Generalized inverse limits and topological entropy

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DOCTORAL THESIS

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Chapter 1

Introduction

Inverse limits have played a crucial role in the development of the theory of continua in the past half century. They have also been important in dynamical systems. One reason for this is that inverse limits with simple spaces and simple bonding maps can produce very complicated spaces as their inverse limits. In dynamical systems, inverse limits are used to "code", in a sense, complicated dynamical systems. The branched one-manifolds introduced by Robert Williams form a prime example of this. These inverse limits are used to model dynamical systems on solenoids, the Lorenz attractor (in the famous Lorenz system), the Plykin attractor (in the Plykin system).

Inverse limits of inverse sequences with upper semicontinous set-valued bonding functions (abbreviated generalized inverse limits) were introduced in 2004 by W. S. Mahavier [31] as inverse limits with closed subsets of the unit square and later in 2006 in [28] by Mahavier and W.T. Ingram. Since then, great emphasis has been placed on the generalized inverse limits over closed intervals with upper semicontinous set-valued functions (abbreviated u.s.c.) as bonding functions because even in that simple case, with same bonding u.s.c. functions, there is much that is not understood, and many kinds of interesting new spaces have emerged as these inverse limits, giving researchers much to investigate. Surprisingly, although many new and interesting spaces have emerged, it has also been shown that many types of spaces cannot occur. This new form of inverse limit has also shown up in applications to economics and in dynamical systems. For instance, certain models in economics, notably in backward economics, can involve two mappings, and the flexibility to study the effects of using either function at each stage in the model is a valuable feature of generalized inverse limits.

Most of the examples that demonstrate important properties of generalized inverse limits are on the closed unit interval and there one can see the important differences between standard and generalized inverse limits. For example, the subsequence theorem, the closed subset theorem, the full projection property, etc., all do not hold in the new setting, while they do in the standard setting [22].

In the first part of a thesis we investigate some categorical properties of generalized inverse systems and generalized inverse limits. In Chapters 2 and 3 we introduce basic notions we need in Chapter 4 where we give all results of the first part.

We introduce a category $\mathcal{CHU}$ with compact Hausdorff spaces as objects and u.s.c. functions as morphisms. Then we introduce the category $\mathcal{CU}$, a full subcategory of $\mathcal{CHU}$ with compact metric spaces as objects. We also introduce the category $\mathcal{ICU}$, a subcategory of inv - $\mathcal{CU}$ which consists of inverse sequences and level morphisms.

Let $(X_n, f_n)_{n=1}^{\infty}$ and $(Y_n, g_n)_{n=1}^{\infty}$ be inverse sequences in $\mathcal{CU}$, let $X = \varprojlim (X_n, f_n)_{n=1}^{\infty}$ and $Y = \varprojlim (Y_n, g_n)_{n=1}^{\infty}$ be their generalized inverse limits, respectively and let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of u.s.c. functions from X_n to 2^{Y_n}. In Chapter 4 we study which u.s.c. function from X to 2^Y can be interpreted as induced by the given sequence $(\varphi_n)_{n=1}^{\infty}$. The special case when all $\varphi_n, n \in \mathbb{N}$ are mappings was studied in [14, 28], therefore we generalize the definition of the induced function with some mild condition and in Example 4.17 it is shown that the condition is indeed necessary. In Theorems 4.13, 4.14 and Corollary 4.15 we give the necessary and sufficient conditions for an existence of induced function and that it is an u.s.c. function i.e. it is a morphism in the category $\mathcal{CU}$ which we denote by $\varprojlim (\varphi_n)$. Then we study a rule $F : \mathcal{ICU} \to \mathcal{CU}$ defined by

$$(X_n, f_n)_{n=1}^{\infty} \stackrel{F}{\longmapsto} \varprojlim (X_n, f_n)_{n=1}^{\infty}$$
$$(\varphi_n) \stackrel{F}{\longmapsto} \varprojlim(\varphi_n)\,.$$

We show that it is not a functor but very close to being one. Finally, in the Section 4.3 we give an application of the mentioned results.

The theory of dynamical systems is a mathematical theory connected with most of main areas of mathematics. It describes phenomena that are common to physical and biological systems throughout science. The impetus for advances in dynamical systems has come from many sources: mathematics, theoretical science, computer simulation, and experimental science.

This theory is inseparably connected with several other areas, primarily ergodic theory, symbolic dynamics, and topological dynamics. Some would say the theory is also deeply connected with the topological area of continuum theory. The modern theory of dynamical systems derives from the work of H.J.Poincare on the three-body problem of celestial mechanics, in the 1890's. The next major progress was due to G.D. Birkoff who early in his career had proved Poncare's "last geometric theorem" on fixed points of the annulus map in the 1920's. In the late 1950's, S. Smale brought a topological approach to the study of dynamical system, defining gradient-like flows that are now called Morse-Smale systems.

A dynamical system is a state space S, a set of times T and a rule R for evolution, $R : S \times T \to S$ that gives the consequent(s) to a state $s \in S$. A dynamical system can be considered to be a model describing the temporal evolution of a system. The central object of a study in topological dynamics is a topological dynamical system, i.e. a topological space, together with a continuous transformation, a continuous flow, or more generally, a semigroup of continuous transformations of that space. So, let (X, T) be a topological dynamical system, i.e., let X be a nonempty compact Hausdorff space and $T : X \to X$ a continuous function. Topological entropy is a nonnegative number which measures the complexity of the system. Roughly speaking, it measures the exponential growth rate of the number of distinguishable orbits as time advances.

The original definition was introduced by Adler, Konheim and McAndrew in 1965 [1]. Their idea was to assign a number to an open cover of a space to measure its size, and was inspired by Kolmogorov and Tihomirov (1961). To define a topological entropy for continuous maps they strictly imitated the definition of Kolmogorov-Sinai entropy of a measure preserving transformation in ergodic theory. In metric spaces a different definition was introduced by Bowen in 1971 [12] and independently by Dinaburg in 1970. It uses the notion of ε-separated points. Equivalence between the above two notions was proved by Bowen [12] in 1971.

The goal in the second part of the thesis is to generalize the notion of topological entropy so it can be used for u.s.c. functions, using the new tool, the Mahavier product defined in [19]. Mahavier products are a convenient way to study subsets of a generalized inverse limit space. They also make it easier to study "finite" generalized inverse limits – which are not interesting at all in inverse limits, but are interesting in their own right in generalized inverse limits.

In Chapter 5 we generalize the definition of topological entropy due to Adler, Konheim, and McAndrew [1] to set-valued functions from a closed subset of the interval to closed subsets of the interval. We view these set-valued functions, via their graphs, as closed subsets of $[0,1]^2$. Motivation for observing graphs and their Mahavier products is that when set-valued functions are iterated in the usual sense, information is lost - and often lost very fast.

In the Sections 5.1 and 5.2 we review Mahavier product and traditional version of topological entropy, together with their properties. In the Section 5.3 first we introduce some background information on the closed sets and open covers we are using and then, in several steps, finalizing with Theorem 5.44 we define the topological entropy of a closed subset of $[0,1]^2$. In Theorem 5.52 we show that, given a continuous function $f : [0,1] \to [0,1]$ and a closed subset G of $[0,1]^2$ as the graph of f^{-1}, then the topological entropy of G is the same as that of the traditional definition of topological entropy of the function f. We further compare the properties of the new definition of topological entropy with those of the traditional one, in Remarks 5.45, Theorem 5.48, and in Theorem 5.57 after generalizing our definition to subsets of $[0,1]^N$, for arbitrary $N \in \mathbb{N}$, in Section 5.4. Finally, in Section 5.5 we compute the topological entropy for various closed subsets G of $[0,1]^2$.

Chapter 2

Preliminaries

In this section we outline basic concepts which we use throughout the thesis.

Definition 2.1 *A topological space X is said to be disconnected if there exists separation of X-a pair U, V of disjoint, nonempty open subsets of X whose union is X. If X is not disconnected, we say it is connected.*

Definition 2.2 *A topological space is said to be compact if every open covering $\mathcal{U}$ of X contains finite subcovering.*

We state some properties of topological spaces related to the compactness and connectedness in the following theorem.

Theorem 2.3

1. *The continuous image of a connected space is connected.*
2. *The continuous image of a compact space is compact.*
3. *Every closed subspace of a compact space is compact.*
4. *Every compact subspace of a Hausdorff space is closed.*
5. *A subspace A of euclidean space $\mathbb{R}^n$ is compact if and only if A is closed and bounded.*

6. *A Cartesian product* $\prod_{\alpha \in A} X_\alpha$ *with product topology is connected if and only if each* $X_\alpha, \alpha \in A$ *is connected.*

7. *(Tychonoff theorem) A Cartesian product* $\prod_{\alpha \in A} X_\alpha$ *with product topology of compact spaces is compact if and only if each* $X_\alpha, \alpha \in A$ *is compact.*

8. *A Cartesian product* $\prod_{\alpha \in A} X_\alpha$ *with product topology is metrizable space if and only if* $\operatorname{card} A \leq \aleph_0$ *and each* $X_\alpha, \alpha \in A$ *is metrizable.*

Proof. See [16, p. 108, 109, 224] and [34, p. 165, 173, 234] . ■

On a given metric space X with metric d we define new metric $\overline{d}$ defined by the equation $\overline{d}(x, y) = \min\{d(x, y), 1\}$. Metrics d and $\overline{d}$ induce the same topology on X. The metric $\overline{d}$ is called standard bounded metric corresponding to d.

On the product space $\prod_{n=1}^{\infty} X_n$, where (X_n, d_n) is a compact metric space for each n, and the set of all diameters of (X_n, d_n) is majorized by 1, we use the metric

$$D(x, y) = \sup_{n \in \mathbb{N}} \left\{ \frac{d_n(x_n, y_n)}{2^n} \right\},$$

where $x = (x_1, x_2, x_3, \ldots)$, $y = (y_1, y_2, y_3, \ldots)$. It is well known that the metric D induces the product topology on $\prod_{n=1}^{\infty} X_n$.

Definition 2.4 *A continuum is a nonempty, compact, connected, metric space. Subcontinuum is a continuum that is a subspace of a continuum.*

Following theorem is consequence of the previous theorem.

Theorem 2.5 *Metric space which is homeomorphic to some continuum is also continuum.*

Some basic examples of continua:

Example 2.6 *(Arcs) An arc is any space which is homeomorphic to the closed interval* $[0, 1]$. *Since* $[0, 1]$ *is continuum, an arc is continuum.*

Example 2.7 *(Hilbert cube) A Hilbert cube is a space which is homeomorphic to the countable product* $\prod_{i=1}^{\infty}[0,1]$ *equipped with product topology. It follows from Theorem 2.3 (*6., 7. *and* 8.*) and Theorem 2.5 that a Hilbert cube is a continuum.*

Many other important continua will be mentioned in the second chapter.

Definition 2.8 *Let* X *and* Y *be topological spaces,* $f : X \to Y$ *continuous function and let* $f(X)$ *be subspace of* Y*. If* $f : X \to f(X)$ *is a homeomorphism, we say that* f *is an embedding of* X *in* Y*.*

An important property of a Hilbert cube is that every continuum can be embedded in a Hilbert cube as a closed subset. It is consequence of Urysohn metrisation theorem.

2.1 Hyperspaces

Many properties of continua can be studied using sequences of sets and their convergence in topological spaces called hyperspaces. In this section we define those spaces, state their properties which we will use throughout the thesis. All the details and proofs in this sections can be found in [21].

Let $(X, \mathcal{T})$ be a topological space. Let us denote

$$CL(X) = \{A \subseteq X : A \text{ is nonempty and closed in } X\}.$$

We equip the set $CL(X)$ with Vietoris topology defined as follows.

Definition 2.9 *Let* $(X, \mathcal{T})$ *be a topological space. The* ***Vietoris topology****,* $\mathcal{T}_V$*, for* $CL(X)$ *is the smallest topology for* $CL(X)$ *having the following properties:*

(i) $\{A \in CL(X) : A \subseteq U\} \in \mathcal{T}_V$ *whenever* $U \in \mathcal{T}$*;*

(ii) $\{A \in CL(X) : A \subseteq B\}$ *is* $\mathcal{T}_V$*-closed whenever* B *is* $\mathcal{T}$*-closed.*

Definition 2.10 *Let* $(X, \mathcal{T})$ *be a topological space and* $\mathcal{H} \subseteq CL(X)$*. A subspace* $(\mathcal{H}, \mathcal{T}_V|_{\mathcal{H}})$ *of the space* $(CL(X), \mathcal{T}_V)$ *is called* ***hyperspace*** *of the space* X*.*

We have the following theorem about the topological invariance of $CL(X)$.

Theorem 2.11 *If X and Y are homeomorphic, then the hyperspaces $CL(X)$ and $CL(Y)$ are homeomorphic.*

We introduce some hyperspaces that consist of sets having some elementary topological properties such as connectedness, compactness:

(i) $CLC(X) = \{A \in CL(X) : A \text{ is connected}\}$;

(ii) $2^X = \{A \in CL(X) : A \text{ is compact}\}$;

(iii) $C(X) = \left\{A \in 2^X : A \text{ is connected}\right\}$.

Statement analog to the Theorem 2.11 holds for the spaces defined above. Note that $2^X = CL(X)$ when X is compact because each closed subset of X is compact. Further, if X is a Hausdorff space, we have $2^X = \{A \subseteq X : A \text{ is non empty and compact}\}$, because every compact subset of a Hausdorff space is closed. In particular, both of the above hold when X is the unit segment $[0, 1]$ with standard metric.

Proposition 2.12 *Let X be a connected T_1-space. Then $CL(X)$ is connected.*

Let (X, d) be metric space. For any $r > 0$ and any $A \in CL(X)$, let $N_d(r, A) = \{x \in X : d(x, A) < r\}$.

Theorem 2.13 *Let (X, d) be a bounded metric space and $H_d : CL(X) \times CL(X) \to \mathbb{R}$ function defined as follows: for any $A, B \in CL(X)$,*

$$H_d(A, B) = \inf\{r > 0 : A \subseteq N_d(r, B) \text{ and } B \subseteq N_d(r, A)\}.$$

*Then H_d is metric on $CL(X)$ called **Hausdorff metric** for $CL(X)$ induced by d.*

In the previous theorem we assumed that (X, d) is bounded. We can do that without loss of generality as noted in the first section of this chapter.

Theorem 2.14 *If* $(X, \mathcal{T})$ *is a metrizable topological space, then* $\left(2^X, \mathcal{T}_V\right)$ *is metrizable. Moreover, if* d *is any metric for* X *that induces* $\mathcal{T}$*, then* $\mathcal{T}_V = \mathcal{T}_{H_d}$.

Theorem 2.15 *Let* $(X, \mathcal{T})$ *be a* T_1*-space. Then we have the following:*

(i) $(CL(X), \mathcal{T}_V)$ *is metrizable if and only if* $(X, \mathcal{T})$ *is a compact, metrizable space.*

(ii) $\left(2^X, \mathcal{T}_V\right)$ *is metrizable if and only if* $(X, \mathcal{T})$ *is metrizable.*

Definition 2.16 *Let* $(X, \mathcal{T})$ *be a topological space, and let* $(A_i)_{i=1}^{\infty}$ *be a sequence of subsets of* X*. We define the* ***limit inferior*** *of* $(A_i)_{i=1}^{\infty}$ *, denoted by* $\liminf(A_i)$*, and the* ***limit superior*** *of* $(A_i)_{i=1}^{\infty}$ *, denoted by* $\limsup(A_i)$ *as follows:*

$$\begin{aligned}\liminf(A_i) = &\{x \in X : \textit{ for any } U \in \mathcal{T} \textit{ such that } x \in U, U \cap A_i \neq \emptyset \\ &\textit{for all but finitely many } i\} \\ \limsup(A_i) = &\{x \in X : \textit{ for any } U \in \mathcal{T} \textit{ such that } x \in U, U \cap A_i \neq \emptyset \\ &\textit{for infinitely many } i\}\,.\end{aligned}$$

Obviously, $\liminf(A_i) \subseteq \limsup(A_i)$.

