performance of the new approach on damping optimization based on considering minimization of total average energy. In both illustrative examples we consider the mechanical system, the so-called the *n-mass oscillator* which was already mentioned in examples in previous chapters. In all examples, we take $\mathbf{Z_U} = \mathbf{I}$.

Example 4.2 In this example, we consider the system from (1.2) with dimension $n = 20$, where the mass and stiffness matrices are defined as:

$$
\begin{aligned}
M &= \operatorname{diag}(m_1, m_2, \ldots, m_n), \\
m_i &= \begin{cases} 200 - 20(i-1), & i = 1, \ldots, 10 \\ 201 + 20(i-11), & i = 11, \ldots, 20 \end{cases}, \\
K &= \begin{bmatrix}
4 & -1 & -1 & & & & & \\
-1 & 4 & -1 & -1 & & & & \\
-1 & -1 & 4 & -1 & -1 & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & -1 & -1 & 4 & -1 & -1 & \\
 & & & -1 & -1 & 4 & -1 & \\
 & & & & -1 & -1 & 4 &
\end{bmatrix}.
\end{aligned}
$$

The structure of masses and stiffness is shown in Figure 4.6.

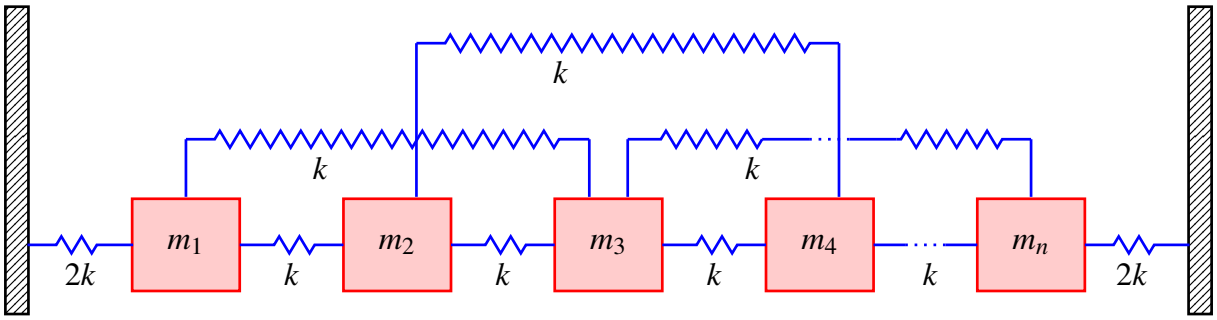

Figure 4.6: n-mass oscillator

Further, the damping matrix $C(\mathbf{v})$ is given as $C(\mathbf{v}) = C_{\text{int}} + C_{\text{ext}}(\mathbf{v})$, where the external damping matrix has a block diagonal structure $C_{\text{ext}}(\mathbf{v}) = \operatorname{diag}(C_1(\mathbf{v}), C_2(\mathbf{v}), \ldots, C_{10}(\mathbf{v}))$, where each block has its own viscosity v_i for $i = 1, \ldots, 10$.

The blocks are defined as:

$$
C_i(\mathbf{v}) = \begin{bmatrix} v_i + v_i p & -v_i p & 0 \\ -v_i p & v_i + 2v_i p & -v_i p \\ 0 & -v_i p & v_i + v_i p \end{bmatrix}, i = 1, 2, 3, \tag{4.70}
$$

$$
C_i(\mathbf{v}) = \begin{bmatrix} v_i + v_i p & -v_i p \\ -v_i p & v_i + v_i p \end{bmatrix}, i = 4, 5, 6, 7, \tag{4.71}
$$

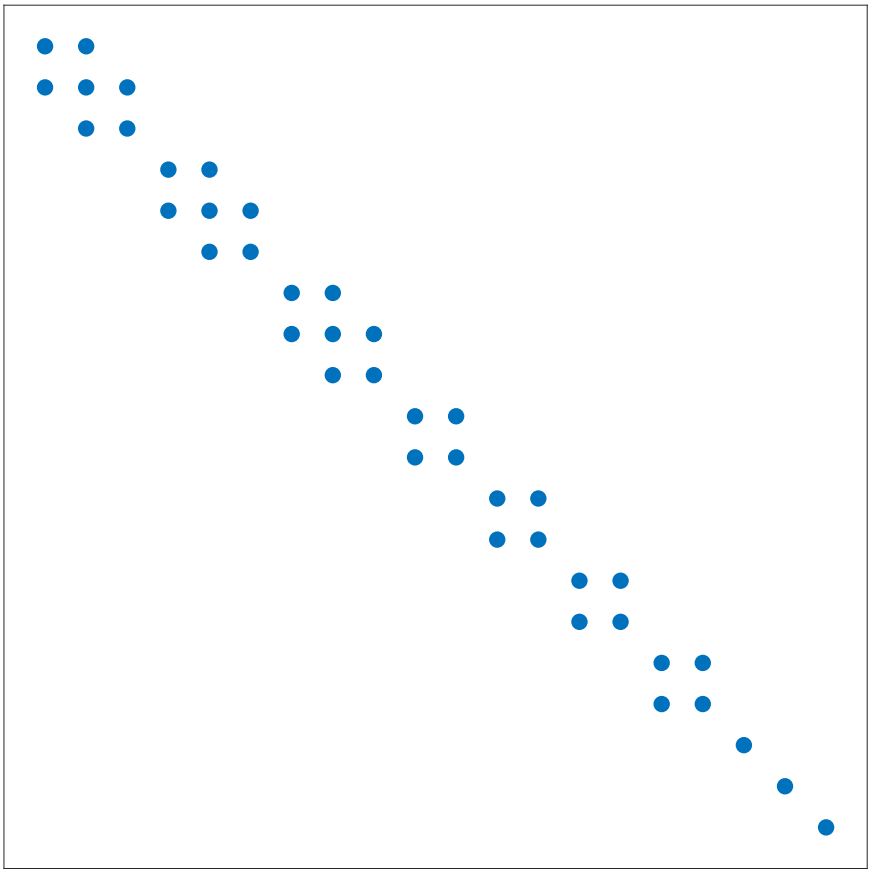

Figure 4.7: Block diagonal structure of matrix $C_{\text{ext}}(\mathbf{v})$

$$C_i(\mathbf{v}) \quad = \quad \left[v_i + v_i p\right], i = 8,9,10, \tag{4.72}$$

where $p = 0.001$.

The block diagonal structure of the matrix $C_{\text{ext}}(\mathbf{v})$ is shown in Figure 4.7, while the structure of the damping for each block $C_i(\mathbf{v})$ in our mechanical system, is shown in Figures 4.8, 4.9 and 4.10. The internal damping is $C_{\text{int}} = \alpha_c C_{\text{crit}}$, where C_{crit} is given by (1.21).

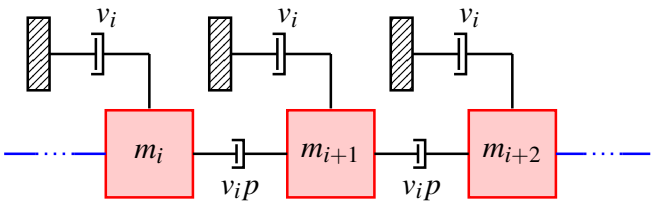

Figure 4.8: Block within n-mass oscillator represented by matrix (4.70)

We calculate optimal viscosities for two different cases:

Case 1 In the first case, we assume that there is no internal damping, i.e., $\alpha_c = 0$.

For the purpose of comparison, we present the optimal viscosity and the corresponding minimal trace, denoted by $(\mathbf{v}^*, \text{tr}(\widetilde{\mathbf{X}}(\mathbf{v}^*)))$, obtained by direct calculations using formula (4.63) and

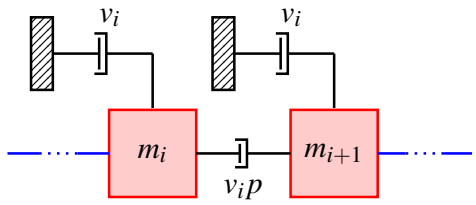

Figure 4.9: Block within n-mass oscillator represented by matrix (4.71)

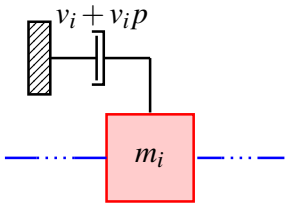

Figure 4.10: Block within n-mass oscillator represented by matrix (4.72)

the optimal viscosity and the corresponding minimal trace denoted by $(\mathbf{v}, \mathrm{tr}(\mathbf{X}(\mathbf{v})))$, obtained by the minimization of the trace of the "dual Lyapunov equation" of the Lyapunov equation (4.17) directly with MATLAB's function `fminsearch`, where we have used MATLAB's function `lyap` for solving Lyapunov equations.

For $(\mathbf{v}^*, \mathrm{tr}(\widetilde{\mathbf{X}}(\mathbf{v}^*)))$ and $(\mathbf{v}, \mathrm{tr}(\mathbf{X}(\mathbf{v})))$, we have obtained the following:

$$\mathbf{v}^* = \begin{bmatrix} 37.9626 \\ 23.3395 \\ 14.7396 \\ 19.4686 \\ 28.6084 \\ 32.6407 \\ 38.6879 \\ 45.7553 \\ 54.7100 \\ 64.6193 \end{bmatrix}, \quad \mathrm{tr}(\widetilde{\mathbf{X}}(\mathbf{v}^*)) = 487.4226, \quad \mathbf{v} = \begin{bmatrix} 38.1249 \\ 23.1773 \\ 14.5789 \\ 17.4601 \\ 28.4168 \\ 32.4962 \\ 38.5573 \\ 45.6625 \\ 55.0314 \\ 65.0329 \end{bmatrix}, \quad \mathrm{tr}(\mathbf{X}(\mathbf{v})) = 484.8125.$$

Thus, relative errors for the obtained approximations are:

$$\mathrm{err}_{\mathbf{v}} = \frac{||\mathbf{v} - \mathbf{v}^*||}{||\mathbf{v}||} = 0.0171, \tag{4.73}$$

$$\mathrm{err}_{\mathrm{tr}} = \frac{||\mathrm{tr}(\widetilde{\mathbf{X}}(\mathbf{v}^*)) - \mathrm{tr}(\mathbf{X}(\mathbf{v}))||}{||\mathrm{tr}(\mathbf{X}(\mathbf{v}))||} = 0.0054. \tag{4.74}$$

Here the residual error from (4.44) is $\mathrm{R_{er}} = 0.3534$. This shows that even if the considered mechanical system is not very close to the modally damped one ($\mathrm{R_{er}}$ is not significantly smaller than 1), formula (4.63) still insures the satisfying result.