Definition 2.17 *Let* $(X, \mathcal{T})$ *be a topological space, let* $(A_i)_{i=1}^{\infty}$ *be a sequence of subsets of* X *and let* $A \subseteq X$*. TWe say that* $(A_i)_{i=1}^{\infty}$ *converges to* A *in* X *and write* $\lim(A_i) = A$ *if*

$$\liminf(A_i) = A = \limsup(A_i).$$

Two final theorems of this section connect convergence of sets in X and convergence in topological space $CL(X)$.

Theorem 2.18 *Let* $(X, \mathcal{T})$ *be a compact Hausdorff space, and let* $(A_i)_{i=1}^{\infty}$ *be a sequence in* $CL(X)$*. Then* $(A_i)_{i=1}^{\infty}$ *converges to* A *in* X *if and only if* $(A_i)_{i=1}^{\infty}$ *converges to* A *in topological space* $CL(X)$.

Theorem 2.19 *Let* $(X, \mathcal{T})$ *be a compact metrizable space, and let* $(A_i)_{i=1}^{\infty}$ *be a sequence in* $CL(X)$*. Then* $(A_i)_{i=1}^{\infty}$ *converges to* A *in* X *if and only if* $(A_i)_{i=1}^{\infty}$ *converges to* A *in the metric space* $(CL(X), H_d)$.

2.2 Categories

In this section we briefly state the basics about categories and functors which we use in the following two chapters.
For more see [20].
A **category** $\mathcal{C}$ consists of:

1. a class $\text{Ob}(\mathcal{C})$ of objects,

2. a class $\text{Mor}(\mathcal{C})$ which consists of sets $\mathcal{C}(X, Y)$, associated to each ordered pair of objects X, Y of $\mathcal{C}$, called morphisms from X to Y,

3. a law of composition $\circ$, which associate to each triple of objects X, Y, Z of $\mathcal{C}$, unique function $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$.

A category must satisfy following axioms:

A1: $X \neq X'$ and $Y \neq Y'$ implies that sets $\mathcal{C}(X, Y)$ and $\mathcal{C}(X', Y')$ are disjoint.

A2: Given $f \in \mathcal{C}(X, Y), g \in \mathcal{C}(Y, Z), h \in \mathcal{C}(Z, W)$, then

$$\circ\left(\circ\left(f, g\right), h\right) = \circ\left(f, \circ(g, h)\right).$$

A3: To each object X there is a morphism $1_X \in \mathcal{C}(X, X)$, called **identity morphism,** such that, for any $f \in \mathcal{C}(X, Y), g \in \mathcal{C}(Z, X)$,

$$\circ(1_X, f) = f,\ \circ(g, 1_X) = g.$$

It is easy to see that the morphism 1_X is uniquely defined. A morphism $f \in \mathcal{C}(X, Y)$ is usually denoted by $f : X \to Y$, and call morphism from **domain** X to **codomain** Y, and composition $\circ\,(f, g)$ with $g \circ f$. We say that a morphism $f : X \to Y$ in $\mathcal{C}$ is **isomorphic** (or is an **isomorphism**) if there exists a morphism $g : Y \to X$ in $\mathcal{C}$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.
We now list several examples of categories:

- The category *Set* of all sets and all functions between sets;

- the category Top of all topological spaces and all continuous functions between topological spaces;
- the category Gr of all groups and homomorphisms between groups;

We say that a category $\mathcal{C}'$ is a **subcategory** of a category $\mathcal{C}$ if it satisfies the following:

1. $\mathrm{Ob}(\mathcal{C}') \subseteq \mathrm{Ob}(\mathcal{C})$
2. for every ordered pair of objects (X', Y') in $\mathcal{C}'$ we have $\mathcal{C}'(X', Y') \subseteq \mathcal{C}(X', Y')$
3. if $f \in \mathcal{C}'(X', Y')$ and $g \in \mathcal{C}'(Y', Z')$ then their composition $g \circ f$ in $\mathcal{C}'$ coincides with the composition $g \circ f$ in $\mathcal{C}$.

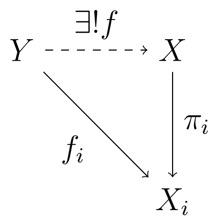
In particular, if $\mathcal{C}'(X', Y') = \mathcal{C}(X', Y')$, for every pair (X', Y') in $\mathcal{C}'$, we say that $\mathcal{C}'$ is **full** subcategory of $\mathcal{C}$.
Full subcategories of the category Top, whose objects are metric spaces, compact Hausdorff spaces, compact metric spaces are denoted with $\mathcal{M}, Cpt, c\mathcal{M}$, respectively. The category $Gr(Ab)$ whose objects are abelian groups is a full subcategory of the category Gr.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ is a rule which associates with every object X of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$ and with every morphism f in $\mathcal{C}(X, Y)$ a morphism $F(f)$ in $\mathcal{D}(F(X), F(Y))$, subject to the rules

$$F(f \circ g) = F(f) \circ F(g), \quad F(1_X) = 1_{F(X)}.$$

The inclusion of a subcategory $\mathcal{C}'$ in a category $\mathcal{C}$ is a functor. With $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ we denote identity functor. If $F : \mathcal{C} \to \mathcal{D}$ is a functor and f is an isomorphism in the category $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$. We define a composition of functors in a natural way. Composition is associative so we have invertible functors and isomorphic categories.

Definition 2.20 *Let $(X_i : i \in I)$ be a family of objects of a category $\mathcal{C}$. Then a* ***product*** *of the family $(X_i : i \in I)$ is an object X, together with the family $(\pi_i : X \to X_i : i \in I)$ of morphisms in $\mathcal{C}$ called* ***projections****, satisfying* *the following universal property. Given any object Y of $\mathcal{C}$ and a family $(f_i : Y \to X_i : i \in I)$ of morphisms in $\mathcal{C}$, there exists a unique morphism $f : Y \to X$ in $\mathcal{C}$, such that $\pi_i f = f_i$, for every $i \in I$.*



If the product $(X, (\pi_i, i \in I))$ of a family $(X_i : i \in I)$ of objects in the category $\mathcal{C}$ exists, then the object X is unique up to an isomorphism and we denote it with $\prod_{i\in I} X_i$.
Categories Set, Gr, Top, Cpt admit products i.e. any family of objects from the named categories admits product X and X is just a Cartesian product of $(X_i : i \in I)$ together with projections π_i from X on $X_i, i \in I$.
Since the topological product of an uncountably many metric spaces is not a metric space, the category $\mathcal{M}$ doesn't admit product.

Chapter 3

Inverse limits and generalized inverse limits

Inverse limits have played very important role in the development of the theory of continua in the past 50 years. They have also been important in dynamical systems. One reason for this is that inverse systems with simple spaces and simple bonding maps can produce very complicated spaces as their inverse limits.

The study of the inverse limits begins in the 1920s and 1930s while inverse limits of inverse sequences with upper semicontinous set-valued bonding functions (abbreviated generalized inverse limits) were introduced in 2004 by W. S. Mahavier [31] as inverse limits with closed subsets of the unit square and later in 2006 in [28] by Mahavier and W.T. Ingram.

Since then, great emphasis has been placed on the inverse limits over closed intervals with upper semicontinous set-valued functions (abbreviated u.s.c.) as bonding functions because even in a simple case, with same bonding u.s.c. functions, there is much that is not understood, and many kinds of interesting new spaces have emerged as these inverse limits, giving researchers much to investigate [4, 5, 7, 17, 22, 27]. Surprisingly, although many new and interesting spaces have emerged, it has also been shown that many types of spaces cannot occur.

Goal of this chapter is to list all notions used in the Chapter 4 where we introduce all the results. In the first section we provide an introduction to inverse systems and inverse limits in the most general way, in the category theory. Then we introduce a definition of inverse limit with topological spaces as terms and continuous functions as bonding maps. At the end we state basic compactness and connectedness theorems. For more details see [32, 33].

In the second section we introduce a generalized inverse limit and various theorems and examples that differ them from inverse limits.

3.1 Inverse limits

Definition 3.1 *A* ***preordering*** *on a set Λ is a binary relation $\leq$ on Λ which is reflexive and transitive. A preordering is called an* ***ordering*** *if it is antisymmetric. A preordered set $(\Lambda, \leq)$ is said to be* ***directed*** *provided for any $\lambda_1, \lambda_2 \in \Lambda$ there exists a $\lambda \in \Lambda$ such that $\lambda_1 \leq \lambda$ and $\lambda_2 \leq \lambda$. We say that preordered set $(\Lambda, \leq)$ is* ***cofinite*** *provided for each $\lambda \in \Lambda$ the set of all predecessors of λ, $\{\lambda' \leq \lambda : \lambda' \in \Lambda\}$, is a finite set.*

If $(\Lambda, \leq)$ is a preordered set and $\Lambda' \leq \Lambda$, then $\leq$ induces a preordering on Λ' and one speaks of the preordered subset $(\Lambda', \leq)$. The preordered subset Λ' is **cofinal** in Λ if each $\lambda \in \Lambda$ admits $\lambda' \in \Lambda'$ such that $\lambda \leq \lambda'$. If Λ' is cofinal in Λ, then Λ is directed if and only if Λ' is directed.

Definition 3.2 *Let $\mathcal{C}$ be an arbitrary category. An* ***inverse system*** *$(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in the category $\mathcal{C}$ consists of a directed set Λ, called the* ***index set****, of an objects X_λ from $\mathcal{C}$ for each $\lambda \in \Lambda$, called the* ***terms****, and of morphisms $p_{\lambda\lambda'} : X'_\lambda \to X_\lambda$ from $\mathcal{C}$ for each pair of indeces $\lambda, \lambda' \in \Lambda, \lambda \leq \lambda'$, called the* ***bonding morphisms****. Moreover, one requires that $p_{\lambda\lambda} = 1_{X_\lambda}$, for all $\lambda \in \Lambda$ and that $\lambda \leq \lambda' \leq \lambda''$ implies $p_{\lambda\lambda'} \circ p_{\lambda'\lambda''} = p_{\lambda\lambda''}$*

An inverse system indexed by the positive integers $\mathbb{N}$ is called an **inverse sequence**. It is often denoted by $(X_n, p_{nn+1}, \mathbb{N})$ or (X_n, p_n), omitting the index set.

Definition 3.3 *A* ***morphism*** *(f, f_μ)* ***of inverse systems*** *$(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $(Y_\mu, p_{\mu\mu'}, M)$ consists of a function $f : M \to \Lambda$, called the index function* *and of morphisms* $f_\mu : X_{f(\mu)} \to Y_\mu$ *in* $\mathcal{C}$, *for each* $\mu \in M$, *such that whenever* $\mu \le \mu'$, *then there is a* $\lambda \in \Lambda, \lambda \ge f(\mu), f(\mu')$, *for which* $f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}$, *i.e. the following diagram commutes.*

$$\begin{array}{ccccc} X_{f(\mu)} & \xleftarrow{p_{f(\mu)\lambda}} & X_\lambda & \xrightarrow{p_{f(\mu')\lambda}} & X_{f(\mu')} \\ \big\downarrow f_\mu & & & & \big\downarrow f_{\mu'} \\ Y_\mu & & \xleftarrow[q_{\mu\mu'}]{\qquad\qquad} & & Y_{\mu'} \end{array}$$

Remark 3.4 *Using the same notation from the previous definition we have the following:*

- *If the index function* f *is increasing then one requires that the mappings* f_μ *satisfy* $f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}$ *and the morphism is called* ***simple*** *(left diagram)*
- *If the inverse systems have same index set* Λ *and the index function is identity* 1_Λ, *the morphism is called* ***level-preserving morphism*** *or shorter a* ***level morphism*** *(right diagram)*

$$\begin{array}{ccc} X_{f(\mu)} & \xleftarrow{p_{f(\mu')f(\mu)}} & X_{f(\mu')} \\ \big\downarrow f_\mu & & \big\downarrow f_{\mu'} \\ Y_\mu & \xleftarrow[q_{\mu\mu'}]{} & Y_{\mu'} \end{array} \qquad\qquad \begin{array}{ccc} X_\lambda & \xleftarrow{p_{\lambda\lambda'}} & X'_\lambda \\ \big\downarrow f_\lambda & & \big\downarrow f_{\lambda'} \\ Y_\lambda & \xleftarrow[q_{\lambda\lambda'}]{} & Y_{\lambda'} \end{array}$$

Level morphisms have very important role in the third chapter.
For every inverse system $(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ we can define identity morphism of systems:

$$(1_\Lambda, 1_{X_\lambda}) : (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to (X_\lambda, p_{\lambda\lambda'}, \Lambda)$$

Definition 3.5 *If* $(f, f_\mu) : \mathbf{X} \to \mathbf{Y}$ *and* $(g, g_\nu) : \mathbf{Y} \to \mathbf{Z}$ *are morphisms of inverse systems, we define the composition* $(g, g_\nu) \circ (f, f_\mu) = (h, h_\nu) : \mathbf{X} \to \mathbf{Z}$ *as follows:*

$$h = f \circ g \text{ and } h_\nu = g_\nu \circ f_{g(\nu)} : X_{h(\nu)} \to Z_\nu.$$

Remark 3.6 *It is easy to see the following:*

(i) *Composition defined in the previous definition is a morphism between inverse systems.*

(ii) *Composition of morphisms between inverse systems is associative operation*

(iii) $(f, f_\mu) \circ (1_\Lambda, 1_{X_\lambda}) = (f, f_\mu)$ *and* $(1_\Lambda, 1_{X_\lambda}) \circ (g, g_\nu) = (g, g_\nu)$

We have thus obtained new category, denoted by inv-$\mathcal{C}$, whose objects are inverse systems in $\mathcal{C}$ and morphisms are morphisms of inverse systems described above.

Remark 3.7 *A property of an index set that has proven useful is cofiniteness because it often makes proof by induction on the number of predecessors possible. It is shown that this condition is not restrictive because, with certain requirements satisfied, one can indeed work with cofinite and ordered index set and level morphisms. For more details see [32] and [33].*

A single object X of $\mathcal{C}$ can be viewed as a **rudimentary system**, i.e., a system indexed by a singleton and will be denoted by (X).

Definition 3.8 *An* ***inverse limit*** *of an inverse system* $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *is a morphism* $\mathbf{p} : (X) \to \mathbf{X}$ *of inv -* $\mathcal{C}$ *satisfying following universal property. If* $\mathbf{g} : (Y) \to \mathbf{X}$ *is an arbitrary morphism of inv -* $\mathcal{C}$*, there exists a unique morphism* $(g) : (Y) \to (X)$ *of inv -* $\mathcal{C}$ *(i.e.* $\mathcal{C}$*) such that* $\mathbf{p}\ (g) = \mathbf{g}$.

Proposition 3.9 *Inverse limit of the inverse system* $\mathbf{X}$, *if it exists, is unique up to a natural isomorphism.*

Proof. See [32, p. 54] ■

Due to the previous proposition we usually call the space X from the Definition 3.8 as the inverse limit and denote it with $X = \lim \mathbf{X} = \lim (X_\lambda, p_{\lambda\lambda'}, \Lambda)$.