Case 2 Within the second case, we assume the existence of internal damping; thus let $\alpha_c = 0.005$.

As emphasized in Remark 4.1 for the case $\alpha_c \neq 0$, one can not use formula (4.63) directly. Thus we use Newton's method for optimization of the trace approximation given by formula

(4.64). Again by $(\mathbf{v}^*, \mathrm{tr}(\widetilde{\mathbf{X}}(\mathbf{v}^*)))$ we denote the obtained approximation for optimal viscosity and the corresponding minimal trace. Similarly, by $(\mathbf{v}, \mathrm{tr}(\mathbf{X}(\mathbf{v})))$ we denote optimal viscosity and the corresponding function value obtained by the minimization of the trace of the dual Lyapunov equation of the equation (4.17) directly with MATLAB's function `fminsearch`, where the Lyapunov equation was solved by MATLAB's function `lyap`.

Here are the obtained quantities:

$$\mathbf{v}^* = \begin{bmatrix} 36.1512 \\ 22.1206 \\ 14.0986 \\ 17.6473 \\ 26.2969 \\ 29.9301 \\ 35.5781 \\ 42.3487 \\ 51.1036 \\ 60.8131 \end{bmatrix}, \quad \mathrm{tr}(\widetilde{\mathbf{X}}(\mathbf{v}^*)) = 486.3990, \qquad \mathbf{v} = \begin{bmatrix} 36.3126 \\ 21.9638 \\ 13.9714 \\ 15.8175 \\ 26.1052 \\ 29.7869 \\ 35.4482 \\ 42.2551 \\ 51.4233 \\ 61.2265 \end{bmatrix}, \quad \mathrm{tr}(\mathbf{X}(\mathbf{v})) = 483.9260.$$

For relative errors defined in (4.73) and (4.74), here we have:

$$\mathrm{err}_{\mathbf{v}} = 0.0169, \quad \mathrm{err}_{\mathrm{tr}} = 0.0051.$$

In this example, the residual error from (4.44) has a similar magnitude, i.e., $\mathrm{R}_{\mathrm{er}} = 0.3049$.

In the second example, we consider a more general structure.

Example 4.3 In this example, we consider the system from (1.2) with a dimension $n = 500$ and the matrices M and K defined as:

$$M = 10^3 \mathrm{diag}(m_1, m_2, \ldots, m_n),$$

$$m_i = \begin{cases} 5000 - 20(i-1), & i = 1, \ldots, 250 \\ 5001 + 20(i-1), & i = 1, \ldots, 250 \end{cases},$$

$$K = \begin{bmatrix} 10 & -1 & & & & & \\ -1 & 10 & -1 & & & & \\ & -1 & 10 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 10 & -1 & \\ & & & & -1 & 10 & \end{bmatrix}.$$

The damping matrix C has the block diagonal structure as follows

$$C = \begin{bmatrix} 0 & & & & \\ & C_1 & & & \\ & & 0 & & \\ & & & C_2 & \\ & & & & 0 \end{bmatrix} + C_{\text{int}}, \tag{4.75}$$

where $\alpha_c = 0.005$ and 0 represents a zero matrix of the corresponding dimension. The matrices C_i are defined as:

$$C_i = \begin{bmatrix} v_{i_1} + v_{i_1} p & -v_{i_1} p & & & \\ -v_{i_1} p & v_{i_1} + 2v_{i_1} p & -v_{i_1} p & & \\ & -v_{i_1} p & v_{i_1} + (v_{i_1} + v_{i_2}) p + v_{i_2} & -v_{i_2} p & \\ & & -v_{i_2} p & v_{i_2} + v_{i_2} p & \end{bmatrix}, \quad i = 1, 2, \tag{4.76}$$

where $p = 0.01$ and the structure of damping for each block C_i in our mechanical system is shown in Figure 4.11. Thus, each block has 2 different viscosities, which means that we have 4 different viscosity parameters to optimize.

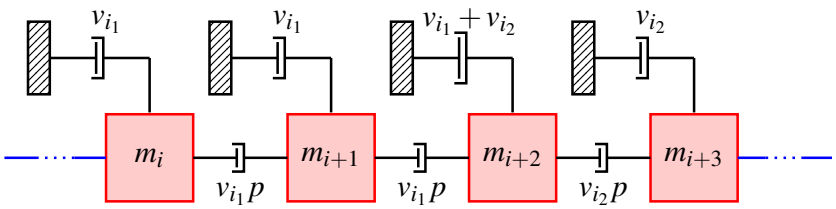

Figure 4.11: Block within n-mass oscillator represented by matrix (4.76)

Again, we compare the approximation of optimal viscosities obtained by our new approach proposed in Section 4.1.2.2 with optimal viscosity obtained by the minimization of the trace of the "dual Lyapunov equation" of the equation (4.17) directly with MATLAB's function `fminsearch`, based on the MATLAB's function `lyap` for solving Lyapunov equations.

This comparison has been performed for different positions of matrices C_1 and C_2, i.e., in each new configuration we change the position of matrices C_1 and C_2. The following configurations are taken into consideration:

$$\begin{aligned} (i,j) \in \{&(2,17),(2,67),(2,117),(2,267),(2,317),(52,67),(52,117),(52,167),(52,267),\\ &(52,317),(52,367),(52,417),(102,117),(102,217),(102,367),(152,167),(152,267),\\ &(152,317),(202,417),(252,267),(252,367),(252,417),(252,467),(302,367),(302,417)\\ &(352,417),(352,467)\}, \end{aligned}$$

where i represents the position of the matrix C_1 and j represents the position of the matrix C_2. Figure 4.12 shows the relative error

$$\mathrm{err}_{\mathrm{tr}} = \frac{||\,\mathrm{tr}(\mathbf{X}(\mathbf{v})) - \mathrm{tr}(\widetilde{\mathbf{X}}(\mathbf{v}^*))||}{||\,\mathrm{tr}(\mathbf{X}(\mathbf{v}))||},$$

for each configuration.

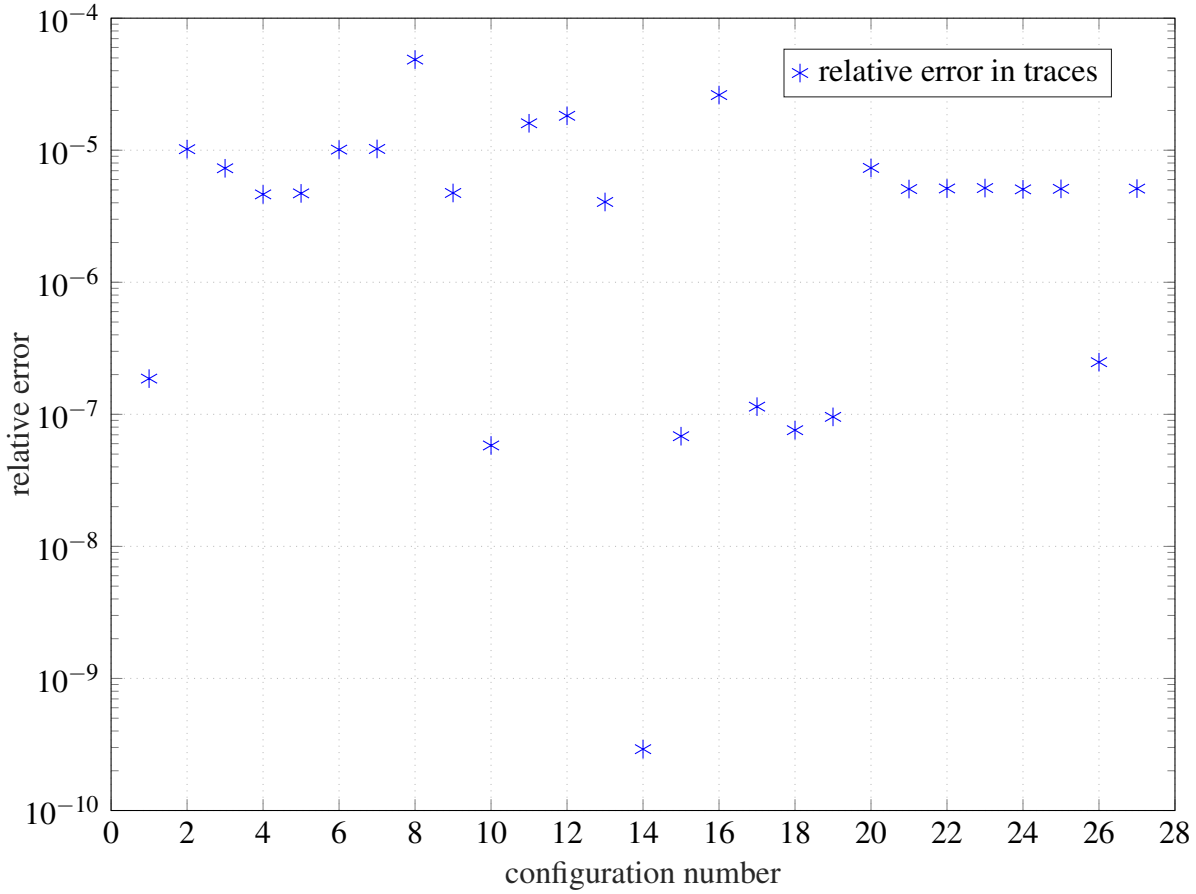

Figure 4.12: Relative errors $\mathrm{err}_{\mathrm{tr}}$ for different positions of matrices C_1 and C_2

During the optimization process, using Algorithm 6, we have calculated the percentage of updates of the matrix U with the tolerance $\mathtt{tol}_{\mathrm{U}} = 10^{-5}$. The number of updates for each configuration is shown in Figure 4.13.

Moreover, in order to illustrate the quality of the new approach using the surface plot in prescribed viscosities, we set $v_1 = v_3$, $v_2 = v_4$, while $v_1 \in [40, 200]$ and $v_2 \in [200, 340]$. The block with viscosities v_1 and v_2 starts at position 242 and the block with v_3 and v_4 starts at 470.

For the first step in iterations we have used the matrix U defined by optimal viscosities $v_1 = 101.4445$ and $v_2 = 268.3622$, while during the iteration process the matrix U has been updated with the tolerance $\mathtt{tol}_{\mathrm{U}} = 10^{-5}$.