All necessary and sufficient conditions for existence of an inverse limit in $\mathcal{C}$ of an inverse system in category $\mathcal{C}$ can be found in [32, Ch. I].

Inverse systems have limits in the following categories: $Set, Gr, Gr(Ab), Top$.

Remark 3.10 *Let inverse systems $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, p_{\mu\mu'}, M)$ admit inverse limits. A morphism $(f, f_\mu) : (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to (Y_\mu, p_{\mu\mu'}, M)$ of inv-$\mathcal{C}$ induces a morphism $f : \lim (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \lim (Y_\mu, p_{\mu\mu'}, M)$ in $\mathcal{C}$. This is a unique morphism f such that for every $\mu \in M$, $f_\mu p_{f(\mu)} = q_\mu f$.*

If in an inverse system $(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ all the bonding morphisms $p_{\lambda\lambda'}$ are isomorphisms, then for any $\lambda \in \Lambda, p_\lambda : X = \lim (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to X_\lambda$ is an isomoprhim in $\mathcal{C}$.

Proof. See [32, p. 57]. ■

Remark 3.11 *In the category Top an inverse limit X of $(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ can be obtained as the subspace of $\prod_{\lambda\in\Lambda} X_\lambda$, which consists of all points x satisfying*

$$\pi_\lambda(x) = p_{\lambda\lambda'}\pi_{\lambda'}(x), \lambda \le \lambda',$$

where $\pi_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \to X_\lambda$ denotes the projection on the $\lambda^{th}-$coordinate. Moreover, the projections $p_\lambda : X \to X_\lambda$ of the limit space X are given by

$$p_\lambda = \pi_\lambda|_X.$$

Theorem 3.12 *Let X be an inverse limit of an inverse system $(X_\lambda, p_{\lambda\lambda'}, \Lambda)$, where all X_λ are Hausdorff spaces, for all $\lambda \in \Lambda$. Then, X is closed subspace of the product $\prod_{\lambda\in\Lambda} X_\lambda$. Moreover, if all terms X_λ are compact and non-empty, then X is compact and non-empty.*

Proof. See [32, p. 58]. ■

From the previous theorem it follows that inverse system in the category Cpt has an inverse limit.

In general, inverse systems in the category $\mathcal{M}$ do not have limits but we have the following.

Corollary 3.13 *Inverse limit of an inverse sequence of metrizable spaces is metrizable space. Inverse limit of an inverse sequence of non-empty compact metric spaces is non-empty compact metric space.*

Proposition 3.14 *Let A be a subspace of the inverse limit $X = \lim(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $A_\lambda = p_\lambda(A)$, for all $\lambda \in \Lambda$. Then $\left(\mathrm{Cl}\, A_\lambda, p_{\lambda\lambda'}|_{\mathrm{Cl}(A_{\lambda'})}, \Lambda\right)$ is an inverse system and $\mathrm{Cl}\,(A) = \lim\left(\mathrm{Cl}\, A_\lambda, p_{\lambda\lambda'}|_{\mathrm{Cl}(A_{\lambda'})}, \Lambda\right)$*

Proof. See [32]. ■

Theorem 3.15 *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, an inverse system of non-empty compact and connected Hausdorff spaces. Then the inverse limit $X = \lim \mathbf{X}$ is non-empty compact and connected space.*

Proof. See [32]. ■

From the previous we have that the limit of inverse sequence consisting of continua is a continuum.
At the end of this section we give some examples.

Example 3.16 *Let $([0,1], f_{nn+1})$ be an inverse sequence where $f_{nn+1}(x) = 1_{[0,1]}$, for each $n \in \mathbb{N}$. It follows from Remark 3.11 that $\lim([0,1], f_{nn+1})$ is an arc.*

Example 3.17 *Let S^1 be the unit circle, i.e. $S^1 = \{z \in \mathbb{C} : \|z\| = 1\}$. For $p \in \mathbb{N}, p \geq 2$, let $q_p : S^1 \to S^1$ be given by $q_p(z) = z^p, z \in S^1$. An inverse limit of an inverse sequence $(S^1, f_{nn+1}), f_{nn+1} = q_p, \forall n \in \mathbb{N}$, is a continuum called p-adic solenoid.*

Example 3.18 *Let $(Y_1, y_1) = (S^1, s_0)$ be a pointed circle and $(Y_n, y_n) = (Y_{n-1}, y_{n-1}) \vee (S^1, s_0)$ wedge of n pointed circles. Let $p_{nn+1} : Y_{n+1} \to Y_n$ be defined by its restrictions $p_{nn+1}|_{Y_n} = 1_{Y_n}$ and $p_{nn+1}|_{(Y_{n+1}\setminus Y_n)} = const_{y_n}$. Then the inverse limit $\lim((Y_n, y_n), p_{nn+1})$ is called Hawaiian earring.*

3.2 Generalized inverse limits

Definition 3.19 *Let X and Y be topological spaces. A function $f : X \to CL(Y)$ is said to be* ***upper semicontinous at the point*** $x_0 \in X$ *if for each open set V in Y containing $f(x_0)$, there is an open set U in X containing x_0 such that if $x \in U$, then $f(x) \subseteq V$.*

Function f is called **upper semicontinous** (abbreviated u.s.c.) if it is upper semicontinous at each point $x \in X$.

Definition 3.20 *The graph $\Gamma(f)$ of a function $f : X \to 2^Y$ is set $\{(x, y) \in X \times Y : y \in f(x)\}$.*

Following theorem is a well known characterisation of u.s.c functions between Hausdorff compacta.

Theorem 3.21 *Let X and Y be compact Hausdorff spaces. A function $f : X \to 2^Y$ is u.s.c. if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.*

Proof. See [28] ■

If $A \subseteq X$ then the image of A under $f : X \to 2^Y$ is $f(A) = \cup_{x \in A} f(x)$. Function $f : X \to 2^Y$ is said to have surjective graph if $f(X) = Y$.

Let $f : X \to Y$ be a function. Then f induces a function $\tilde{f} : X \to 2^Y$ given by $\tilde{f}(x) = \{f(x)\}$.

In that case, following statements are equivalent:

(i) f is continuous

(ii) $\tilde{f}$ is u.s.c.

(iii) $\tilde{f}$ is continuous (with respect to Vietoris topology).

Often, instead of $\tilde{f} : X \to 2^Y$ we write $f : X \to 2^Y$ and identify f with $\tilde{f}$. For instance, if $f = id_X : X \to X$, we will refer to $\tilde{f}$ as the identity on X.

A general example of u.s.c. functions is following. If X and Y are compact metric space and $f : X \to Y$ is a continuous surjective function, then $f^{-1} : Y \to 2^X$ is an u.s.c. function where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

Proofs of statements given above and other general facts about u.s.c. functions can be found in [29], [30] and [35].

Proposition 3.22 *Let X and Y be topological spaces. If $f : X \to \mathrm{CL}(Y)$ is continuous with respect to the topology $\mathcal{T}_V$ then f is an u.s.c. function.*

Proof. Let $x_0 \in X$ and W open set in Y such that $f(x_0) \subseteq V$. Then $f(x_0) \in \{A \in CL(X) : A \subseteq W\}$, which is open set in $CL(Y)$ by Definition 2.9. Since f is continuous, there exists open set U such that $x_0 \in U$ and $f(x) \in f(U) \subseteq \{A \in CL(X) : A \subseteq W\}$. Therefore, for each $x \in U$, $f(x) \subseteq W$ so f is u.s.c. ■

Converse doesn't hold. For example, function $f : [0,1] \to 2^{[0,1]}$ given by $f(0) = [0,1], f(x) = \{0\}, x \in (0,1]$ is u.s.c. by Theorem 3.21 and it is not continuous in respect with $\mathcal{T}_V$. Set $W = \left\{A \in 2^{[0,1]} : A \subseteq \langle \frac{1}{4}, \frac{3}{4} \rangle\right\} \in \mathcal{T}_V$ and $f^{-1}(W) = \{0\}$ which is not open in $[0,1]$. Therefore, f is not continuous with respect to $\mathcal{T}_V$.

Now we introduce the notion of inverse limits with u.s.c. bonding function, denoted by generalized inverse limits, as introduced by Mahavier in [31] and Ingram and Mahavier in [28].

Definition 3.23 *A **generalized inverse sequence** of compact Hausdorff spaces X_k with u.s.c. bonding functions f_k is a sequence $(X_k, f_k)_{k=1}^{\infty}$, where $f_k : X_{k+1} \to 2^{X_k}$ for each k.*

Let (X_k, f_k) be an inverse sequence. Using $\tilde{f}_k$ as defined above, it induces unique generalized inverse sequence $(X_k, \tilde{f}_k)_{k=1}^{\infty}$. Therefore every inverse sequence can be observed as a generalized one.

Definition 3.24 *The **generalized inverse limit** of a generalized inverse sequence $(X_k, f_k)_{k=1}^{\infty}$, denoted by $\varprojlim (X_k, f_k)_{k=1}^{\infty}$, is the subspace of the product space $\prod_{k=1}^{\infty} X_k$ defined by*

$$\varprojlim (X_k, f_k)_{k=1}^{\infty} = \left\{ x \in \prod_{k=1}^{\infty} X_k : \pi_k(x) \in f_k(\pi_{k+1}(x)), \forall k \in \mathbb{N} \right\}.$$

If $X_k = X, f_k = f, \forall k \in \mathbb{N}$, to simplify notation, we denote generalized inverse limit $\varprojlim (X_k, f_k)_{k=1}^{\infty}$ with $\varprojlim \mathbf{f}$.

Let (X_k, f_k) be an inverse sequence and $(X_k, \tilde{f}_k)_{k=1}^{\infty}$ an induced generalized inverse sequence. Since we have (using notations from Definition 3.24 and Remark 3.11) $\pi_k(x) \in \tilde{f}_k(\pi_{k+1}(x)) \Leftrightarrow \pi_k(x) \in \{f_k(\pi_{k+1}(x))\} \Leftrightarrow \pi_k(x) = f_k(\pi_{k+1}(x))$, we get that $\varprojlim \left(X_k, \tilde{f}_k\right)_{k=1}^{\infty} = \lim(X_k, f_k)$. Therefore, an inverse limit of an inverse sequence can be viewed as a generalized inverse limit of an induced generalized inverse sequence. Hence the term "generalized" both for inverse sequences and inverse limits really makes sense.

Many properties that inverse limits have, do not hold for generalized inverse limits. Some theorems do extend, such as the Existence theorem which is analogous to the Theorem 3.15 for inverse limits and Topology conjugacy theorem (Corollary 3.27), analogous to the Remark 3.10 (applied to inverse sequences).

Theorem 3.25 *If $(X_k, f_k)_{k=1}^{\infty}$ is an inverse sequence of non-empty compact Hausdorff spaces with u.s.c. bonding functions then the generalized inverse limit $\varprojlim (X_k, f_k)$ is nonempty and compact.*

Proof. See [28]. ■

Following theorem and corollary are from [28].

Theorem 3.26 *Let (X_k) and (Y_k) be sequences of compact Hausdorff spaces and, for each positive integer $i, f_i : X_{i+1} \to 2^{X_i}$ and $g_i : Y_{i+1} \to 2^{Y_i}$, are u.s.c. functions. Suppose further that, for each positive integer $i, \varphi_i : X_i \to Y_i$ is a mapping such that $\varphi_i \circ f_i = g_i \circ \varphi_{i+1}$. The function $\varphi : \varprojlim \mathbf{f} \to \varprojlim \mathbf{g}$ given by $\varphi(\mathbf{x}) = (\varphi_1(x_1), \varphi_2(x_2), \varphi_3(x_3), \ldots)$ is continuous. Furthermore, φ is one-to-one (and surjective) if each φ_i is one-to-one (and surjective).*

Proof. The function $\Phi : \prod_{i=1}^{\infty} X_i \to \prod_{i=1}^{\infty} Y_i$ given by $\Phi(x) = (\varphi_1(x_1), \varphi_2(x_2), \varphi_3(x_3), \ldots)$ is continuous and is one-to-one if each φ_i is one-to-one. Because $\varphi = \Phi|_{\varprojlim \mathbf{f}}, \varphi$ inherits continuity from Φ, and it is one-to-one if each φ_i is one-to-one. Thus, there are only two things to show: (1) for $x \in \varprojlim \mathbf{f}$,that $\varphi(x) \in \varprojlim \mathbf{g}$, and (2) if each φ_i is a homeomorphism then φ is surjective. To show (1), we need to know $\varphi_i(x_i) \in g_i(\varphi_{i+1}(x_{i+1}))$ for each positive integer i. Suppose i is a positive integer. Because $\varphi_i \circ f_i =$

$g_i \circ \varphi_{i+1}$, we have $\varphi_i(f_i(x_{i+1})) \in g_i(\varphi_{i+1}(x_{i+1}))$. From $x \in \varprojlim \mathbf{f}$, it follows $x_i(i) \in f_i(x_{i+1})$ and, thus, $\varphi_i(x_i) \in \varphi_i(f_i(x_{i+1})) \in g_i(\varphi_{i+1}(x_{i+1}))$. It follows that $\varphi(x) \in \varprojlim \mathbf{g}$. To see (2), suppose $y \in \varprojlim \mathbf{g}$. Because each φ_i is one-to-one, $x = \big(\varphi_1^{-1}(y_1), \varphi_2^{-1}(y_2), \varphi_3^{-1}(y_3), \ldots\big)$ is a point of $\prod_{i=1}^{\infty} X_i$ such that $\varphi(x) = y$. We now observe that $x \in \varprojlim \mathbf{f}$. Let $i \in \mathbf{N}$. Because $g_i \circ \varphi_{i+1} = \varphi_i \circ f_i$ and $\varphi_{i+1}(x_{i+1}) = y_{i+1}, g_i(y_{y+1} = \varphi_i\,(f_i(x_{i+1})))$. Because $y_i \in g_i(y_{i+1})$, there is a point t of $f_i(x_{i+1})$ such that $y_i = \varphi_i(t)$. But, $\varphi_i(x_i) = y_i$ and φ_i is one-to-one, so $x_i = t$. Thus, $x_i \in f_i(x_{i+1})$ and we have $x \in \varprojlim \mathbf{f}$. ■

Suppose X is a compact Hausdorff space. If $f : X \to 2^X$ and $g : X \to 2^X$ are u.s.c. function and g is surjective, f and g are said to be **topologically conjugate** provided there is homeomorphism $g : X \to X$ such that $fh = hg$.

Following corollary is an immediate consequence of the previous theorem.

Corollary 3.27 *If $f : X \to 2^X$ and $g : X \to 2^X$ are topologically conjugate upper semicontinuous functions, then $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{g}$ are homeomorphic.*

Many important continua can be constructed as inverse limits with just one bonding function. Bellow we give several examples. All proofs can be found in [22] but some of them are very important in the last chapter of the thesis so we list them here together with proofs.

Example 3.28 *(The Hilbert cube)*
Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(t) = [0,1]$ for each $t, 0 \leq t \leq 1$. Then $\varprojlim \mathbf{f}$ is homeomorphic to Hilbert cube.

Several examples in the thesis (including the following one) involve the cantor set, which we now define.

Let $X = (\{0,1\}, \mathcal{D})$ be a discrete topological space and $X_n = X, n \in \mathbb{N}$. Then a space $\prod_{n=1}^{\infty} X_n = \{0,1\}^{\infty}$ is called the Cantor set.

Example 3.29 *(The Cantor set) Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(t) = \{0,1\}$ for $t, 0 \leq t \leq 1$. Then $\varprojlim \mathbf{f}$ is homeomorphic to Cantor set.*

Following example is very important in the Chapter 5 so we give it with a proof.

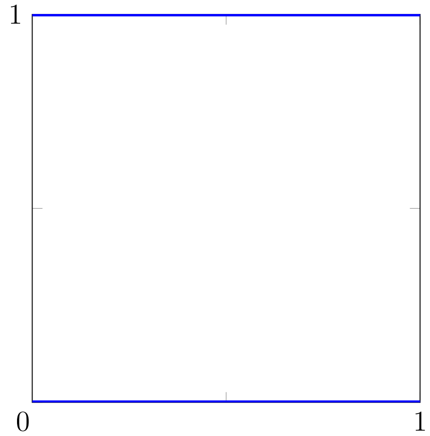


Figure 3.1: The graph of f in Example 3.29

Example 3.30 *(The Cantor fan)*
Let $f : [0,1] \to 2^{[0,1]}$ *be given by* $f(t) = \{t, 1-t\}$ *for* $0 \le t \le 1$. *Then* $\varprojlim \mathbf{f}$ *is the Cantor fan (i.e. cone over Cantor set) with the vertex* $\mathbf{v} = (1/2, 1/2, 1/2, \ldots)$

Proof. There are four homeomorphisms whose union is f. They are $g_1 : [0,1/2] \to [0,1/2]$ given by $g_1(t) = t, g_2 : [0,1/2] \to [1/2,1]$ given by $g_2(t) = 1-t, g_3 : [1/2,1] \to [0,1/2]$ given by $g_3(t) = 1-t$, and $g_1 : [1/2,1] \to [1/2,1]$ given by $g_4(t) = t$. A point x is in $\varprojlim \mathbf{f}$ if and only if there is a sequence $h_1, h_2, h_3, \ldots$ such that $h_i \in \{g_1, g_2, g_3, g_4\}$ for each i and $x_i = h_i(x_{i+1})$ for each positive integer i. Each such inverse limit is an arc having $\mathbf{v} = (1/2, 1/2, 1/2, \ldots)$ as one endpoint and the other endpoint in the Cantor set $\{0,1\}^\infty$. Moreover, if $\mathbf{p} \in \{0,1\}^\infty$, there is a sequence $\mathbf{h}$ such $h_i \in \{g_1, g_2, g_3, g_4\}$ for each i and $\mathbf{p} \in \varprojlim \mathbf{h}$. ■

Example 3.31 *(Hurewicz continuum)*
Let $g_1 : [0,1] \to [0,1]$ *be the mapping given by* $g_1(t) = t + 1/2$ *for* $0 \le t \le 1/2$ *and* $g_1(t) = 3/2 - t$ *for* $1/2 \le t \le 1$. *Let* $g_2 : [0,1] \to [0,1]$ *be the mapping given by* $g_2(t) = 1/2 - t$ *for* $0 \le t \le 1/2$ *and* $g_2(t) = t - 1/2$ *for* $1/2 \le t \le 1$. *Let* $f : [0,1] \to 2^{[0,1]}$ *be an upper semicontinuous function whose graph is the set-theoretic union of graphs of* g_1 *and* g_2. *Then,* $\varprojlim \mathbf{f}$ *is a nonplanar continuum called Hurewicz continuum.*

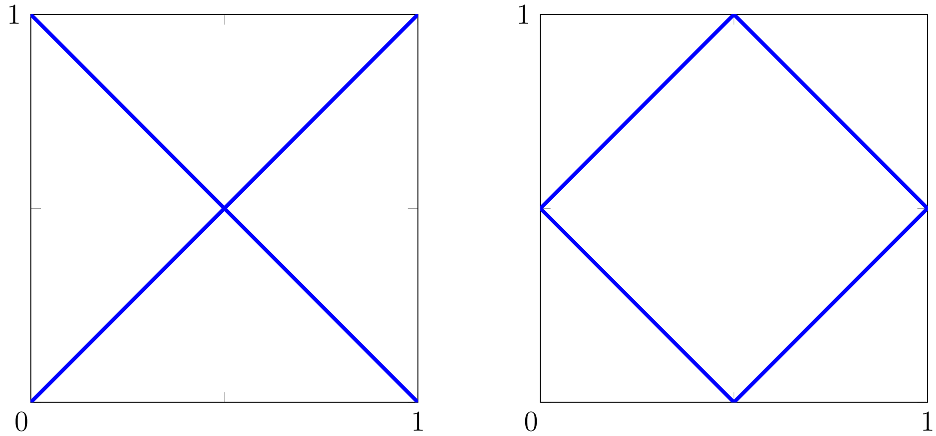


Figure 3.2: The graph of the functions from Example 3.30 and Example 3.31

For standard inverse limits, taking cofinal subsets of the index set in inverse systems we get homeomorphic inverse limits. For generalized inverse systems that is not the case as the following example shows.

Example 3.32 *Let $f : [0,1] \to 2^{[0,1]}$ be the upper semicontinous function given by $f(t) = \left\{1-t, \frac{1}{2}\right\}$ for $0 \leq t \leq \frac{1}{2}$, $f(t) = \frac{1}{2}$ for $\frac{1}{2} < t < 1$ and $f(1) = [0, \frac{1}{2}]$. Then $f^2 : [0,1] \to 2^{[0,1]}$ is given by $f^2(0) = [0, \frac{1}{2}], f^2(t) = \frac{1}{2}$ for $0 < t < 1$, and $f(1) = [\frac{1}{2}, 1]$. Then $\varprojlim ([0,1], f)$ contains a triod and $\varprojlim ([0,1], f^2)$ is an arc.*

Proof. Let $M = \varprojlim ([0,1], f)$. Let $\alpha = \{x \in M : x_1 \in [1/2, 1], x_2 = 1 - x_1,$ and $x_{2j-1} = 1,$ while $x_{2j} = 0$ for each integer$j > 1\}$. Let $\beta = \{x \in M : x_1 = x_2 = 1/2, x_3 \in [1/2, 1], x_4 = 1 - x_3$ and $x_{2j-1} = 1,$ while $x_{2j} = 0$ for each integer $j > 2\}$. Let $\gamma = \{x \in M : x_1, x_2 \in [0, 1/2],$ and $x_{2j-1} = 1,$ while $x_{2j} = 0$ for each integer $j > 1\}$. Then $(1/2, 1/2, 1, 0, 1, 0, \ldots)$ is the only point common to any of two arcs α, β, and γ, so $\alpha \cup \beta \cup \gamma$ is a triod. It is not difficult to see that $\varprojlim ([0,1], f^2)$ is an arc. ■

From the definition of inverse limit it follows that bonding functions and projections commute. For generalized inverse limits that is not the case:

Theorem 3.33 *Suppose $\mathbf{f} = (f_n)$ is a sequence of upper semicontinuous functions $f_n : [0,1] \to 2^{[0,1]}$ and $M = \varprojlim \mathbf{f}$. If $m, n \in \mathbf{N}$ with $m \leq n$ and $H \subseteq M$, then $\pi_m(H) \subseteq f_{mn}(\pi_n(H))$.*

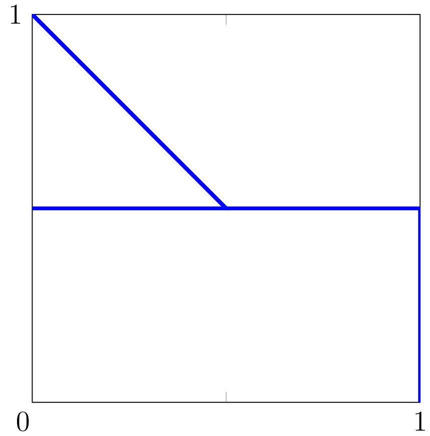


Figure 3.3: Graph of f in Example 3.32

In case $\mathbf{f}$ is a sequence of upper semicontinuous functions, if $H \subseteq \varprojlim \mathbf{f}$ and i is a positive integer such that $f_i|_{\pi_{i+1}(H)}$ is a continuous function, it is true that $\pi_i(H) = f_i(\pi_{i+1}(H))$. In general, equality is not necessary as seen in the next example.

Example 3.34 *Let* $f : [0,1] \to 2^{[0,1]}$ *be given by* $f(t) = [0,1]$ *for each* $t \in [0,1]$. *Then, for* $[0,1/2]^\infty$, *we have* $H \subseteq \varprojlim f$ *and, if* $i \in \mathbf{N}, \pi_i(H) = [0,1/2]$, *but* $f_i(\pi_{i+1}(H)) = [0,1]$.

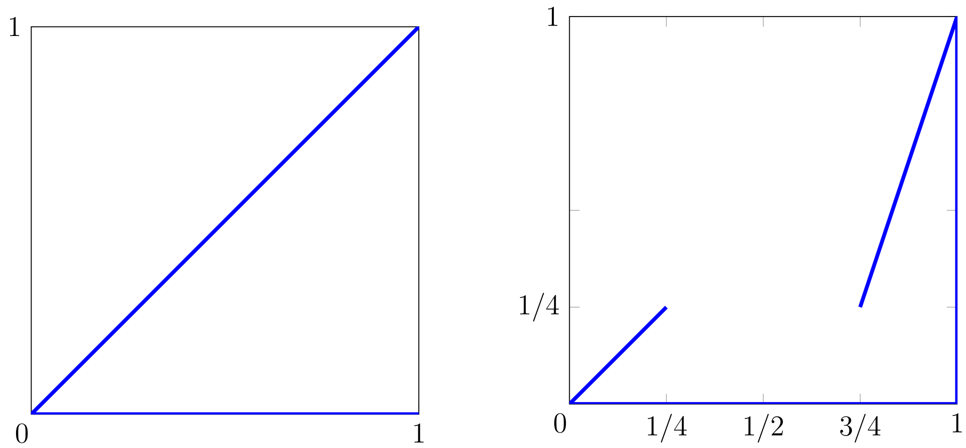


Figure 3.4: The graph of the functions from Example 3.35 and 3.36

Example 3.35 *Let* $f : [0,1] \to C([0,1])$ *be given by* $f(t) = \{0, t\}$ *for* $0 \leq t \leq 1$. *Then* $H = \left\{\mathbf{x} \in \varprojlim \mathbf{f} : x_i = x_1 \text{ for each positive integer } i\right\}$ *is a closed proper subset of* $\varprojlim \mathbf{f}$ *such that* $\pi_n(H) = [0,1]$ *for each positive integer* n.

Using notation from the previous example-but if f is continuous function i.e. the inverse limit case, we have that there exists $n \in \mathbb{N}$ such that $\pi_n(H)$ is a proper subset of $[0,1]$ - this statement is an immediate consequence of Proposition 3.14.

In the following example we have continua as terms but generalized inverse limit is not connected. That can not happen for inverse limits, as seen in Proposition 3.15.

Example 3.36 *Let* $f : [0,1] \to 2^{[0,1]}$ *be given by* $f(t) = \{0, t\}$ *for* $t, 0 \leq t \leq \frac{1}{4}$, $f(t) = 0$ *for* $\frac{1}{4} < t < \frac{3}{4}$, $f(t) = \{3t - 2, 0\}$ *for* $\frac{3}{4} \leq t < 1$, *and* $f(1) = [0,1]$. *Then,* $\Gamma(f)$ *is connected, but* $\varprojlim \mathbf{f}$ *is not connected.*

Chapter 4

Categories with u.s.c. functions as morphisms

In this chapter we introduce the results from the paper by I. Banič and T. Sovič [10] and from the [8].

We introduce a category $\mathcal{CHU}$ and generalize the notion of generalized inverse limit when the index set is not sequence of integers. We prove that generalized inverse limit is not inverse limit in the category $\mathcal{CHU}$ but it is a weak inverse limit.

Then we introduce the category $\mathcal{CU}$ in which the compact metric spaces are objects and upper semicontinuous functions from X to 2^Y are morphisms from X to Y. We show that it is a full subcategory of $\mathcal{CHU}$. We also introduce the category $\mathcal{ICU}$ of inverse sequences in $\mathcal{CU}$. Then we investigate the induced functions between inverse limits of compact metric spaces with upper semicontinuous bonding functions. We provide criteria for their existence and prove that under suitable assumptions they have surjective graphs. We also show that taking such inverse limits is very close to being a functor (but is not a functor) from $\mathcal{ICU}$ to $\mathcal{CU}$, if morphisms are mapped to induced functions. At the end of the paper we give a useful application of the mentioned results.

4.1 The category $\mathcal{CHU}$

In this section, including corresponding subsections 4.1.1 and 4.1.2 we present results from the paper by I. Banič and T. Sovič [10] and due to the completeness, we state some results together with proofs.

The category $\mathcal{CHU}$ of compact Hausdorff spaces and u.s.c. functions consists of the following objects and morphisms:

(1) $\mathrm{Ob}\,(\mathcal{CHU})$ - compact Hausdorff spaces;

(2) $\mathrm{Mor}\,(\mathcal{CHU})$ - the set of morphisms from X to Y, denoted by $\mathrm{Mor}\,(\mathcal{CHU})\,(X,Y)$, is the set of u.s.c. functions from X to 2^Y

We denote a morphism $f \in \mathrm{Mor}\,(\mathcal{CHU})\,(X,Y)$ by $f : X \to Y$, same as a function but to avoid confusion we emphasise - a morphism $f : X \to Y$ (i.e. an u.s.c. function $f : X \to 2^Y$).

We also define the partial binary operation $\circ$ (composition) as follows. For each $f \in \mathrm{Mor}\,(\mathcal{CHU})\,(X,Y)$ and each $g \in \mathrm{Mor}\,(\mathcal{CHU})\,(Y,Z)$, $g \circ f \in \mathrm{Mor}\,(\mathcal{CHU})\,(X,Z)$ is defined by

$$(g \circ f)\,(x) = g\,(f(x)) = \bigcup_{y \in f(x)} g(y)$$

for each $x \in X$.

Set $(g \circ f)\,(x)$ is indeed an element of 2^Z (see [29]).

Theorem 4.1 *$\mathcal{CHU}$ is a category.*

Proof. First, we show that $\circ$ is well defined. Let $f : X \to Y$ and $g : Y \to Z$ be any morphisms. Let also $x \in X$ be arbitrary and let U be an open set in Z such that $(g \circ f)\,(x) \subseteq U$. Since g is u.s.c. and $f(x) \subseteq Y$, for each $y \in f(x)$ there is an open set W_y in Y such that:

(1) $y \in W_y$,

(2) for all $w \in W_y, g(w) \subseteq U$.

Let $W = \bigcup_{y \in f(x)} W_y$. Since W is open in $Y, f(x) \subseteq W$, and since f is u.s.c. function, there is an open set V in X such that:

(1) $x \in V$,

(2) for all $v \in V, f(v) \subseteq W$.

Let $v \in V$ be arbitrary. Then

$$(g \circ f)(v) = g(f(v)) = \bigcup_{z \in f(v)} g(z) \subseteq U$$

since for each $z \in f(v), g(z) \subseteq U$. Therefore $\circ$ is well defined. It is obvious that the composition $\circ$ of u.s.c. function is an associative operation. All that is left to show is that for each $X \in \mathrm{Ob}(\mathcal{CHU})$ there is a morphism $1_X : X \to X$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any morphisms $f : Y \to X$ and $g : X \to Z$. We can easily see that the identity map $1_X : X \to X$, defined by $1_X(x) = \{x\}$ for each $x \in X$, is the u.s.c. function satisfying the above conditions. ■

4.1.1 Generalized inverse systems and generalized inverse limits

In this section we define objects in the category $\mathcal{CHU}$ called generalized inverse limits. If $(X_\alpha, f_{\alpha\beta}, A)$ is an inverse system of compact Hausdorff spaces and u.s.c. set-valued bonding functions, then the generalized inverse limit is denoted by

$$\varprojlim (X_\alpha, f_{\alpha\beta}, A)$$

and we show that, together with the projections, is not necessarily an inverse limit in the category $\mathcal{CHU}$.