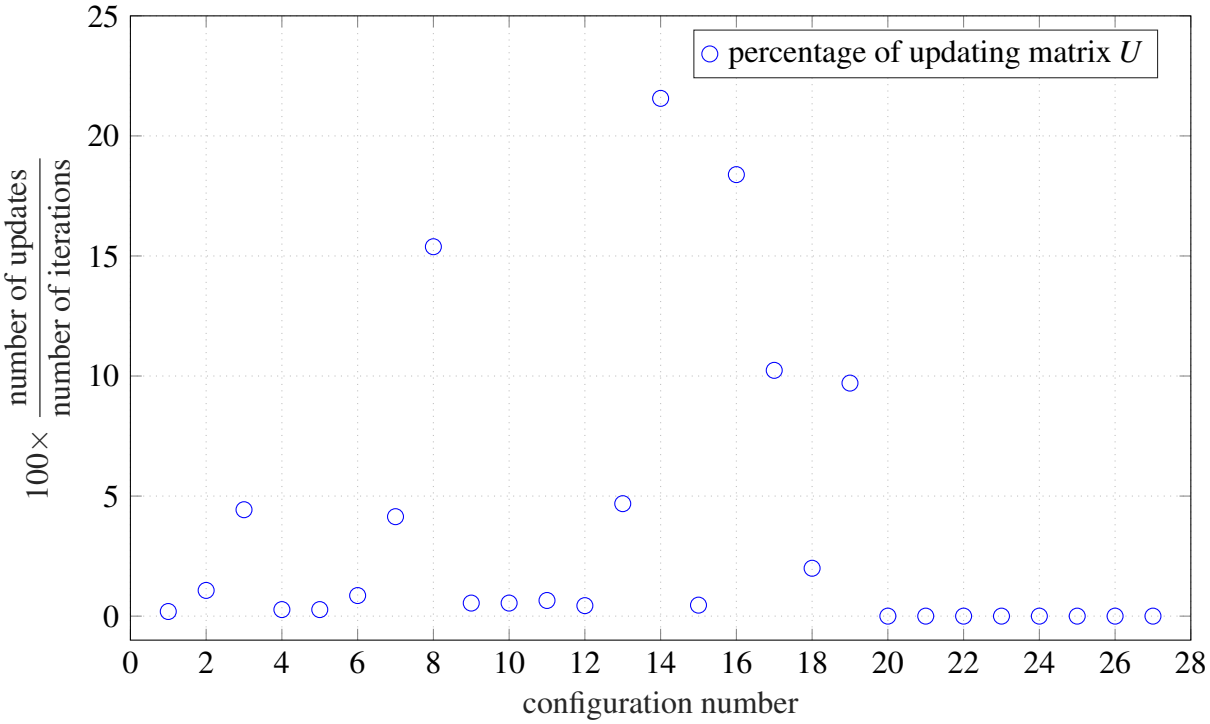

Figure 4.13: Percentage of updating matrix U for different positions of matrices C_1 and C_2

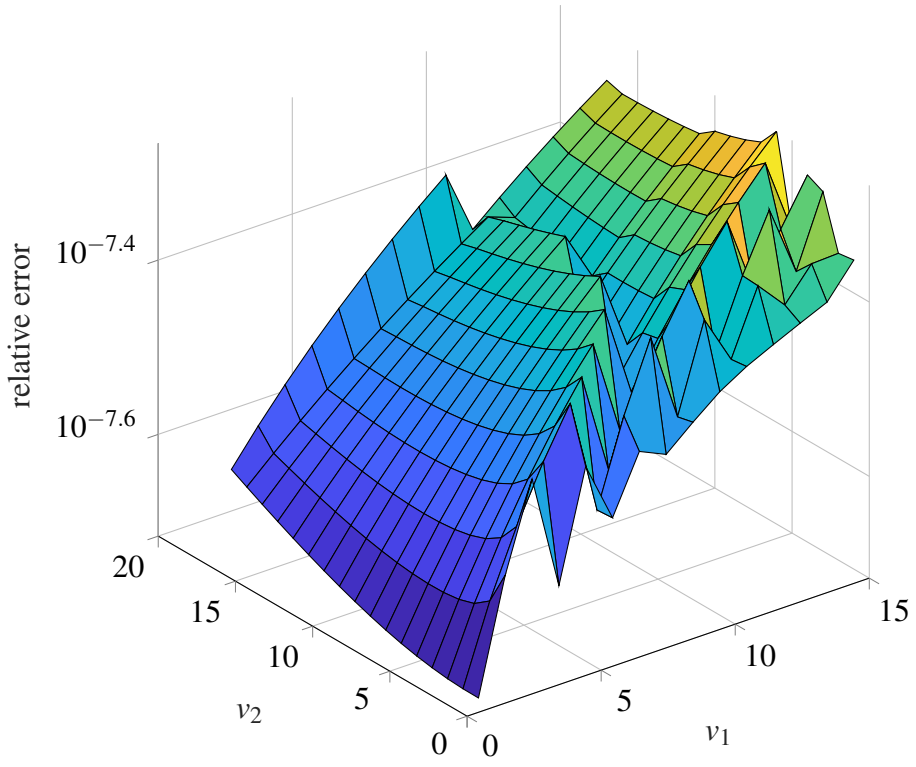

Figure 4.14: Relative error of the function value

In Figure 4.14, we can see the relative error

$$\mathrm{err}_{\mathrm{rel}} = \frac{||f_{\mathrm{lyap}} - f_{\mathrm{approx}}||}{||f_{\mathrm{lyap}}||},$$

in which f_{lyap} represents the trace of the solution of the Lyapunov equation for certain viscosities v_1 and v_2, by MATLAB's function `lyap` and f_{approx} represents the trace calculated by our algorithm. The relative error is less than 10^{-7}.

In the next section we present eigenvalue-based approach to damping optimization. This approach is based on frequency isolation, i.e., we are focused on dangerous natural frequencies, which we want to avoid.

4.2 Frequency-weighted damping optimization

We consider vibrational mechanical system described by (4.2), where damping matrix $C(\mathbf{v})$ depends on parameters encoded in vector $\mathbf{v} = [v_1, \ldots, v_s]^T \in \mathbb{R}^s_+$. We assume that $C(\mathbf{v})$ is given by

$$C(\mathbf{v}) = C_{\mathrm{int}} + G\,\mathrm{diag}(\mathbf{v})G^T, \tag{4.77}$$

where v_i for $i = 1, \ldots, s$ represent damping viscosities and geometry of damping positions is described by the matrix G. Usually, there are small numbers of damping parameters compared to full order dimension which means that $s \ll n$. Since we are interested in damping optimization, our main focus is on the damping matrix $C(\mathbf{v})$ given by (1.2) that can be written in the following form

$$C(\mathbf{v}) = C_{\mathrm{int}} + \sum_{i=1}^{s} v_i g_i g_i^T, \tag{4.78}$$

where g_i for $i = 1 \ldots s$, are columns of matrix G. When $C(\mathbf{v})$ has the structure from (4.78) we can easily apply our approach from Section 2.3 for fast eigenvalue computation.

In this section we would like to present a new criteria based on eigenvalues, that are motivated by a frequency isolation problem and ensures that spectral abscissa is bounded or minimized.

The frequency isolation problem corresponds to the problem of parameter optimization in such a way that the new system has no eigenvalues close to the dangerous frequencies (dangerous areas). In damping optimization setting this can be achieved by optimizing damping positions and damping viscosities. This problem has been previously studied in [49] and [32]. In [49], the author proposed a Newton-type method for structures vibrating at low frequencies, while in [32], the authors proposed a less costly inverse eigenvalue method: a target spectrum away from the resonance band is fixed in advance. The frequency isolation problem in undamped vibrational systems was considered in [69], but we would like to use it in a more general case which means

that we vary parameters also in the external damping matrix.

The main interest of this section is to present a new approach for the frequency isolation problem. First we will formulate a new optimization criteria that takes into account the eigenvalue behaviour with respect to dangerous areas as well as the magnitude of spectral abscissa. Additionally we propose that dangerous frequencies are weighted in order to ensure importance of certain frequency areas. Since that damping optimization process requires repeated solutions of QEP for different damping parameters, we use the efficient approximation approach given Section 2.3, which gives us the fast solution of the QEP.

We consider two different approaches to the frequency isolation problem, where the undesirable areas are ellipses with centers on the imaginary axis:

Approach 1:
We minimize spectral abscissa such that frequencies are isolated from a priori determined ellipses. Furthermore we need to emphasize that this case depends on choice of the value of the major axis, i.e., if we choose too big major axis the optimization problem might be infeasible or if we choose too small major axis this case comes down to basic minimization of the spectral abscissa.

Approach 2:
The centers of the ellipses are fixed and we bound a spectral abscissa by $\texttt{tol}_{\text{sa}}$, while we maximize (scaled) major axis of the considered ellipses and make sure there are no eigenvalues in them. One can notice that with one ellipse we can push away more then one eigenvalue.

First we will focus on the single ellipse case for both approaches, and in Subsection 4.2.2 we will extend this to the multiple ellipse case. The multiple case will be illustrated in Subsection 4.2.3, and single case will be illustrated in Section 4.3 where we will compare these two approaches on Example 4.1

4.2.1 Single ellipse case

Let $E = (c, a(\mathbf{v}), b)$ be the ellipse in the complex plane, where $c \in \mathbf{i}\mathbb{R}_+$, $b \in \mathbb{R}_+$ are respectively constants for its center and minor axis and $a(\mathbf{v})\colon \mathbb{R}^s_+ \to \mathbb{R}_+$ is the major axis function, which will be maximized. The center determines which frequency is undesirable. Since eigenvalues $\lambda(\mathbf{v}) \in \mathbb{C}$, and $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic we define the ellipse on $\mathbb{C}$ by

$$E \quad \dots \quad \frac{x^2}{a^2(\mathbf{v})} + \frac{(y - \mathrm{Im}(c))^2}{b^2} = 1, \quad \text{where} \quad x + y\mathbf{i} \in \mathbb{C}. \tag{4.79}$$

In both approaches described above we need to make sure that no eigenvalues are inside the ellipse, so we need to define the distance function $d(\lambda(\mathbf{v}), E)$ that measures the distance of eigen-

value $\lambda(\mathbf{v})$ to the ellipse E. There is no explicit formula for distance from point to ellipse. One can use Mahalanobis distance-like function from [86], but we will define distance as algebraic distance, for more see [108]. Let $d(\lambda(\mathbf{v}),E)$ be defined as:

$$d(\lambda(\mathbf{v}),E) = \frac{\mathrm{Re}(\lambda(\mathbf{v}))^2}{a^2(\mathbf{v})} + \frac{(\mathrm{Im}(\lambda(\mathbf{v})-c))^2}{b^2} - 1, \tag{4.80}$$

which means that

$$d(\lambda(\mathbf{v}),E) \begin{cases} < 0, & \text{if } \lambda(\mathbf{v}) \text{ is in the ellipse } E, \\ = 0, & \text{if } \lambda(\mathbf{v}) \text{ is on the ellipse } E, \\ > 0, & \text{if } \lambda(\mathbf{v}) \text{ is outside of the ellipse } E. \end{cases} \tag{4.81}$$

In Approach 1 we minimize spectral abscissa such that there are no eigenvalues in a priori determined ellipse. For the Approach 1 the major axis is fixed, i.e., $a(\mathbf{v}) = a$ and $E = (c,a,b)$ and the optimization problem is defined by

$$\begin{aligned} &\min_{\mathbf{v}\in\mathbb{R}^s_+} \quad \alpha_{\mathrm{MCK}}(\mathbf{v}) \\ &\text{such that} \\ &\qquad d(\lambda(\mathbf{v}),E) \geq 0, \quad \forall \lambda(\mathbf{v}) \in \Lambda(\mathbf{v}). \end{aligned} \tag{4.82}$$

In Approach 2 we want to maximize the major axis of the ellipse $a(\mathbf{v})$, such that there are no eigenvalues in the ellipse. We can see that $a(\mathbf{v})$ needs to satisfy the following condition

$$d(\lambda(\mathbf{v}),E) \geq 0, \quad \forall \lambda(\mathbf{v}) \in \Lambda(\mathbf{v}), \tag{4.83}$$

i.e.,

$$a(\mathbf{v}) \quad \leq \quad \frac{b\,|\mathrm{Re}(\lambda(\mathbf{v}))|}{\sqrt{b^2 - (\mathrm{Im}(\lambda(\mathbf{v})-c))^2}}, \quad \forall \lambda(\mathbf{v}) \in \Lambda(\mathbf{v}), \text{ s.t. } b > |\mathrm{Im}(\lambda(\mathbf{v})-c))|. \tag{4.84}$$

Here we would like to note that in cases when there are no eigenvalues such that $\mathrm{Im}(\lambda(\mathbf{v})) \in (\mathrm{Im}(c)-b, \mathrm{Im}(c)+b)$, i.e., when $b \leq |\mathrm{Im}(\lambda(\mathbf{v})-c))|$ then $a(\mathbf{v}) = \infty$. This means that we obtained strip $\mathrm{St}(c,b)$ which is parallel to the real axis and there are no eigenvalues in it, i.e.,

$$\Lambda(\mathbf{v}) \cap \mathrm{St}(c,b) = \emptyset, \tag{4.85}$$

where

$$\mathrm{St}(c,b) = \{x + y\mathbf{i} \in \mathbb{C} \colon \mathrm{Im}(c) - b \leq y \leq \mathrm{Im}(c) + b\}. \tag{4.86}$$

Thus, we define $a(\mathbf{v})$ as:

$$a(\mathbf{v}) = \min_{\lambda(\mathbf{v})\in\Lambda(\mathbf{v})} \begin{cases} \dfrac{b\,|\mathrm{Re}(\lambda(\mathbf{v}))|}{\sqrt{b^2-(\mathrm{Im}(\lambda(\mathbf{v})-c))^2}}, & b > |\mathrm{Im}(\lambda(\mathbf{v})-c)| \\ \infty, & b \le |\mathrm{Im}(\lambda(\mathbf{v})-c)|. \end{cases} \tag{4.87}$$

Let $d_{\mathrm{obj}}(\mathbf{v}) = a(\mathbf{v})$, where $a(\mathbf{v})$ defined in (4.87), then in the Approach 2 with single ellipse the optimization problem is defined by

$$\begin{aligned} &\max_{\mathbf{v}\in\mathbb{R}_+} \quad d_{\mathrm{obj}}(\mathbf{v}) \\ &\text{such that} \\ &\qquad \alpha_{\mathrm{MCK}}(\mathbf{v}) \le \mathtt{tol}_{\mathrm{sa}}. \end{aligned} \tag{4.88}$$

In following section we rewrite both approaches for the case of multiple ellipse.

4.2.2 Multiple ellipse case

Given k ellipses, $k \in \mathbb{N}$, let $E_i = (c_i, a_i(\mathbf{v}), b_i)$ be ith ellipse, where $c_i \in \mathbf{i}\mathbb{R}_+$, $b_i \in \mathbb{R}_+$, are respectively constants for its center, minor axis. Since some frequency bands are less desirable than others, we can introduce scaling $w_i \in \mathbb{R}_+$ to help optimally balance how the different resonant bands are damped. Product $a_i(\mathbf{v}) = a(\mathbf{v})w_i$ represents the scaled major axis. This means that our ith ellipse is defined by

$$E_i \quad \ldots \qquad \frac{x^2}{a^2(\mathbf{v})w_i^2} + \frac{(y-\mathrm{Im}(c_i))^2}{b_i^2} = 1. \tag{4.89}$$

For $i = 1,\ldots,k$ the function $a_i(\mathbf{v}) : \mathbb{R}_+^s \to \mathbb{R}_+$ calculates the maximal length of unscaled major axis such that there are no eigenvalues in the ellipse E_i, which major axis is scaled, i.e.,

$$d(\lambda(\mathbf{v}), E_i) \ge 0, \quad \forall\lambda(\mathbf{v}) \in \Lambda(\mathbf{v}), i = 1,\ldots,k, \tag{4.90}$$

where d is defined in (4.80). Similarly, as in the single ellipse case

$$a(\mathbf{v}) \;=\; \min_{i=1,\ldots,k} \frac{a_i(\mathbf{v})}{w_i}, \tag{4.91}$$

where

$$a_i(\mathbf{v}) \;=\; \min_{\lambda(\mathbf{v})\in\Lambda(\mathbf{v})} \begin{cases} \dfrac{b_i\,|\mathrm{Re}(\lambda(\mathbf{v}))|}{\sqrt{b_i^2-(\mathrm{Im}(\lambda(\mathbf{v})-c_i))^2}}, & b_i > |\mathrm{Im}(\lambda(\mathbf{v})-c_i)|, \\ \infty, & b_i \le |\mathrm{Im}(\lambda(\mathbf{v})-c_i)|. \end{cases} \tag{4.92}$$

Now that we have described our undesirable areas, we can formulate the optimization prob-

lem for both cases Approach 1 and Approach 2. Let $d_{\text{obj}}(\mathbf{v}) = a(\mathbf{v})$ where $a(\mathbf{v})$ is defined by (4.91).

Approach 1

$$\begin{aligned} &\min_{v\in\mathbb{R}_+} \quad \alpha_{\text{MCK}}(\mathbf{v}) \\ &\text{such that} \\ &\qquad d(\lambda(\mathbf{v}), E_i) \geq 0, \\ &\qquad \forall \lambda(\mathbf{v}) \in \Lambda(\mathbf{v}), \\ &\qquad i = 1, \ldots, k, \end{aligned} \tag{4.93}$$

Approach 2

$$\begin{aligned} &\max_{v\in\mathbb{R}_+} \quad d_{\text{obj}}(\mathbf{v}) \\ &\text{such that} \\ &\qquad \alpha_{\text{MCK}}(\mathbf{v}) \leq \texttt{tol}_{\text{sa}}(\mathbf{v}). \end{aligned} \tag{4.94}$$

For the optimization we will use gradient-based nonsmooth optimization solvers, such as [25], and thus we will derive the gradients for all eigenvalues $\lambda(\mathbf{v})$ and for function $a_j(\mathbf{v})$, such that $a(\mathbf{v}) = \frac{a_j(\mathbf{v})}{w_j}$, i.e. j is the index for which we obtain minimum over all $\frac{a_i(\mathbf{v})}{w_i}$, $i = 1, \ldots, k$. Since these functions are smooth almost everywhere, we assume that these functions are evaluated at a point where the function is smooth. Thus, the partial derivatives for the functions are given by:

$$\frac{\partial \lambda}{\partial v_l}(\mathbf{v}) = -\frac{y^*(\mathbf{v})\left(\lambda(\mathbf{v}) g_l g_l^T\right) x(\mathbf{v})}{y^*(\mathbf{v})(2\lambda(\mathbf{v})M + C(\mathbf{v}))x(\mathbf{v})}, \tag{4.95}$$

$$\begin{aligned} \frac{\partial a_j}{\partial v_l}(\mathbf{v}) = \texttt{sgn}(\text{Re}(\lambda(\mathbf{v}))) \Bigg(& \frac{b_j \text{Re}\left(\frac{\partial \lambda}{\partial v_l}(\mathbf{v})\right)}{\sqrt{b_j^2 - (\text{Im}(\lambda(\mathbf{v}) - c_j))^2}} \\ & + \frac{b_j \text{Im}\left(\frac{\partial \lambda}{\partial v_l}(\mathbf{v})\right) \text{Re}(\lambda(\mathbf{v}))(\text{Im}(\lambda(\mathbf{v}) - c_j))}{\sqrt[3]{b_j^2 - (\text{Im}(\lambda(\mathbf{v}) - c_j))^2}} \Bigg), \quad \text{for} \quad b_j > |\text{Im}(\lambda(\mathbf{v}) - c_j)|, \end{aligned} \tag{4.96}$$

and

$$\frac{\partial d}{\partial v_l}(\lambda(\mathbf{v}), E) = \frac{2\,\text{Re}(\lambda(\mathbf{v}))\text{Re}\left(\frac{\partial \lambda}{\partial v_l}(\mathbf{v})\right)}{a^2} + \frac{2\,\text{Im}(\lambda(\mathbf{v}) - c)\text{Im}\left(\frac{\partial \lambda}{\partial v_l}(\mathbf{v})\right)}{b^2}, \tag{4.97}$$

where $l = 1, \ldots, s$. When $a(\mathbf{v}) = \infty$, we reached the maximum, so we don't need the derivative of the function in that point.

One can notice that partial derivative of distance function is computed only for Approach 1, since only in that case we need to compute the distance, and in Approach 2 positive distance is ensured by the definition of the major axis.

Now we only need an efficient algorithm for eigenvalue computation so we can summarise the optimization algorithm. Since the damping matrix (4.77) has the same structure as the damping matrix from Section 2.3 we can use Algorithm 5 for eigenvalue computation.