Definition 4.2 *Let $(X_\alpha, f_{\alpha\beta}, A)$ be an object in the category* inv-$\mathcal{CHU}$. *We say that $(X_\alpha, f_{\alpha\beta}, A)$ is an* ***generalized inverse system****.*

Definition 4.3 *Let $(X_\alpha, f_{\alpha\beta}, A)$ be a generalized inverse system. A subset*

$$\left\{ x \in \prod_{\alpha \in A} X_\alpha : \pi_\alpha(x) \in f_{\alpha\beta}(\pi_\beta(x)), \forall \alpha < \beta \right\}$$

of the Cartesian product $\prod_{\alpha \in A} X_\alpha$ *is called the* ***generalized inverse limit*** *and is denoted by* $\varprojlim (X_\alpha, f_{\alpha\beta}, A)$.

Let $(X_\alpha, f_{\alpha\beta}, A)$ be an inverse system (defined in Definition 3.2) and $\left(X_\alpha, \tilde{f}_{\alpha\beta}, A\right)$ an induced generalized inverse system (defined in the same way as a sequence in the comment after Definition 3.23). Again, in the same way we get $\varprojlim \left(X_\alpha, \tilde{f}_{\alpha\beta}, A\right) = \lim (X_\alpha, f_{\alpha\beta}, A)$. Therefore, each inverse limit of an inverse system is a generalized inverse limit of an induced generalized inverse system.

In the following theorem we prove that $\varprojlim (X_\alpha, f_{\alpha\beta}, A)$ is really an object of $\mathcal{CHU}$.

Theorem 4.4 *Let* $(X_\alpha, f_{\alpha\beta}, A)$ *a generalized inverse system. Then the generalized inverse limit*

$$\varprojlim (X_\alpha, f_{\alpha\beta}, A)$$

is a compact Hausdorff space, i.e. $\varprojlim (X_\alpha, f_{\alpha\beta}, A) \in \mathrm{Ob}(\mathcal{CHU})$.

Proof. For each $\gamma \in A$, X_γ is a compact Hausdorff space, and therefore the product $\prod_{\gamma \in A} X_\gamma$ is a compact Hausdorff space. Since $\varprojlim (X_\alpha, f_{\alpha\beta}, A)$ is a subspace of the Hausdorff space, it is also a Hausdorff space. We show that $\varprojlim (X_\alpha, f_{\alpha\beta}, A)$ is a closed subset of the compact space $\prod_{\gamma \in A} X_\gamma$ to show that it is compact. For all $\alpha, \beta \in A, \alpha < \beta$, let

$$G_{\alpha\beta} = \Gamma(f_{\alpha\beta}) \times \prod_{\gamma \in A \setminus \{\alpha, \beta\}} X_\gamma = \left\{ x \in \prod_{\gamma \in A} X_\gamma : x_\alpha \in f_{\alpha\beta}(x_\beta) \right\}$$

Since the graph $\Gamma(f_{\alpha\beta})$ of $f_{\alpha\beta}$ is by Theorem 3.21 a closed subset of $X_\beta \times X_\alpha$, $G_{\alpha\beta}$ is also a closed subset of $\prod_{\gamma \in A} X_\gamma$. It is obvious that

$$\varprojlim (X_\alpha, f_{\alpha\beta}, A) = \bigcap_{\alpha, \beta \in A, \alpha < \beta} G_{\alpha\beta}$$

and hence $\varprojlim (X_\alpha, f_{\alpha\beta}, A)$ is a closed subset of $\prod_{\gamma \in A} X_\gamma$. ■

Now, we have a natural question: Is the generalized inverse limit an inverse limit in the category $\mathcal{CHU}$? Answer is negative as shown by the following example (see [10, Example 4.3]).

Example 4.5 *Let* $A = \mathbb{N}, X_k = [0,1]$, *and let* $f_{kk+1} = f$ *for each* $k \in \mathbb{N}$, *where* $f : [0,1] \to 2^{[0,1]}$ *is the function on* $[0,1]$ *defined by its graph*

$$\Gamma(f) = \{(t,t) \in [0,1] \times [0,1] : t \in [0,1]\} \cup (\{1\} \times [0,1]).$$

Also let $X = \varprojlim ([0,1], f_{kk+1}, \mathbb{N})$ *and let* $\mathbf{p} = (p_k : X \to X_k : k \in \mathbb{N}) : (X) \to (X, f_{kk+1}, \mathbb{N})$ *be any morphism in* inv-$\mathcal{CHU}$. *Then* $\mathbf{p} : (X) \to (X, f_{kk+1}, \mathbb{N})$ *is not an inverse limit in* $\mathcal{CHU}$.
Let $Y = [0,1]$ *be an object in* $\mathcal{CHU}$ *and let* $\varphi = (\varphi_k : Y \to X_k : k \in \mathbb{N})$ *be a morphism in* inv-$\mathcal{CHU}$ *where* $\varphi_k(t) = [0,1]$ *for each* k *and each* $t \in Y$. *We distinguish the following two cases.*

(1) If there is a positive integer i_0,*such that* $1 \notin p_{i_0}(x)$ *for each* $x \in X$, *then suppose that* Φ *is any morphism* $Y \to X$. *Then* $\varphi_{i_0}(t) = [0,1]$ *but* $1 \notin p_{i_0}(\Phi(t))$ *for any* $t \in Y$. *Therefore* $\mathbf{p}\ \Phi \neq \varphi$.

(2) If for each positive integer i *there is* $x^i \in X$ *such that* $1 \in p_i(x^i)$, *then let* $s \in X$ *be an accumulation point of the sequence* $(x^i)_{i=1}^{\infty}$. *We show first that* $p_i(s) = [0,1]$ *for each* i. *Let* k *be any positive integer. Then for each* $l > k$, *it follows from*

$$[0,1] \supseteq p_k\left(x^l\right) = f_{kl}\left(p_l\left(x^l\right)\right) \supseteq f_{kl}(1) \supseteq [0,1]$$

that $p_k\left(x^l\right) = [0,1]$. *Let* $(n_i)_{i=1}^{\infty}$ *be any increasing sequence of positive integers such that:*

- $n_i > k$ *for each* i;
- $\lim\limits_{i\to\infty} x^{n_i} = s$.

It follows from $p_k(x^{n_i}) = [0,1]$ *that* $\{x^{n_i}\} \times [0,1] \subseteq \Gamma(p_k)$ *for each* i. *This means that for each* $t \in [0,1]$, *the point* $(x^{n_i}, t) \in \Gamma(p_k)$. *Therefore* $\lim\limits_{i\to\infty}(x^{n_i}, t) = (s,t) \in \Gamma(p_k)$ *and hence* $p_k(s) = [0,1]$.

Next, let $\Phi, \Psi : Y \to X$ *be the morphisms in* $\mathcal{CHU}$, *defined by*

$$\Phi(t) = X, \Psi(t) = \{s\}$$

for each $t \in Y$. It follows from

$$p_k(\Phi(t)) = p_k(X) = [0,1] = \varphi_k(t)$$

and

$$p_k(\Psi(t)) = p_k(\{s\}) = [0,1] = \varphi_k(t)$$

that $\mathbf{p}\ \Phi = \varphi$ and $\mathbf{p}\ \Psi = \varphi$. Therefore we don't have unique morphism which satisfies the equation from the definition on an inverse limit.

Note that in the second part of the previous example, $\Psi(t) \subseteq \Phi(t) = \left(\prod_{k=1}^{\infty} \varphi_k(t)\right) \cap X$ holds true for each $t \in Y$. The following lemma shows that such an inclusion is not accidental. It will be used in the proof of Theorem 4.11.

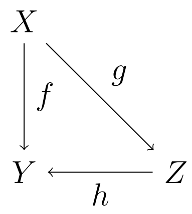
Lemma 4.6 *Let $(X_\alpha, f_{\alpha\beta}, A)$ be a generalized inverse system and let $X = \varprojlim (X_\alpha, f_{\alpha\beta}, A)$. Suppose $(\varphi_\alpha : Y \to X_\alpha : \alpha \in A) : (Y) \to (X_\alpha, f_{\alpha\beta}, A)$ is a morphism in* inv-$\mathcal{CHU}$*. Then $\varphi : Y \to 2^X$, defined by $\varphi(y) = \left(\prod_{\gamma \in A} \varphi_\gamma(y)\right) \cap X$ for each $y \in Y$, is a morphism in $\mathcal{CHU}$ such that for any morphism $\Psi : Y \to X$ such that $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$ for each $\alpha \in A$ and for each $y \in Y, \Psi(y) \subseteq \varphi(y)$ holds true for all $y \in Y$.*

Proof. See [10]. ■

4.1.2 Weak inverse limits in $\mathcal{CHU}$

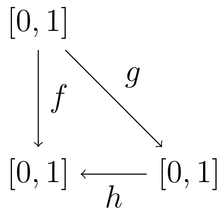
In this section we introduce the notion of weak inverse limit in $\mathcal{CHU}$, as defined in [10], and show that $\varprojlim (X_\alpha, f_{\alpha\beta}, A)$, together with the projections, is a weak inverse limit in $\mathcal{CHU}$. In the following definition we define a weak commutation of a diagram in the category $\mathcal{CHU}$.

Definition 4.7 *Let $X, Y, Z \in$ Ob$(\mathcal{CHU})$ and let $f : X \to Y, g : X \to Z$ and $h : Z \to Y$ be any morphisms in $\mathcal{CHU}$. We say that the diagram*



weakly commutes if, for any $x \in X, f(x) \subseteq (h \circ g)(x)$ *and denote it by* $f \overset{w}{=} h \circ g$.

Example 4.8 *Let* $f : [0,1] \to 2^{[0,1]}, g : [0,1] \to 2^{[0,1]}$ *be identity functions on* $[0,1]$ *and let* $h : [0,1] \to 2^{[0,1]}$ *be defined by* $h(x) = [0,1]$ *for all* $x \in [0,1]$. *Then the diagram*



weakly commutes but does not commute.

In the following definition we generalise the notion of inverse limits in $\mathcal{CHU}$.

Definition 4.9 *Let* $(X_\alpha, f_{\alpha\beta}, A)$ *be a generalized inverse system. An object* $X \in \mathrm{Ob}(\mathcal{CHU})$, *together with the family* $(p_\alpha : X \to X_\alpha : \alpha \in A)$ *of morphisms* p_α *in* $\mathcal{CHU}$ *is called a* ***weak inverse limit*** *of* $(X_\alpha, f_{\alpha\beta}, A)$ *provided*

(i) for each $\alpha, \beta \in A, \alpha \leq \beta, f_\alpha \overset{w}{=} p_{\alpha\beta} f_\beta$;

(ii) for any morphism $(\varphi_\alpha : Y \to X_\alpha : \alpha \in A) : (Y) \to (X_\alpha, f_{\alpha\beta}, A)$ *in* inv-$\mathcal{CHU}$ *and any morphism* $\Psi : Y \to X$ *such that* $p_\alpha \Psi = \varphi_\alpha$ *for each* $\alpha \in A$ *it holds* $\Psi(y) \subseteq \left(\prod_{\gamma \in A} \varphi_\gamma(y)\right) \cap X$, *for each* $y \in Y$.

Note that each inverse limit in $\mathcal{CHU}$ is always a weak inverse limit in $\mathcal{CHU}$.

Example 4.10 *Let* $X = \varprojlim([0,1], f_{kl}, \mathbb{N})$ *be the generalized inverse limit that we defined in Example 4.5. Then* X, *together with the projection mappings, is obviously not an inverse limit but it is a weak inverse limit in* $\mathcal{CHU}$.

We show in the following theorem that the generalized inverse limits together with projections are always weak inverse limits in $\mathcal{CHU}$.

Theorem 4.11 *Let $(X_\alpha, f_{\alpha\beta}, A)$ be a generalized inverse system. Then the generalized inverse limit*

$$\varprojlim (X_\alpha, f_{\alpha\beta}, A),$$

together with the projections

$$p_\gamma : \varprojlim (X_\alpha, f_{\alpha\beta}, A) \to X_\gamma,\ p_\gamma \left((x_\alpha)_{\alpha\in A}\right) = \{x_\gamma\},$$

is a weak inverse limit of $(X_\alpha, f_{\alpha\beta}, A)$ in $\mathcal{CHU}$.

Proof. Let $X = \varprojlim (X_\alpha, f_{\alpha\beta}, A)$. First, we prove (i) i.e. that $f_\alpha \overset{w}{=} p_{\alpha\beta} f_\beta$ for each $\alpha, \beta \in A, \alpha \leq \beta$. Choose any $x \in X$ and let $\alpha < \beta$. Then

$$p_\alpha(x) = \{x_\alpha\} \subseteq f_{\alpha\beta}(\{x_\beta\}) = (f_{\alpha\beta} \circ p_\beta)(x).$$

(ii) follows from Lemma 4.6. ■

4.2 Induced functions and induced morphisms

Let $\mathcal{CU}$ be a full subcategory of $\mathcal{CHU}$ whose objects are compact metric spaces. Let $\mathcal{ICU}$ be a subcategory of inv-$\mathcal{CHU}$ which consists of inverse systems and level morphisms.

Recall the following:

If $\varphi = (\varphi_n : X_n \to Y_n, n \in \mathbb{N}) : (X_n, f_n)_{n=1}^\infty \to (Y_n, g_n)_{n=1}^\infty$ and $\psi = (\psi_n : Y_n \to Z_n), n \in \mathbb{N}) : (Y_n, g_n)_{n=1}^\infty \to (Z_n, h_n)_{n=1}^\infty$ are two morphisms of $\mathcal{ICU}$, then the composition $\psi \circ \varphi : (X_n, f_n)_{n=1}^\infty \to (Z_n, h_n)_{n=1}^\infty, \psi \circ \varphi = (\psi_n \circ \varphi_n : X_n \to Z_n, \mathbb{N})$ is a morphism of $\mathcal{ICU}$.

We continue with the definition of functions induced by sequences of u.s.c. functions.

Definition 4.12 *Let $(X_n, f_n)_{n=1}^\infty$ and $(Y_n, g_n)_{n=1}^\infty$ be inverse sequences in $\mathcal{CU}$), $X = \varprojlim (X_n, f_n)_{n=1}^\infty$ and $Y = \varprojlim (Y_n, g_n)_{n=1}^\infty$ their generalized inverse limits* *and let* (φ_n) *be a sequence of u.s.c. functions* $\varphi_n : X_n \to 2^{Y_n}$*. For each* $x = (x_n) \in X$*, put*

$$\Phi(x) = \left(\prod_{n=1}^{\infty} \varphi_n(x_n) \right) \cap Y \in 2^Y. \tag{4.1}$$

If Φ *is an u.s.c. function from* X *to* 2^Y*, we say that* Φ *is* ***induced*** *by* (φ_n)*.*

The next theorem provides a simple criterion for recognizing induced functions.