Algorithm 7: Frequency-weighted damping optimization algorithm

Require: $M, K \in \mathbb{R}^{n\times n}$, $\alpha \in \mathbb{R}_+$ for C_{int}, $G = [g_1, \ldots, g_s] \in \mathbb{R}^{n\times s}$, $c \in i\mathbb{R}^k$, centers of ellipses, $\tilde{b} \in \mathbb{R}^k_+$ minor axes, $w_i \in \mathbb{R}^k_+$ scaling of the ellipse, $\texttt{tol}_{\text{sa}}$ minimal spectral abscissa, initial viscosity v_0

Ensure: vector of parameters v such that goal function is minimized

Off-line stage:

1: Determine $\Phi \in \mathbb{C}^{n\times n}$ such that (1.4) hold.
2: Do perfect shuffle and construct $\widehat{A}$ from (2.73).
3: Calculate matrix Ψ from (2.77).

Online stage:

4: Compute $D - \sum_{l=1}^{s} v_l \hat{g}_l \hat{g}_l^T$ as in (2.79)
5: Run GRANSO on (4.94) using Algorithm 5 for evaluations of eigenvalues.

Notice that the off-line stage of Algorithm 7 is done in $O(n^3)$ operations, but we only have to incur this expense once. Since, in Step 5 we repeatedly compute eigenvalues and eigenvectors for different viscosities v, which is the most time consuming part of our optimization, we use Algorithm 5 for eigenvalue computation which is efficient for our structured low-rank updates, i.e., one computation of eigenvalues and eigenvectors is done in $O(n^2)$.

For optimization, we use GRANSO[1] which is optimization software, intended to be efficient for nonsmooth constrained optimization problems. For details on GRANSO see [25], also there are other methods that could have been used here, such as SQP-GS [26], DIRECT method [48], Nelder-Mead [59], etc.

The quality of these optimization approaches is illustrated in the following subsection.

4.2.3 Numerical experiment

In this subsection we present an example for each optimization criteria we formulated in Subsection 4.2.2, i.e., Approach 1 and Approach 2. In both examples we consider mechanical system shown in Figure 2.14 with dimension $n = 1000$, $k_i = 5, i = 1, \ldots, n+1$ and $m_i = m_{n+1-i} = \frac{n-i}{200}, i = 1, \ldots, \frac{n}{2}$, while parameter in the critical damping (1.21) is $\alpha_c = 0.002$.

Example 4.4 Let the positions of dampers be $(100, 200, 300)$. For initial viscosity in Algorithm 7 we took $\mathbf{v}^0 = [1, 1, 1]^T$.

The centers of ellipses are set at

$$c_1 = 0.11\,\mathbf{i},\ c_2 = 0.75\,\mathbf{i},\ c_3 = 0.95\,\mathbf{i},$$

while the minor axes of all ellipses are

$$\hat{b}_1 = \hat{b}_2 = \hat{b}_3 = \frac{\max\limits_{j=1,\ldots,2n} \text{Im}(\lambda_j(\mathbf{0}))}{n} \cdot 10 = 1.6293 \cdot 10^{-2},$$

[1] software is available at https://gitlab.com/timmitchell/GRANSO/

and tolerance on the spectral abscise is

$$\texttt{tol}_{\text{sa}} = \alpha_{\text{MCK}}(\mathbf{0}) = -4.8879 \cdot 10^{-6}.$$

Furthermore, since in this case ellipses are fixed there is no need for the scaling of the major axes, i.e.,

$$a_1 = 0.0004,\, a_2 = 0.0016,\, a_3 = 0.00205 \quad \text{and} \quad w_1 = w_2 = w_3 = 1.$$

Figure 4.15 shows the upper half of the complex plane, since everything is symmetric on the lower half of the complex plane. In Figure 4.15 are shown three sets of eigenvalues, black dots represent eigenvalues of the system with no external damping, red pluses represent eigenvalues of the system with initial viscosity $\mathbf{v}^0 = [1,1,1]^T$ and blue circles represent eigenvalues of the system with optimal viscosity, i.e., the viscosity for which the spectral abscissa is minimized and no eigenvalues are in the interior of the ellipses. In this case the optimal viscosity is the following:

$$\mathbf{v}_{\text{FI1}} = \begin{bmatrix} 8.1970 \\ 1.3131 \\ 26.2544 \end{bmatrix}.$$

The spectral abscise of the system with no external damping is shown with black line, the spectral abscise of the system with initial viscosity is shown with red dash-dotted line, while the obtained optimal spectral abscise of the system is shown with blue dashed line in Figure 4.15. The values of these spectral abscissae are following:

$$\begin{aligned} \alpha_{\text{MCK}}(\mathbf{0}) &= -4.8879 \cdot 10^{-6}, \\ \alpha_{\text{MCK}}(\mathbf{v}^0) &= -5.8676 \cdot 10^{-5}, \\ \alpha_{\text{MCK}}(\mathbf{v}_{\text{FI1}}) &= -2.1991 \cdot 10^{-4}. \end{aligned}$$

On three plots in the lower part of Figure 4.15 one can see enlarged ellipses and that blue circles only touch ellipses, while some of red dots are in the interior of the ellipses, which means that we pushed all those eigenvalues that were in the interior further left in the complex plane. One can notice that eigenvalues denoted by blue circles became closer to the ellipse with the center $c_3 = 0.95$, but they are still outside of the ellipse. At the same time we pushed six eigenvalues, denoted by red plus, from the ellipse with the center $c_1 = 0.11$ and one eigenvalue from the ellipse with the center $c_2 = 0.75$.

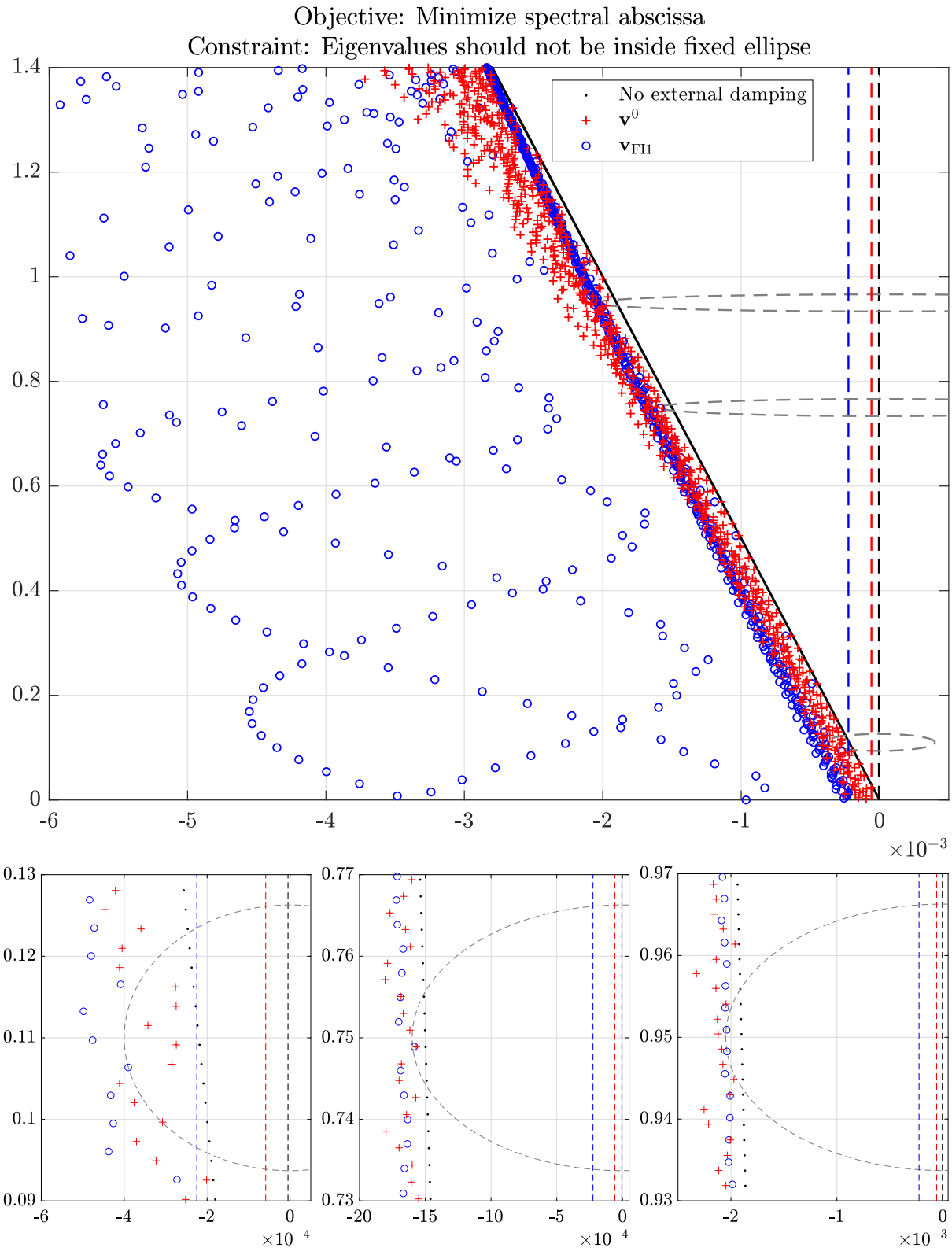

Figure 4.15: In the upper half of the complex plane are shown the eigenvalues of the system with no external damping, eigenvalues of the damped system with v_0 and eigenvalues of the damped system with minimized spectral abscissa, respectively represented by black dots, red pluses and blues circles. On the three plots in the lower part are enlarged ellipses.

Example 4.5 In this example the positions of dampers are following $(100, 200, 900)$. For initial viscosity in Algorithm 7 we also took $\mathbf{v}^0 = [1,1,1]^T$.

The centers of ellipses are set at

$$c_1 = 0.1\,\mathbf{i},\ c_2 = 0.55\,\mathbf{i},\ c_3 = 1.1\,\mathbf{i},$$

while the minor axes of all ellipses are

$$\hat{b}_1 = \hat{b}_2 = \hat{b}_3 = \frac{\max\limits_{j=1,\ldots,2n} \mathrm{Im}(\lambda_j(\mathbf{0}))}{n} \cdot 10 = 1.6293 \cdot 10^{-2},$$

and the tolerance on spectral abscissa is also taken as a spectral abscissa of the system with no external damping, i.e.,

$$\mathtt{tol}_{\mathrm{sa}} = \alpha_{\mathrm{MCK}}(\mathbf{0}) = -4.8879 \cdot 10^{-6}.$$

Furthermore, the scaling for this example is chosen as follows:

$$w_i = |\mathrm{Re}(\lambda_k(\mathbf{0}))|, \quad \text{such that} \quad [\mathtt{min}_{\mathrm{value}}, k] = \min_{j=1,\ldots,2n} |\mathrm{Im}(\lambda_j(\mathbf{0}) - c_i)|,$$

which means that we took the real part of the eigenvalue $\lambda_k(\mathbf{0})$ whose imaginary part is closest to the center c_i, for $i = 1,\ldots,3$. This ensures that each ellipse moves at least few eigenvalues further from imaginary axis. In this example the scaling was:

$$w_1 = 1.9932 \cdot 10^{-4},\ w_2 = 1.0997 \cdot 10^{-3},\ w_3 = 2.1992 \cdot 10^{-3}.$$

Figure 4.16 shows the upper half of the complex plane, since everything is symmetric on the lower half of the complex plane. In Figure 4.16 are shown three sets of eigenvalues, black dots represent eigenvalues of the system with no external damping, red pluses represent eigenvalues of the system with initial viscosity $\mathbf{v}^0 = [1,1,1]^T$ and blue circles represent eigenvalues of the system with optimal viscosity, i.e., the viscosity for which the major axes of ellipse are maximized and there are no eigenvalues inside ellipse. In this case optimal viscosity is the following:

$$\mathbf{v}_{\mathrm{FI2}} = \begin{bmatrix} 5.0865 \\ 1.2139 \\ 7.4170 \end{bmatrix},$$

while the obtained maximal unscaled major axis is $2.7671 \cdot 10^{-3}$.

The spectral abscise of the system with no external damping is shown with black line, the spectral abscise of the system with initial viscosity is shown with red dash-dotted line, while the obtained optimal spectral abscise of the system is shown with blue dashed line in Figure 4.15.

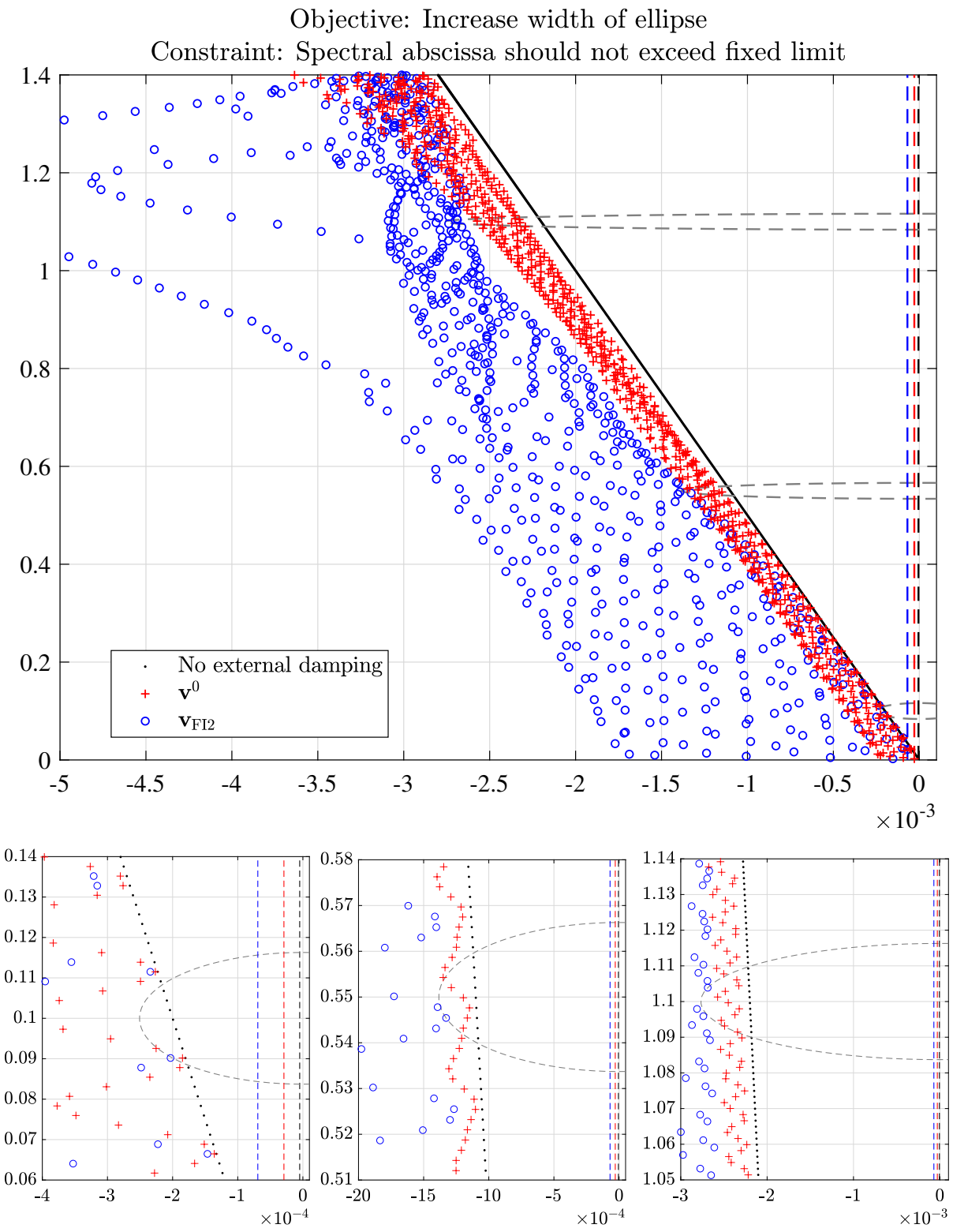

Figure 4.16: In the upper half of the complex plane are shown eigenvalues of the system with no external damping, eigenvalues of the damped system with v_0 and eigenvalues of the damped system with maximized major ellipse axes, respectively represented by black dots, red pluses and blues circles. On the three plots in the lower part are enlarged ellipses.

The values of these spectral abscissae are following:

$$\begin{aligned} \alpha_{\mathrm{MCK}}(\mathbf{0}) &= -4.8879 \cdot 10^{-6}, \\ \alpha_{\mathrm{MCK}}(\mathbf{v}^0) &= -2.9116 \cdot 10^{-5}, \\ \alpha_{\mathrm{MCK}}(\mathbf{v}_{\mathrm{FI2}}) &= -6.9199 \cdot 10^{-5}. \end{aligned}$$

On three plots in the lower part of Figure 4.16 one can see enlarged ellipses and that the blue circles only touch ellipses, while some of the red dots are in the interior of the ellipses, which means that we pushed all those eigenvalues that were in the interior further left in the complex plane and in that way we got the maximal major axes of the ellipses.

4.3 The frequency isolation vs. the minimization of the total average energy

In this section we compare the frequency isolation approaches to damping optimization, Approach 1 and Approach 2 from Section 4.2, with two approaches illustrated in Example 4.1, i.e., with minimization of spectral abscissa and minimization of total average energy. Difference between Approach 1 and Approach 2 from Section 4.2 is that Approach 1 requires the value of the major axis of the ellipse a priori. Since the dangerous frequencies of the external function are

$$\omega_2 = 0.0490, \quad \omega_3 = 0.0651, \quad \omega_4 = 0.0885, \quad \omega_5 = 0.1071.$$

we consider the single ellipse case with $c = 0.1\mathbf{i}$ and $b = 0.0698$, but since everything is symmetric with respect to the real axis we actually consider two ellipses with same major and minor axis but with centers that are conjugate pair.

For Approach 1 we need to set the value of the major axis and since

$$\alpha_{\mathrm{MCK}}(\mathbf{v}_{\mathrm{sa}}) = -0.0244,$$

we set $a = 0.05$.

From Figure 4.17 we can see that we managed to isolate eigenvalues (blue circles) from the given ellipses and obtained optimal viscosities $\mathbf{v}_{\mathrm{FI1}}$ with

$$\alpha_{\mathrm{MCK}}(\mathbf{v}_{\mathrm{FI1}}) = -0.0085,$$

black dots represent eigenvalues of the system with no external damping.

For the Approach 2 we obtained optimal viscosities $\mathbf{v}_{\mathrm{FI2}}$ and the corresponding major axis