Theorem 4.13 *We use the notation from Definition 4.12. Then* Φ *is induced by* (φ_n) *if and only if* $\Phi(x) \neq \emptyset$ *for each* $x = (x_n) \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$*.*

Proof. If $\Phi : X \to 2^Y$ is induced by (φ_n), then for each $x = (x_n) \in X$, $\Phi(x) \in 2^Y$. Therefore $\Phi(x) \neq \emptyset$.

Now assume that $\Phi(x) \neq \emptyset$ for each $(x) \in X$. Since $\Phi(x)$ is compact it is closed in Y. Therefore the function $\Phi : X \to 2^Y$ is well-defined.

Next we prove that Φ is a u.s.c. function. It is sufficient to prove that the graph $\Gamma(\Phi)$ of Φ is a closed subset of $X \times Y$. It follows from the definition of Φ that

$$\Gamma(\Phi) = \{((x_n), (y_n)) \in X \times Y \mid \forall i \in \mathbb{N}, y_i \in \varphi_i(x_i)\}.$$

Let $\alpha : X \times Y \to (X_1 \times Y_1) \times (X_2 \times Y_2) \times \cdots$ be defined by

$$\alpha(x, y) = ((x_1, y_1), (x_2, y_2), \ldots),$$

for all $x = (x_n) \in X$ and $y = (y_n) \in Y$.

Note that $A : X \times Y \to \operatorname{Im} \alpha$ defined by $A(x, y) = \alpha(x, y)$ is a homeomorphism.

We prove that $A(\Gamma(\Phi))$ is closed in $\operatorname{Im} A = A(X \times Y)$.

$$
\begin{aligned}
A(\Gamma(\Phi)) &= \{((x_1,y_1),(x_2,y_2),\ldots) \mid x \in X, y \in Y, \forall i \in \mathbb{N}, y_i \in \varphi_i(x_i)\} \\
&= \{((x_1,y_1),(x_2,y_2),\ldots) \mid x \in X, y \in Y, \forall i \in \mathbb{N}, (x_i,y_i) \in \Gamma(\varphi_i)\} \\
&= \{((x_1,y_1),(x_2,y_2),\ldots) \in \Gamma(\varphi_1) \times \Gamma(\varphi_2) \times \cdots \mid x \in X, y \in Y\} \\
&= (\Gamma(\varphi_1) \times \Gamma(\varphi_2) \times \cdots) \cap \{((x_1,y_1),(x_2,y_2),\ldots) \mid x \in X, y \in Y\} \\
&= (\Gamma(\varphi_1) \times \Gamma(\varphi_2) \times \cdots) \cap A(X \times Y).
\end{aligned}
$$

The product

$$\Gamma(\varphi_1) \times \Gamma(\varphi_2) \times \Gamma(\varphi_3) \times \cdots$$

is a closed subset of $(X_1 \times Y_1) \times (X_2 \times Y_2) \times \cdots$, therefore $A(\Gamma(\Phi))$ is closed in $A(X \times Y)$. It follows that $\Gamma(\Phi)$ is closed in $X \times Y$. ■

The next theorem presents a commutativity-like condition under which Φ is induced.

Theorem 4.14 *Let $(X_n, f_n)_{n=1}^{\infty}$ and $(Y_n, g_n)_{n=1}^{\infty}$ be inverse sequences in $\mathcal{CU}$, and let (φ_n) be a sequence of u.s.c. functions $\varphi_n : X_n \to 2^{Y_n}$. If*

$$(\varphi_n \circ f_n)(x) \subseteq (g_n \circ \varphi_{n+1})(x)$$

for each positive integer n and each $x \in X_{n+1}$, then Φ defined by (4.1) is induced by (φ_n).

Proof. By Theorem 4.13 it suffices to prove that $\Phi(x)$ is nonempty for arbitrary $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$. For arbitrary $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$ we construct a point $y = (y_1, y_2, y_3, \ldots) \in \Phi(x)$ by an inductive construction of coordinates y_i. More precisely, by induction on $i \in \mathbb{N}$, we construct a sequence (y_i) in Y_i, satisfying $y_i \in \varphi_i(x_i)$ and $y_i \in g_i(y_{i+1})$ for each i.

We choose any $y_1 \in \varphi_1(x_1)$; it can be done since $\varphi_1(x_1)$ is nonempty.

Assume next that we have already constructed $y_i \in \varphi_i(x_i)$. Now we construct $y_{i+1} \in \varphi_{i+1}(x_{i+1})$ such that $y_i \in g_i(y_{i+1})$.

It follows from $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$ that $x_i \in f_i(x_{i+1})$. Therefore

$$y_i \in \varphi_i(x_i) \subseteq (\varphi_i \circ f_i)(x_{i+1}) \subseteq (g_i \circ \varphi_{i+1})(x_{i+1}) = \bigcup_{t \in \varphi_{i+1}(x_{i+1})} g_i(t).$$

Hence, there exists a point $t_0 \in \varphi_{i+1}(x_{i+1})$ such that $y_i \in g_i(t_0)$. We take any such t_0 for y_{i+1}. ■

This immediately leads to the following corollary.

Corollary 4.15 *Let $(X_n, f_n)_{n=1}^{\infty}$ and $(Y_n, g_n)_{n=1}^{\infty}$ be any objects of $\mathcal{ICU}$, and let the sequence (φ_n) be any morphism of $\mathcal{ICU}$ from $(X_n, f_n)_{n=1}^{\infty}$ to $(Y_n, g_n)_{n=1}^{\infty}$. Then $\Phi : \varprojlim (X_n, f_n)_{n=1}^{\infty} \to 2^{\varprojlim (Y_n, g_n)_{n=1}^{\infty}}$, defined by (4.1), is induced by (φ_n), meaning that $\Phi : \varprojlim (X_n, f_n)_{n=1}^{\infty} \to \varprojlim (Y_n, g_n)_{n=1}^{\infty}$ is a morphism in $\mathcal{CU}$.*

Definition 4.16 *The function Φ from Corollary 4.15 is called the morphism of $\mathcal{CU}$ induced by the morphism (φ_n) of $\mathcal{ICU}$ and is denoted by $\Phi = \varprojlim(\varphi_i)$.*

Note that if (φ_n) is not a morphism of $\mathcal{ICU}$ but the induced function Φ is a morphism of $\mathcal{CU}$ (that happens when (φ_n) satisfies the conditions of Theorem 4.13) $\varprojlim \varphi_i$ does not exist.

The induced morphism $\varprojlim(\varphi_i) : \varprojlim (X_n, f_n)_{n=1}^{\infty} \to \varprojlim (Y_n, g_n)_{n=1}^{\infty}$ cannot be defined simply by the formula

$$\left(\varprojlim(\varphi_i)\right)(x_1, x_2, x_3, \ldots) = \prod_{i=1}^{\infty} \varphi_i(x_i) \qquad (4.2)$$

since the right hand side product of (4.2) is not necessarily a subset of $\varprojlim (Y_n, g_n)_{n=1}^{\infty}$, as we show in the following example.

Example 4.17 *Let $X_i = Y_i = [0,1]$, let $f_i = g_i = 1_{[0,1]}$, where $1_{[0,1]} : [0,1] \to 2^{[0,1]}$ is the u.s.c. function, defined by $1_{[0,1]}(x) = \{x\}$ for each $x \in [0,1]$, and let $\varphi_i : [0,1] \to 2^{[0,1]}$ be defined by its graph: $\Gamma(\varphi_i) = [0,1] \times [0,1]$, for each positive integers i. Then $\varphi_1(x_1) \times \varphi_2(x_2) \times \varphi_3(x_3) \times \cdots$ is not a subset of $\varprojlim (Y_n, g_n)_{n=1}^{\infty}$.*

Proof. Obviously, $g_i \circ \varphi_{i+1} = \varphi_i \circ f_i$ holds true for any positive integer i, and therefore (φ_n) is a morphism of $\mathcal{ICU}$.

Also, $\varprojlim (X_n, f_n)_{n=1}^{\infty} = \varprojlim (Y_n, g_n)_{n=1}^{\infty} = \{(t,t,t,\ldots) \mid t \in [0,1]\}$, and therefore

$$\varphi(x_1, x_2, x_3, \ldots) = \varphi_1(x_1) \times \varphi_2(x_2) \times \varphi_3(x_3) \times \cdots = [0,1] \times [0,1] \times [0,1] \times \cdots$$

is not a subset of $\varprojlim (Y_n, g_n)_{n=1}^{\infty}$ (and therefore it is not an element of $2^{\varprojlim (Y_n, g_n)_{n=1}^{\infty}}$). ■

This example shows also that (4.2) cannot replace (4.1) in the definition of induced functions.

In the following theorem we prove that if each of the φ_i's has a surjective graph, then also $\varprojlim(\varphi_i)$ has a surjective graph. Note that it is not required that any of the functions f_n and g_n have a surjective graph.

Theorem 4.18 *Let $(X_n, f_n)_{n=1}^{\infty}$ and $(Y_n, g_n)_{n=1}^{\infty}$ be any objects of $\mathcal{ICU}$ and let the sequence (φ_n) be a morphism of $\mathcal{ICU}$ from $(X_n, f_n)_{n=1}^{\infty}$ to $(Y_n, g_n)_{n=1}^{\infty}$ such that $\varphi_i : X_i \to 2^{Y_i}$ has a surjective graph for each positive integer i. Then $\varprojlim \varphi_i$ has a surjective graph.*

Proof. Let $y = (y_1, y_2, y_3, \ldots) \in \varprojlim (Y_n, g_n)_{n=1}^{\infty}$ be arbitrary. We construct a point $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$ such that $y \in \left(\varprojlim(\varphi_i)\right)(x)$.

Let n be any positive integer. Since $\varphi_n : X_n \to 2^{Y_n}$ has a surjective graph, there is a point $x_n^n \in X_n$ such that $y_n \in \varphi_n(x_n^n)$. We choose and fix such an x_n^n. Then by downwards induction we prove that for any $k \in \{1, 2, 3, \ldots, n-1\}$ there is $x_k^n \in X_k$ such that $y_k \in \varphi_k(x_k^n)$ and $x_k^n \in f_k(x_{k+1}^n)$.

Let k be any integer from $\{1, 2, 3, \ldots, n-1\}$. Assume that x_{k+1}^n has already been chosen in such a way that $y_{k+1} \in \varphi_{k+1}(x_{k+1}^n)$. Note that this assumption is fulfilled for $k = n - 1$.

Since $y_k \in g_k(y_{k+1})$ and $y_{k+1} \in \varphi_{k+1}(x_{k+1}^n)$, it follows that

$$y_k \in (g_k \circ \varphi_{k+1})(x_{k+1}^n) = (\varphi_k \circ f_k)(x_{k+1}^n).$$

Therefore there is a point $x_k^n \in X_k$ such that $x_k^n \in f_k(x_{k+1}^n)$ and $y_k \in \varphi_k(x_k^n)$ and we fix one such x_k^n.

This construction yields

$$x^n = (x_1^n, x_2^n, x_3^n, \ldots, x_{n-1}^n, x_n^n, z_{n+1}^n, z_{n+2}^n, z_{n+3}^n, \ldots) \in \prod_{i=1}^{\infty} X_i,$$

where $z_i^n \in X_i$ is arbitrarily chosen for each $i > n$. Then $\{x^n\}_{n=1}^{\infty}$ is a sequence in the compact metric space $(\prod_{i=1}^{\infty} X_i, D)$. Let $x = (x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X_i$ be any accumulation point of the sequence $\{x^n\}_{n=1}^{\infty}$.

Next we prove that $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$ and that $y \in \left(\varprojlim \varphi_i\right)(x)$. Let $\{i_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers such that

$$\lim_{n\to\infty} x^{i_n} = x.$$

First we prove that $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$. Let m be any positive integer. Then $(x_{m+1}^{i_k}, x_m^{i_k}) \in \Gamma(f_m)$ for each positive integer $i_k > m$. Since $\lim\limits_{k\to\infty} (x_{m+1}^{i_k}, x_m^{i_k}) = (x_{m+1}, x_m)$ and since the graph $\Gamma(f_m)$ is closed in $X_{m+1} \times X_m$, it follows that $(x_{m+1}, x_m) \in \Gamma(f_m)$. Therefore $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$.

Finally we prove that $y \in \left(\varprojlim \varphi_i\right)(x)$. Let m be any positive integer. Then $y_m \in \varphi_m(x_m^{i_k})$ for each positive integer $i_k > m$. Therefore $(x_m^{i_k}, y_m) \in \Gamma(\varphi_m)$ for each $i_k > m$. Since $\lim\limits_{k\to\infty} (x_m^{i_k}, y_m) = (x_m, y_m)$ and since the graph $\Gamma(\varphi_m)$ is closed in $X_m \times Y_m$, it follows that $(x_m, y_m) \in \Gamma(\varphi_m)$, and therefore $y_m \in \varphi_m(x_m)$.

It follows that $(y_1, y_2, y_3, \ldots) \in \left(\varprojlim \varphi_i\right)(x_1, x_2, x_3, \ldots)$ and hence $\varprojlim(\varphi_i)$ has a surjective graph. ■

Next example shows that the function Φ induced by (φ_n) need not have a surjective graph if (φ_n) is not a morphism of $\mathcal{ICU}$, even if each φ_i, f_i, and g_i has a surjective graph and if $g_i \circ \varphi_{i+1} = \varphi_i \circ f_i$ holds true for any positive integer $i > 1$.

Example 4.19 *Let for each positive integer i and $j > 1$, $X_i = Y_i = [0,1]$, $f_i = g_j = \varphi_i = 1_{[0,1]}$, and let $g_1 : [0,1] \to 2^{[0,1]}$ be defined by its graph: $\Gamma(g_1) = [0,1] \times [0,1]$. Then the function Φ induced by (φ_n) does not have a surjective graph.*

Proof. Obviously, $g_i \circ \varphi_{i+1} = \varphi_i \circ f_i$ holds true for any positive integer $i > 1$,

and $\varphi_1 \circ f_1(t) \subseteq g_1 \circ \varphi_2(t)$ for any $t \in [0,1]$. Therefore $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ induces Φ defined by (4.1) according to Theorem 4.14.
Obviously $(0,1,1,1,\ldots) \in \varprojlim (Y_n, g_n)_{n=1}^{\infty}$ and $\varprojlim (X_n, f_n)_{n=1}^{\infty} = \{(t,t,t,\ldots) : t \in [0,1]\}$. But $\Phi(t,t,t,\ldots) = \{(t,t,t,\ldots)\}$, and therefore $(0,1,1,1,\ldots) \notin \Phi(t,t,t,\ldots)$ for any $t \in [0,1]$. ■

In the rest of the section we study a rule $F : \mathcal{ICU} \to \mathcal{CU}$, defined by

$$(X_n, f_n)_{n=1}^{\infty} \xmapsto{F} \varprojlim (X_n, f_n)_{n=1}^{\infty}$$
$$(\varphi_n) \xmapsto{F} \varprojlim(\varphi_n).$$

In the following theorem we show that the rule F is very close to being a functor from $\mathcal{ICU}$ to $\mathcal{CU}$.

Theorem 4.20 *Let $(X_n, f_n)_{n=1}^{\infty}$, $(Y_n, g_n)_{n=1}^{\infty}$ and $(Z_n, h_n)_{n=1}^{\infty}$ be any objects of $\mathcal{ICU}$, and*

$$\varphi = (\varphi_n) : (X_n, f_n)_{n=1}^{\infty} \to (Y_n, g_n)_{n=1}^{\infty}$$

and

$$\psi = (\psi_n) : (Y_n, g_n)_{n=1}^{\infty} \to (Z_n, h_n)_{n=1}^{\infty}$$

morphisms in $\mathcal{ICU}$. Then

1. $F((1_{X_n})) = 1_{\varprojlim (X_n, f_n)_{n=1}^{\infty}}$;
2. $(F(\psi) \circ F(\varphi))(x) \subseteq F(\psi \circ \varphi)(x)$ *for all* $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$.

Proof. To prove (1), choose arbitrary $x = (x_1, x_2, x_3, \ldots) \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$. Then

$$F((1_{X_n}))(x) = \prod_{n=1}^{\infty} 1_{X_n}(x_n) \cap \varprojlim (X_n, f_n)_{n=1}^{\infty} = \{x\} = 1_{\varprojlim (X_n, f_n)_{n=1}^{\infty}}(x).$$

To prove (2), let $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$ and let

$$z \in (F(\psi) \circ F(\varphi))(x) = F(\psi)[F(\varphi)(x)] = \bigcup_{y \in F(\varphi)(x)} F(\psi)(y)$$

be arbitrary. Then

$$z \in \bigcup_{y \in (\varphi_1(x_1) \times \varphi_2(x_2) \times \cdots) \cap \varprojlim (Y_n, g_n)_{n=1}^{\infty}} (\psi_1(y_1) \times \psi_2(y_2) \times \cdots) \cap \varprojlim (Z_n, h_n)_{n=1}^{\infty}$$

and therefore there is a point $y \in \varprojlim (Y_n, g_n)_{n=1}^{\infty}$ such that $y_n \in \varphi_n(x_n)$ and $z_n \in \psi_n(y_n)$ for each positive n. It follows that $z_n \in \bigcup_{t \in \varphi_n(x_n)} \psi_n(t) = (\psi_n \circ \varphi_n)(x_n)$ for each positive integer n and hence $z \in F(\psi \circ \varphi)(x)$. ■

F is a functor if and only if $(F(\psi) \circ F(\varphi))(x) = F(\psi \circ \varphi)(x)$ holds true for all $x \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$ and all objects $(X_n, f_n)_{n=1}^{\infty}$, $(Y_n, g_n)_{n=1}^{\infty}$ and $(Z_n, h_n)_{n=1}^{\infty}$ and all morphisms $\varphi = (\varphi_n) : (X_n, f_n)_{n=1}^{\infty} \to (Y_n, g_n)_{n=1}^{\infty}$ and $\psi = (\psi_1, \psi_2, \psi_3, \ldots) : (Y_n, g_n)_{n=1}^{\infty} \to (Z_n, h_n)_{n=1}^{\infty}$ of $\mathcal{ICU}$. In the following example shows that this is not the case, hence F is not a functor. We use the notation from Theorem 4.20.

Example 4.21 *Let for each positive integer n, $X_n = Y_n = Z_n = [0,1]$ and let $f, g : [0,1] \to 2^{[0,1]}$ be u.s.c. functions defined by $f(t) = \{t\}$ and $g(t) = [0,1]$ for each $t \in [0,1]$. Also let $f_1 = h_1 = \psi_1 = \varphi_n = g$ for each $n \geq 2$ and let $\varphi_1 = f_{n+1} = g_n = h_{n+1} = \psi_{n+1} = f$ for each $n \geq 1$. Let $x = (1, 0, 0, 0, \ldots) \in \varprojlim (X_n, f_n)_{n=1}^{\infty}$. Then $(F(\psi) \circ F(\varphi))(x) \neq F(\psi \circ \varphi)(x)$.*

Proof. Let $z = (1, 0, 0, 0, \ldots) \in \varprojlim (Z_n, h_n)_{n=1}^{\infty}$. Obviously $z \in F(\psi \circ \varphi)(x)$. Then, since $\varphi_1(t) = \{t\}$ for each $t \in [0,1]$, $y = (1, 1, 1, \ldots)$ is the only element in $\varprojlim (Y_n, g_n)_{n=1}^{\infty}$ such that $y \in F(\varphi)(x)$. But, since $F(\psi)(y) = [0,1] \times \{1\} \times \{1\} \times \cdots$ and $z_2 = 0$ it follows that $z \notin F(\psi)(y)$. Therefore $z \notin (F(\psi) \circ F(\varphi))(x)$ and hence $(F(\psi) \circ F(\varphi))(x) \neq F(\psi \circ \varphi)(x)$. ■

4.3 An application

In the final section we study the following diagram.

$$\begin{array}{ccccccccccc}
X_1^1 & \xleftarrow{f_1^1} & X_2^1 & \xleftarrow{f_2^1} & X_3^1 & \xleftarrow{f_3^1} & X_4^1 & \xleftarrow{f_4^1} & \cdots & \qquad & L^1 \\
\uparrow g_1^1 & & \uparrow g_2^1 & & \uparrow g_3^1 & & \uparrow g_4^1 & & & & \uparrow g^1 \\
X_1^2 & \xleftarrow{f_1^2} & X_2^2 & \xleftarrow{f_2^2} & X_3^2 & \xleftarrow{f_3^2} & X_4^2 & \xleftarrow{f_4^2} & \cdots & & L^2 \\
\uparrow g_1^2 & & \uparrow g_2^2 & & \uparrow g_3^2 & & \uparrow g_4^2 & & & & \uparrow g^2 \\
X_1^3 & \xleftarrow{f_1^3} & X_2^3 & \xleftarrow{f_2^3} & X_3^3 & \xleftarrow{f_3^3} & X_4^3 & \xleftarrow{f_4^3} & \cdots & & L^3 \\
\uparrow g_1^3 & & \uparrow g_2^3 & & \uparrow g_3^3 & & \uparrow g_4^3 & & & & \uparrow g^3 \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots
\end{array}$$

$$K_1 \xleftarrow{f_1} K_2 \xleftarrow{f_2} K_3 \xleftarrow{f_3} \cdots \qquad\qquad K \xrightarrow{H} L$$

Theorem 4.22 *Let X_i^j be compact metric spaces, and let $f_i^j : X_{i+1}^j \to 2^{X_i^j}$, $g_i^j : X_{i+1}^j \to 2^{X_i^j}$ be u.s.c. functions, for all positive integers i and j. Let also for each j*

$$L^j = \varprojlim \left(X_i^j, f_i^j\right)_{i=1}^{\infty}$$

and for each i

$$K_i = \varprojlim \left(X_i^j, g_i^j\right)_{j=1}^{\infty}.$$

If for each integer n, f_n is the function induced by $(f_n^1, f_n^2, f_n^3, \ldots)$ and g^n is the function induced by $(g_1^n, g_2^n, g_3^n, \ldots)$, then

$$L = \varprojlim \left(L^j, g^j\right)_{j=1}^{\infty}$$

and

$$K = \varprojlim \left(K_i, f_i\right)_{i=1}^{\infty}.$$

are homeomorphic.

Proof. Define the function $H : K \to L$ as follows:

$$H\left(\left(x_1^1, x_1^2, x_1^3, \ldots\right), \left(x_2^1, x_2^2, x_2^3, \ldots\right), \left(x_3^1, x_3^2, x_3^3, \ldots\right), \ldots\right) = \left(\left(x_1^1, x_2^1, x_3^1, \ldots\right), \left(x_1^2, x_2^2, x_3^2, \ldots\right), \left(x_1^3, x_2^3, x_3^3, \ldots\right), \ldots\right), \tag{3}$$

where $(x_i^1, x_i^2, x_i^3, \ldots) \in K_i$ and $(x_i^1, x_i^2, x_i^3, \ldots) \in f_i\left(x_{i+1}^1, x_{i+1}^2, x_{i+1}^3, \ldots\right)$ for each positive integer i.

We will prove that H is a homeomorphism.

First, we prove that H is well-defined. We need to show that the right side of (3) is a point of L. The proof is in the following steps.

1. $\left(x_1^j, x_2^j, x_3^j, \ldots\right) \in L^j$, for arbitrary $j \in \mathbb{N}$;
2. $\left(x_1^j, x_2^j, x_3^j, \ldots\right) \in g^j\left(x_1^{j+1}, x_2^{j+1}, x_3^{j+1}, \ldots\right)$, for arbitrary $j \in \mathbb{N}$.

Let us prove (1).

Since $(x_i^1, x_i^2, x_i^3, \ldots) \in f_i\left(x_{i+1}^1, x_{i+1}^2, x_{i+1}^3, \ldots\right) = \left(\prod_{j=1}^{\infty} f_i^j(x_{i+1}^j)\right) \cap K_i$, it follows that $x_i^j \in f_i^j\left(x_{i+1}^j\right)$ for each i and j. Hence, $\left(x_1^j, x_2^j, x_3^j, \ldots\right) \in L^j$.

It remains to prove (2).

Since $(x_i^1, x_i^2, x_i^3, \ldots) \in K_i$, it follows that for each i and j, $x_i^j \in g_i^j(x_i^{j+1})$. Therefore $\left(x_1^j, x_2^j, x_3^j, \ldots\right) \in \left(\prod_{i=1}^{\infty} g_i^j(x_i^j)\right) \cap L^j = g^j\left(x_1^{j+1}, x_2^{j+1}, x_3^{j+1}, \ldots\right)$ for all j.

Hence, $((x_1^1, x_2^1, x_3^1, \ldots), (x_1^2, x_2^2, x_3^2, \ldots), (x_1^3, x_2^3, x_3^3, \ldots), \ldots) \in L$. So we have proved that $H : K \to L$ is well defined.

In the same manner we prove that $H' : L \to K$ defined by

$$H'\left(\left(x_1^1, x_2^1, x_3^1, \ldots\right), \left(x_1^2, x_2^2, x_3^2, \ldots\right), \left(x_1^3, x_2^3, x_3^3, \ldots\right), \ldots\right) = \left(\left(x_1^1, x_1^2, x_1^3, \ldots\right), \left(x_2^1, x_2^2, x_2^3, \ldots\right), \left(x_3^1, x_3^2, x_3^3, \ldots\right), \ldots\right),$$

is well defined. Since obviously H and H' are both continuous and inverses to each other, it follows that they are homeomorphisms. ∎

Corollary 4.23 *We use the notation of Theorem 4.22. If for all positive integers i and j*

$$g_i^j \circ f_i^{j+1} = f_i^j \circ g_{i+1}^j,$$

then the spaces L and K are homeomorphic.

Proof. The claim follows by Theorem 4.22 since by Corollary 4.15 there are induced functions f_n and g^n for each n. ■

We conclude the section with the following theorem.

Theorem 4.24 *Let X be any compact metric space and let $f : X \to X$ be a surjective mapping. Let $L' = \varprojlim (X, f^{-1})_{n=1}^{\infty}$, where f^{-1} is the u.s.c. function $f^{-1} : X \to 2^X$ defined by its graph*

$$\Gamma(f^{-1}) = \{(x, y) \in X \times X \mid (y, x) \in \Gamma(f)\}.$$

Let $\sigma : L' \to L'$ be the shift map, defined by

$$\sigma(t_1, t_2, t_3, \ldots) = (t_2, t_3, t_4, \ldots)$$

for each $(t_1, t_2, t_3, \ldots) \in L'$.

Then the inverse limit $\varprojlim (L', \sigma)_{n=1}^{\infty}$ is homeomorphic to $\varprojlim (X, f)_{n=1}^{\infty}$.

Proof.

We show first that the mapping $(t_1, t_2, t_3, \ldots) \mapsto \{\sigma(t_1, t_2, t_3, \ldots)\}$ can be interpreted as an induced function and then we use Theorem 4.22 to prove that the inverse limit $\varprojlim (L', \sigma)_{n=1}^{\infty}$ is homeomorphic to $\varprojlim (X, f)_{n=1}^{\infty}$.

We use the notation that is used in Theorem 4.22. Let for all positive integers i, j, $X_i^j = X$, $g_i^j(t) = \{f(t)\}$, and $f_i^j(t) = f^{-1}(t)$ for each $t \in X$.

Then, $g^n(t_1, t_2, t_3, \ldots) = (\{f(t_1)\} \times \{f(t_2)\} \times \{f(t_3)\} \times \ldots) \cap L' = \{(t_2, t_3, t_4, \ldots)\} = \{\sigma(t_1, t_2, t_3, \ldots)\}$ for any $(t_1, t_2, t_3, \ldots) \in L'$. It follows that $L = \varprojlim (L^n, g^n)_{n=1}^{\infty} = \varprojlim (L', \sigma)_{n=1}^{\infty}$.

Let $K' = K_n = \varprojlim (X, f)_{n=1}^{\infty}$ for each positive integer n. Next we show that $K = \varprojlim (K_n, f_n)_{n=1}^{\infty} = \varprojlim (K', \sigma'^{-1})_{n=1}^{\infty}$, where σ' is the shift map from K' to K'. Note that σ' is a homeomorphism, since f is single-valued, and that $\sigma'^{-1}(t_1, t_2, t_3, \ldots) = (f(t_1), t_1, t_2, t_3, \ldots)$ for each $(t_1, t_2, t_3, \ldots) \in K'$.

Then $f_n(t_1, t_2, t_3, \ldots) = (\{f^{-1}(t_1)\} \times \{f^{-1}(t_2)\} \times \{f^{-1}(t_3)\} \times \ldots) \cap K' = \{(f(t_1), t_1, t_2, t_3, \ldots)\} = \{\sigma'^{-1}(t_1, t_2, t_3, \ldots)\}$ for any $(t_1, t_2, t_3, \ldots) \in K'$. It follows that $K = \varprojlim (K_n, f_n)_{n=1}^{\infty} = \varprojlim (K', \sigma'^{-1})_{n=1}^{\infty}$.

Since σ'^{-1} is a homeomorphism it follows that $K = \varprojlim (K', \sigma'^{-1})_{n=1}^{\infty}$ is homeomorphic to $K' = \varprojlim (X, f)_{n=1}^{\infty}$. By Theorem 4.22 K is homeomorphic to L, and that proves that $\varprojlim (X, f)_{n=1}^{\infty}$ is homeomorphic to $\varprojlim (L', \sigma)_{n=1}^{\infty}$.
■

Chapter 5

Topological entropy for set-valued functions

In this chapter we generalize the idea of topological entropy to closed subsets of $[0,1]^2$. We show that when the closed subset of $[0,1]^2$ is the graph of a continuous function $f : [0,1] \to [0,1]$, then the topological entropy of f with our new definition of topological entropy is the same as that of the traditional definition of topological entropy. We further compare the properties of the new definition of topological entropy with those of the traditional definition, and find that many of the same properties hold, but not all. We reduce the problem of computing topological entropy in our context to one of counting the "boxes" (elements of our grid covers) that certain sets generated by our closed subset of $[0,1]^2$ intersect.

Closed subsets of $[0,1]^2$ can be viewed as a closed relation, or a multi-valued function, from a subset of $[0,1]$ to itself. We use techniques from inverse limits and generalizations of inverse limits to define and investigate topological entropy on closed subsets of $[0,1]^2$.

5.1 Mahavier product

All the results in this section are from [19].

Definition 5.1 *Let* $X_0, X_1, \ldots, X_n, n \in \mathbb{N}, n \geq 2$ *be topological spaces and*

$A_1 \subseteq X_0 \times X_1, \ldots, A_n \subseteq X_{n-1} \times X_n$. *We define*

$$A_1 \star \cdots \star A_n = \star_{i=1}^n A_i = \left\{ (x_0, \ldots, x_n) \in \prod_{i=0}^n X_i : (x_{i-1}, x_i) \in A_i, \forall i = 1, \ldots, n \right\}$$

to be the ***Mahavier product*** *of* $A_i, i = 1, \ldots, n$.