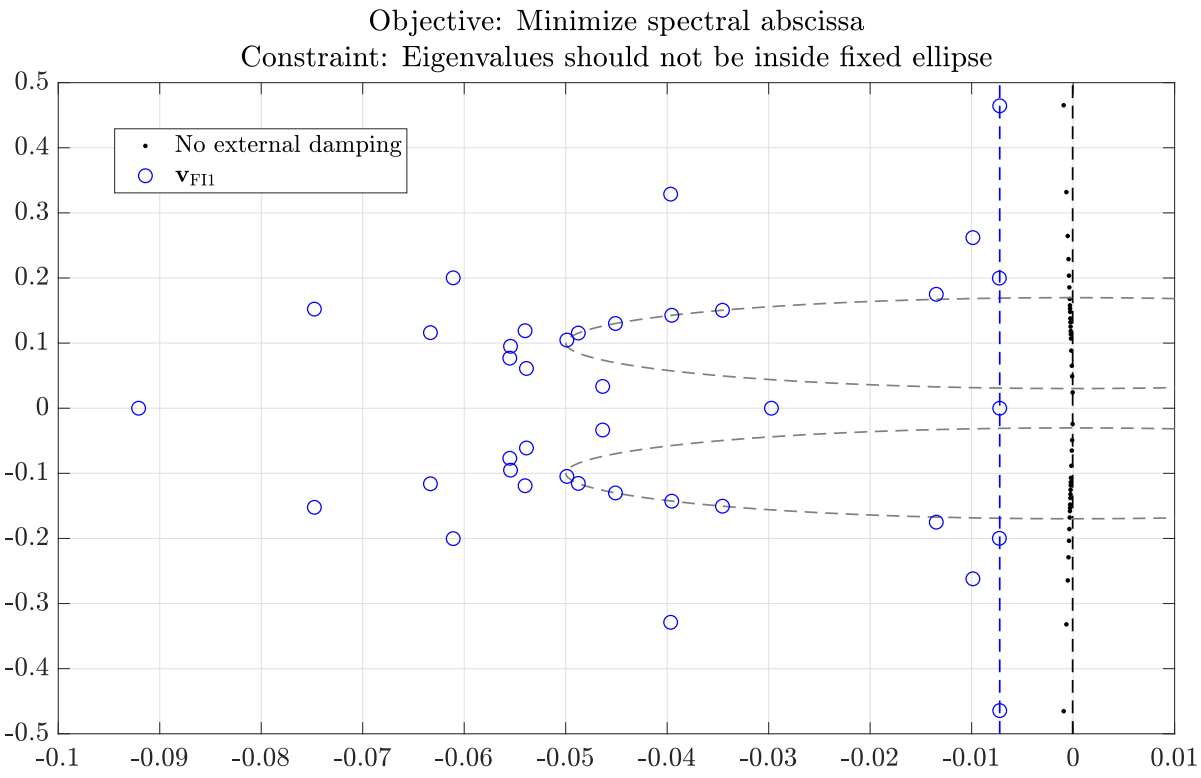

Figure 4.17: Frequency isolation: Approach 1

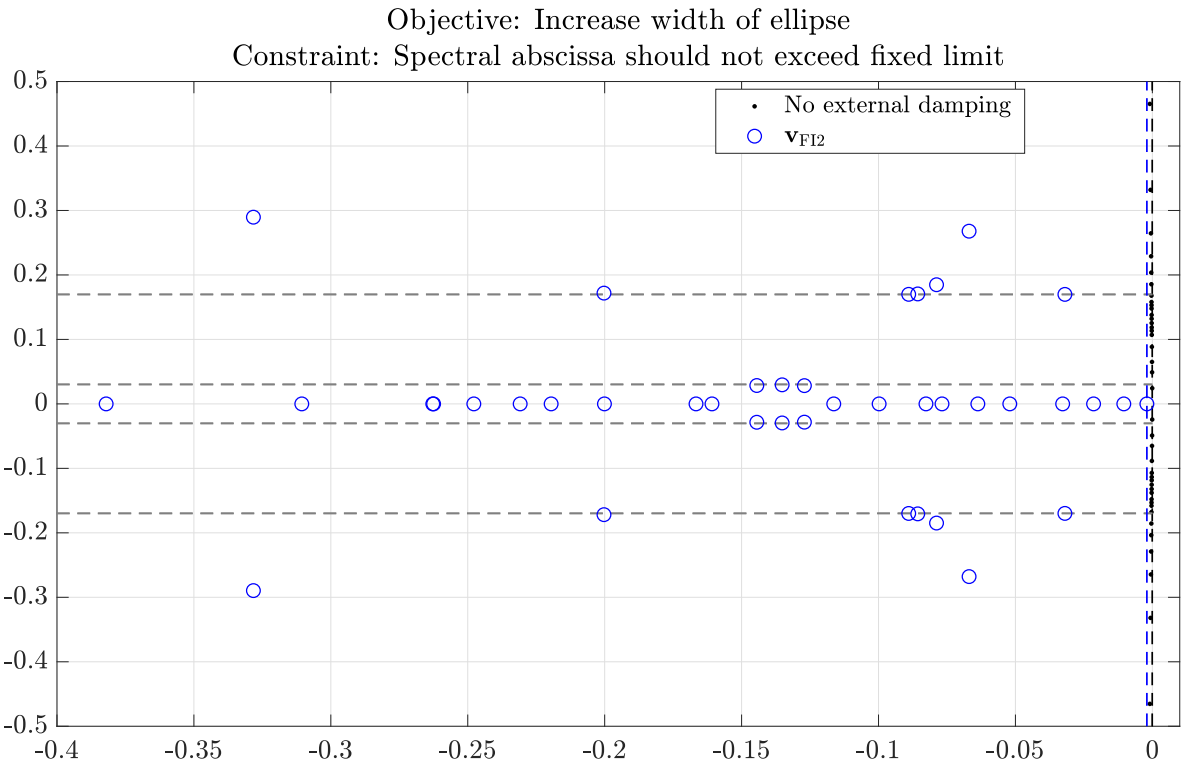

Figure 4.18: Frequency isolation: Approach 2

and spectral abscissa are:

$$a(\mathbf{v}_{\mathrm{FI2}}) = \infty, \qquad \alpha_{\mathrm{MCK}}(\mathbf{v}_{\mathrm{FI2}}) = -0.002.$$

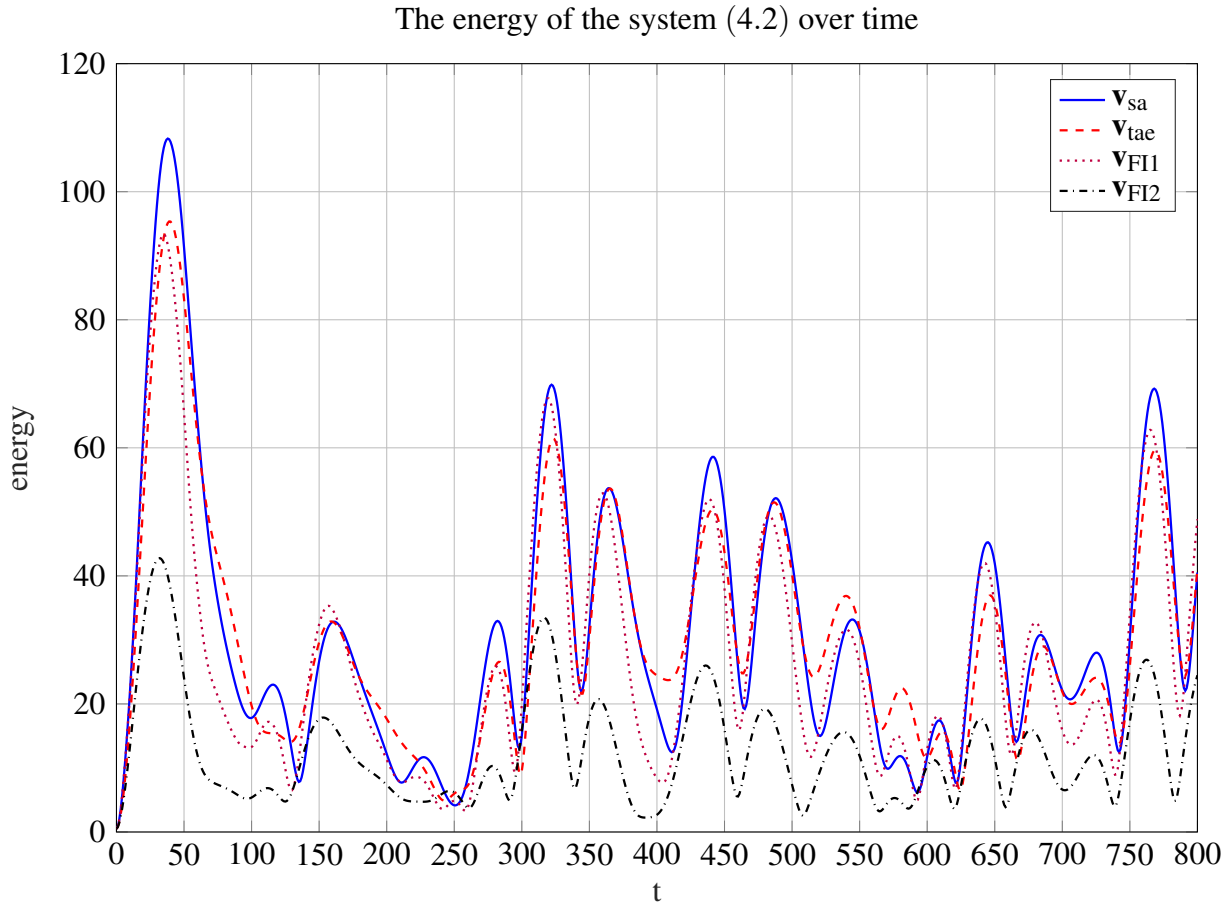

Figure 4.19: The energy of the non-stationary system (4.2)

In Figure 4.19 we compared the energies of the system (4.2) obtained for each viscosity parameter $\mathbf{v}_{\mathrm{sa}}$, $\mathbf{v}_{\mathrm{tae}}$, $\mathbf{v}_{\mathrm{FI1}}$, $\mathbf{v}_{\mathrm{FI2}}$. The blue line show the total energy for the case when the set of viscosities correspond to the optimal viscosities for the minimization of the spectral abscissa, i.e., $\mathbf{v}_{\mathrm{sa}}$, the red dashed line show the total energy for the case when the set of viscosities correspond to the optimal viscosities for the minimization of the total average energy, i.e., $\mathbf{v}_{\mathrm{tae}}$. The purple dotted line show the total energy for the case when the set of viscosities correspond to the optimal viscosities for the frequency isolation Approach 1, i.e., $\mathbf{v}_{\mathrm{FI1}}$, while the black dashed dotted line show the total energy for the case when the set of viscosities correspond to the optimal viscosities for the frequency isolation Approach 2, i.e., $\mathbf{v}_{\mathrm{FI2}}$. One can see that the spectral abscissa criterion resulted in faster decay of the total energy of the system. From Figure 4.19 we can see that the viscosity $\mathbf{v}_{\mathrm{FI2}}$ obtained in frequency isolation Approach 2 resulted in lower energy then other considered optimization criteria. The energy of the system over time is very similar for the other three optimization criteria.

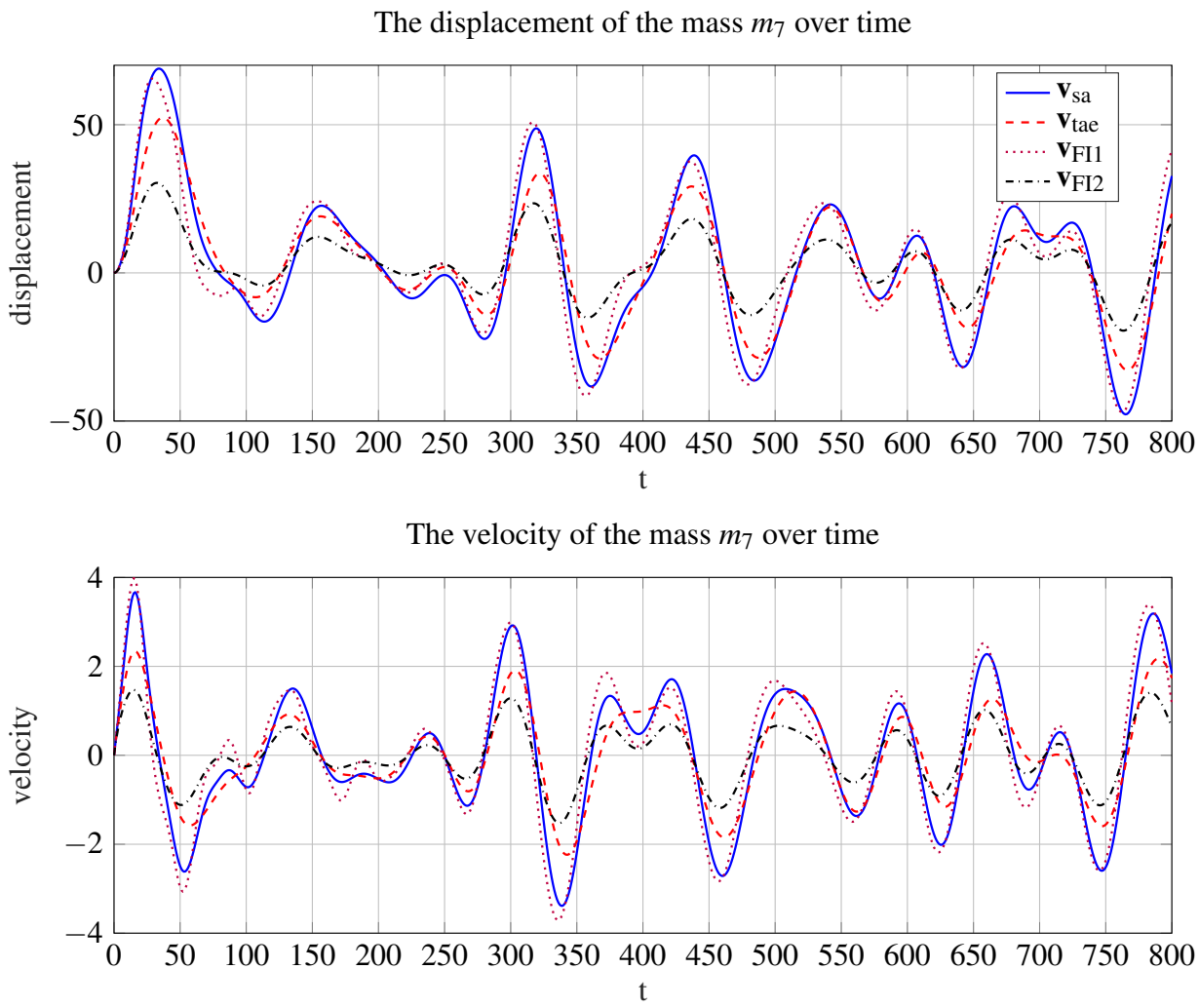

Figure 4.20: The behaviour of the displacement and velocity of the mass m_7 over time for two different sets of viscosities in non-stationary case

The Figures 4.20 and 4.21 show respectively the behaviour of the displacement and the velocity of the masses m_7 and m_{17} over time for the non-stationary system. The masses m_7 and m_{17} are chosen just for illustration and we see that the system with viscosity parameter $\mathbf{v}_{FI2}$ results in lowest amplitude in displacement of the mass and in its velocity, in both cases.

4.4 Conclusion

Throughout this chapter we have considered damping optimization for the stationary mechanical system $M\ddot{q}(\mathbf{v};t)+C(\mathbf{v})\dot{q}(\mathbf{v};t)+Kq(\mathbf{v};t)=0$ and non-stationary mechanical system $M\ddot{q}(\mathbf{v};t)+C(\mathbf{v})\dot{q}(\mathbf{v};t)+Kq(\mathbf{v};t)=f(t)$. We used two different approaches to damping optimization:

1. minimization of total average energy,

2. frequency isolation.

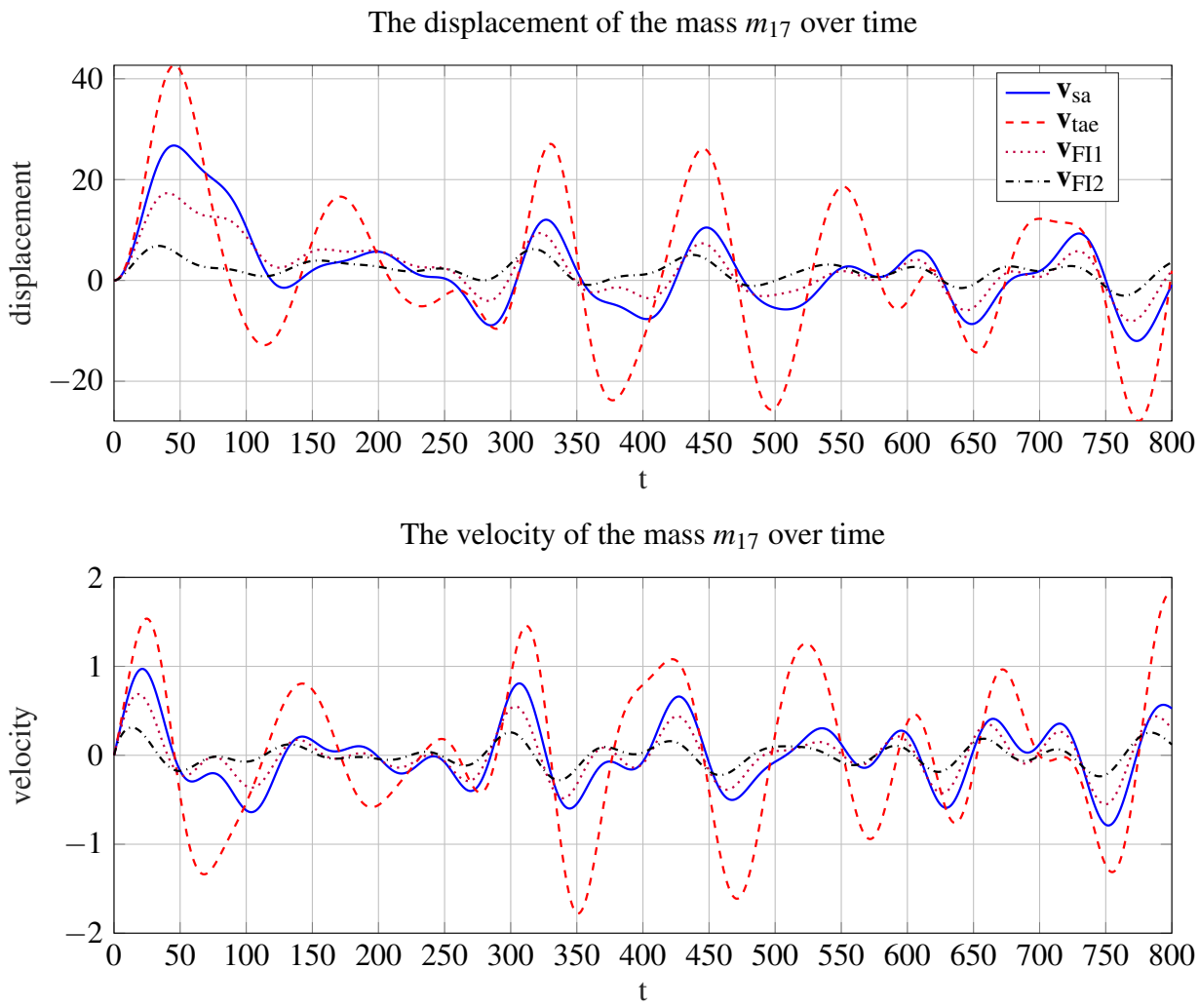

Figure 4.21: The behaviour of the displacement and velocity of the mass m_{17} over time for two different sets of viscosities in stationary case

Since only the damping matrix $C(\mathbf{v})$ depends on parameters, the typical (or often used, standard) approach for viscosity optimization ($\mathbf{v}$), when dealing with minimization of total average energy, assumes preprocessing based on the diagonalization of the mass and stiffness matrices, M and K. Contrary to this approach, we proposed the new approach, which is based on the diagonalization of the damping matrix $C(\mathbf{v})$, and then calculation of optimal viscosities. This is the main contribution of Chapter 4.1, i.e., we have shown that a slight change in the paradigm of damping optimization, for a certain structure, can significantly improve the performance of optimization methods. Although, in general, the new approach can not be more efficient than the standard one, we have shown that in the case when M, $C(\mathbf{v})$ and K are close to the case when all three can be simultaneously diagonalized (or when the mechanical system is close to modally damped one) we can derive optimal viscosities, explicitly or numerically, very efficiently.

We have also provided the bounds which can be easily used to determine whether the considered mechanical system is suitable for applying the new approach, i.e., if the mechanical system

under consideration is close to a modally damped one or not.

Our numerical examples from Subsection 4.1.3 show that with the proposed approach we can obtain satisfactory approximation for optimal parameters. Moreover, we illustrate that with our approach we can significantly accelerate the optimization process for the structured systems.

On the other hand, when dealing with frequency isolation in damping optimization, our approach was based on frequency-weighted damping optimization, i.e., the main problem was to achieve that eigenvalues of damped system are not close to dangerous weighted areas. We defined the areas as the ellipses with the centers on the imaginary axis and in the first optimization process we maximize the major axis of the ellipse such that there are no eigenvalues in the ellipse. Major axes were weighted since some of the areas are more significant then others. The other optimization process is based on minimization of spectral abscissa while making sure no eigenvalues are in the fixed ellipses. In these cases we used eigenvalue approximations from Section 2.3 since we needed both eigenvalues and eigenvectors for this approach and we needed them to be computed efficiently. The quality and efficiency of this method was illustrated in the examples. Last we compared all four approaches to damping optimization in Section 4.3 and we can conclude that the system behaves in the best way if we use frequency isolation Approach 2, i.e., if we maximize the major axis of the ellipse which center is close to the dangerous frequencies.

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Curriculum vitae

Matea Ugrica, was born on July 2, 1991 in Osijek, Croatia. After finishing high school in Osijek, she enroled in the undergraduate program in Mathematics at the Department of Mathematics, J. J. Strossmayer University of Osijek. In September 2015, she obtained her master's degree Mathematics and Computing Science at the Department of Mathematics, University of Osijek. In November 2015, she enroled Croatian doctoral program in Mathematics at the Department of Mathematics, Faculty of Science at the University of Zagreb.

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List of publications

Journal Publications:

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2. Truhar, N., Tomljanović, Z., Puvača, M. (2018), Approximation of damped quadratic eigenvalue problem by dimension reduction, *Applied mathematics and computation*, 374, 40-53.

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4. Puvača, M., Truhar, N., Tomljanović, Z. (2019), Efficient Approximation of Novel Residual Bounds for Parameter Dependent Quadratic Eigenvalue Problem, *Submitted in Journal of Computational and Applied Mathematics*

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1. Tomljanović, Z., Ugrica, M. (2015), QR dekompozicija koristeći Givensove rotacije i primjene, *Osječki matematički list* 14, 117-141.

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Potpis