In particular, for $n = 1$ we put $\star_{i=1}^1 A = A$.

Now we extend the definition to the Mahavier product of infinitely many sets.

Definition 5.2 *Let* $X_0, X_1, \ldots, X_n, \ldots$ *be a sequence of topological spaces and* $A_0, A_1, \ldots, A_n, \ldots$ *a sequence of sets such that* $A_i \subseteq X_{i-1} \times X_i, \forall i \in \mathbb{N}$. *We define*

$$\star_{i=1}^\infty A_i = \left\{ (x_0, \ldots, x_n, \ldots) \in \prod_{i=0}^\infty X_i : (x_0, \ldots, x_n) \in \star_{i=1}^n A_i, \forall n \in \mathbb{N} \right\}$$

to be the Mahavier product of a sequence $A_i, i \in \mathbb{N}$.

We see that

$$\star_{i=1}^\infty A_i = \left\{ (x_0, \ldots, x_n, \ldots) \in \prod_{i=0}^\infty X_i : (x_{i-1}, x_i) \in A_i, \forall i \in \mathbb{N} \right\}.$$

If $A = \{(a, b)\}$ and $B = \{(b, c)\}$, then we write $(a, b) \star (b, c)$ to mean $A \star B = \{(a, b)\} \star \{(b, c)\}$.

Mahavier products have nice algebraic and topological properties, which we list in the following propositions.

Here, X, Y, Z and W are topological spaces. A **basic set** in $X \times Y \times Z$, $X \times Y$ or $Y \times Z$ is a set of the form $A_X \times A_Y \times A_Z$, $A_X \times A_Y$ or $A_Y \times A_Z$, respectively, for some sets $A_X \subset X$, $A_Y \subset Y$ and $A_Z \subset Z$.

The "middle" space Y in the definition has a special role and we observe projection maps from $X \times Y$, $Y \times Z$ and $X \times Y \times Z$ to Y as seen in the following propositions. We denote those projection maps with $\pi_Y^{X \times Y}, \pi_Y^{Y \times Z}, \pi_Y^{X \times Y \times Z}$ respectively, but use π_Y to denote all of them, if there is no confusion, to simplify notation. We also define the following: If $G \subset X \times Y$, then $\pi_Y^G : G \to Y$ is the map defined by $\pi_Y^G((x,y)) = y$ for $(x,y) \in G$, i.e. π_Y^G is the restriction of π_Y on G. For $H \subset Y \times Z$, we define π_Y^H in the same way.

Proposition 5.3 *Suppose G is a subset of $X \times Y$ and H is a subset of $Y \times Z$. Then $G \star H = (G \times Z) \cap (X \times H)$.*

Proposition 5.4 *Suppose G is a subset of $X \times Y$ and H is a subset of $Y \times Z$. Then $\pi_Y (G \star H) = \pi_Y (G) \cap \pi_Y (H)$.*

Proposition 5.5 *Suppose G, J are subsets of $X \times Y$ and H, K are subsets of $Y \times Z$. Then $(G \cap J) \star (H \cap K) = (G \star H) \cap (J \star K) = (G \star K) \cap (J \star H)$.*

Proposition 5.6 *Suppose G, J are subsets of $X \times Y$ and H, K are subsets of $Y \times Z$. Then $(G \cup J) \star (H \cup K) = (G \star H) \cup (J \star K) \cup (G \star K) \cup (J \star H)$.*

Proposition 5.7 *Suppose G, J are subsets of $X \times Y$ and H, K are subsets of $Y \times Z$. Then*

$$(G \cup J) \star (H \cap K) = [(G \star H) \cap (G \star K)] \cup [(J \star H) \cap (J \star K)]$$

and

$$(G \cap J) \star (H \cup K) = [(G \star H) \cap (J \star H)] \cup [(G \star K) \cap (J \star K)] .$$

Proposition 5.8 *Suppose $G \subset X \times Y, H \subset Y \times Z$ and $L \subset Z \times W$. Then $(G \star H) \star L = G \star (H \star L)$. Hence, we may simply write $G \star H \star L$.*

Proposition 5.9 *Suppose $G := G_1 \times G_2$ is a basic open subset of $X \times Y$ and $H := H_2 \times H_3$ is a basic open subset of $Y \times Z$. Then $G \star H = G_1 \times (G_2 \cap H_2) \times H_3$ is a basic open subset of $X \times Y \times Z$ (which is empty if $G_2 \cap H_2 = \emptyset$).*

Proposition 5.10 *Suppose G is an open subset of $X \times Y$ and H is an open subset of $Y \times Z$. Then $G \star H$ is an open subset of $X \times Y \times Z$ (which may be empty).*

Proposition 5.11 *Suppose G is a subset of $X \times Y$ and H is a subset of $Y \times Z$. Then*

$$(X \times Y \times Z) \setminus (G \star H) = [((X \times Y) \setminus G) \times Z] \cup [X \times ((Y \times Z) \setminus H)] \cup [X \times (Y \setminus (\pi_Y(G) \cap \pi_Y(H))) \times Z].$$

Proposition 5.12 *Suppose X, Y and Z are compact Hausdorff spaces, G is a closed subset of $X \times Y$ and H is a closed subset of $Y \times Z$. Then $G \times H$ is a closed subset of $X \times Y \times Z$ (which may be empty)*

Proof. Since G and H are closed, $(X \times Y) \backslash G$ is open in $X \times Y$, $((X \times Y) \setminus G) \times Z$ is open in $X \times Y \times Z$, $(Y \times Z) \setminus H$ is open in $Y \times Z$, and $X \times ((Y \times Z) \setminus H)$ is open in $X \times Y \times Z$. Since G and H are compact, we have that $\pi_Y(G)$ and $\pi_Y(H)$ are compact subsets of Hausdorff space and therefore closed. Hence, $Y \setminus (\pi_Y(G) \cap \pi_Y(H))$ is open in Y, and $X \times (Y \setminus (\pi_Y(G) \cap \pi_Y(H))) \times Z$ is open in $X \times Y \times Z$. The result then follows. ■

Proposition 5.13 *Suppose X, Y and Z are compact Hausdorff spaces, U is open in the closed set $G \subset X \times Y$ and V is open in the closed set $H \subset Y \times Z$. Then $U \star V$ is open relative to the closed set $G \star H$.*

Proof. There are (1) an open set U' of $X \times Y$ such that $U' \cap G = U$, and (2) an open set V' of $Y \times Z$ such that $V' \cap H = V$. Now $U \star V = (U' \cap G) \star (V' \cap H) = (U' \star V') \cap (G \star H)$, $U' \star V'$ is open in $X \times Y \times Z$, $G \star H$ is closed in $X \times Y \times Z$, so $(U' \star V') \cap (G \star H) = U \star V$ is open relative to $G \star H$. ■

Proposition 5.14 *Suppose X, Y and Z are compact Hausdorff spaces. Further, suppose G and H are closed subsets of $X \times Y$ and $Y \times Z$, respectively, and K is a closed subset of Y. Then*

$$\left(G \cap \pi_Y^{-1}(K)\right) \star \left(H \cap \pi_Y^{-1}(K)\right) = \left(\pi_Y^G\right)^{-1}(K) \star \left(\pi_Y^H\right)^{-1}(K).$$

Proof. Suppose $\mathbf{x} = (x, y, z) \in \left(G \cap \pi_Y^{-1}(K)\right) \star \left(H \cap \pi_Y^{-1}(K)\right)$. Then $y \in K, (x, y) \in G$ and $(y, z) \in H$, so we get $(x, y) \in \left(\pi_Y^G\right)^{-1}(K)$ and $(y, z) \in$

$\left(\pi_Y^H\right)^{-1}(K)$. Then $\mathbf{x} = (x, y, z) \in \left(\pi_Y^G\right)^{-1}(K) \star \left(\pi_Y^H\right)^{-1}(K)$. It follows that

$$\left(G \cap \pi_Y^{-1}(K)\right) \star \left(H \cap \pi_Y^{-1}(K)\right) \subset \left(\pi_Y^G\right)^{-1}(K) \star \left(\pi_Y^H\right)^{-1}(K)$$

Suppose $\mathbf{x} = (x, y, z) \in \left(\pi_Y^G\right)^{-1}(K) \star \left(\pi_Y^H\right)^{-1}(K)$. Then we have $y \in K, (x, y) \in G$, and $(y, z) \in H$, so $(x, y) \in G \cap \pi_Y^{-1}(K)$ and $(y, z) \in H \cap \pi_Y^{-1}(K)$. Then $\mathbf{x} \in \left(G \cap \pi_Y^{-1}(K)\right) \star \left(H \cap \pi_Y^{-1}(K)\right)$ and

$$\left(\pi_Y^G\right)^{-1}(K) \star \left(\pi_Y^H\right)^{-1}(K) \subset \left(G \cap \pi_Y^{-1}(K)\right) \star \left(H \cap \pi_Y^{-1}(K)\right)$$

The result follows. ∎

Proposition 5.15 *Suppose that $S \subset Y$. Then*

$$\left(\pi_Y^{X\times Y\times Z}\right)^{-1}(S) = \left(\pi_Y^{X\times Y}\right)^{-1}(S) \star \left(\pi_Y^{Y\times Z}\right)^{-1}(S).$$

Proposition 5.16 *Suppose that X, Y and Z are compact metric spaces. Suppose that G and H are closed subsets of $X \times Y$ and $Y \times Z$, and d_1 and d_2 are (compatible) metrics on $X \times Y$ and $Y \times Z$, respectively. If $\varepsilon > 0$, let $N_\varepsilon(G) = \{\mathbf{x} \in X \times Y : \text{there is some } \mathbf{z} \text{ in } G \text{ such that } d_1(\mathbf{x}, \mathbf{z}) < \varepsilon\}$ and $N_\varepsilon(H) = \{\mathbf{x} \in Y \times Z : \text{there is some } \mathbf{z} \text{ in } H \text{ such that } d_2(\mathbf{x}, \mathbf{z}) < \varepsilon\}$. Then $\bigcap_{\varepsilon>0}(N_\varepsilon(G) \star N_\varepsilon(H)) = G \star H$.*

Proof. Suppose $\mathbf{x} \in \bigcap_{\varepsilon>0}(N_\varepsilon(G) \star N_\varepsilon(H))$. If $\mathbf{x}_G = \pi_{X\times Y}(\mathbf{x})$ and $\mathbf{x}_H = \pi_{Y\times Z}(\mathbf{x})$, then $\mathbf{x}_G \star \mathbf{x}_H = \mathbf{x}$. Then for each $\varepsilon > 0, \mathbf{x}_G \in N_\varepsilon(G)$ and $\mathbf{x}_H \in N_\varepsilon(H)$. Thus, $\mathbf{x}_G \in G, \mathbf{x}_H \in H$, and $\mathbf{x} \in G \star H$. Then $\bigcap_{\varepsilon>0}(N_\varepsilon(G) \star N_\varepsilon(H)) \subset G \star H$. It is easy to see that $\bigcap_{\varepsilon>0}(N_\varepsilon(G) \star N_\varepsilon(H)) \supset G \star H$. The result follows. ∎

Proposition 5.17 *Suppose $G \subset X \times Y, H \subset Y \times Z$, and T is a topological space. If $g : T \to G$ and $h : T \to H$ are continuous and $\pi_Y(g(t)) = \pi_Y(h(t))$ for each $t \in T$, then $g \star h : T \to G \star H$, where $(g \star h)(t) = g(t) \star h(t)$ for each $t \in T$, is continuous.*

Proof. Since for each $t \in T, \pi_Y(g(t)) = \pi_Y(h(t)), g \star h$ is well defined. Suppose $t \in T$, and $U := U_X \times U_Y \times U_Z$ is a basic open set in $X \times Y \times Z$ that contains $g(t) \star h(t)$. Then $U \cap (G \star H)$ is open in $G \star H$. Since g and h are continuous, there are open sets V_1 and V_2 in T, both of which contain t, such that $g(V_1) \subset G \cap (U_X \times U_Y)$ and $h(V_2) \subset H \cap (U_Y \times U_Z)$. Then $t \in V_1 \cap V_2$, an open set in T, and, if $s \in V_1 \cap V_2, g(s) \star h(s) \in U \cap (G \star H)$. The result follows. ■

At the end of this section we outline the connection between generalized inverse limits defined in the Section 3.2 and Mahavier products.

Let $(X_i, f_i)_{i=1}^{\infty}$ be a generalized inverse sequence and put $G_i := \Gamma(f_i^{-1}) = \{(x_i, x_{i+1}) : (x_{i+1}, x_i) \in \Gamma(f_i)\} \subseteq X_i \times X_{i+1}$. It follows directly from the definitions of a generalized inverse limit and a Mahavier product that

$$\varprojlim (X_i, f_i)_{i=1}^{\infty} = \star_{i=1}^{\infty} G_i.$$

Hence, the generalized inverse limit itself is a Mahavier product.

Mahavier products also make it easy to discuss finite generalized inverse limits (something that is not observed in the standard case) and their subsets.

5.2 Topological entropy using open covers

In this section, the traditional version of topological entropy (due to Adler, Konheim, and McAndrew [1]) is reviewed and together with its properties, following to a large extent the discussion in Peter Walters' book [41].

First, we list all notions we need for this section.

Definitions 5.18

- *If α is a finite collection of sets, define $N^*(\alpha)$ to be the cardinality of the collection α. If α is an open cover of the compact topological space X, let $N(\alpha)$ denote the number of sets in a finite subcover of α of smallest cardinality. Define the* ***entropy*** *$H(\alpha)$ by $H(\alpha) = \log N(\alpha)$.*

- *If α is a finite collection of open sets that covers the set G, subset of a topological space X, then a subcover α' of G in α is* ***minimal*** *if there does not exist a subcover of G in α of smaller cardinality.*

- *If α and β are open covers of a space X, define the **join** $\alpha \vee \beta$ to be the collection*
$$\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}$$
of open sets. The join $\alpha \vee \beta$ is also an open cover of the space X. We can likewise define, for a finite collection $(\alpha_i : i = 1, \ldots, n)$ of open covers of X, the join $\vee_{i=1}^{n} \alpha_i$, for any $n \in \mathbb{N}, n \geq 2$.

- *If α and β are open covers of the compact topological space X, then α is a refinement of β if each $A \in \alpha$ is contained in some $B \in \beta$. We will say that $\beta < \alpha$ and also that $\alpha > \beta$. Note that if α is a subcover of X in β, then α is both a subcollection of β and a refinement of β, and $\beta < \alpha$.*

- *If X is a compact topological space, α is an open cover of X, and $f : X \to X$ is continuous, then $f^{-1}(\alpha)$ is the open cover consisting of all sets $f^{-1}(A)$ where $A \in \alpha$. Also,*
$$f^{-1}(\alpha \vee \beta) = f^{-1}(\alpha) \vee f^{-1}(\beta) \qquad (*)$$
and
$$\alpha < \beta \text{ implies } f^{-1}(\alpha) < f^{-1}(\beta).$$
We denote $\alpha \vee f^{-1}(\alpha) \vee \cdots \vee f^{-n}(\alpha)$ by $\vee_{i=0}^{n} f^{-i}(\alpha)$.

Remarks 5.19 *Suppose α and β are open covers of the compact topological space X. Then*

1. *$H(\alpha) \geq 0$.*
2. *$H(\alpha) = 0$ if and only if $N(\alpha) = 1$ if and only if $X \in \alpha$.*
3. *If $\alpha < \beta$, then $H(\alpha) \leq H(\beta)$.*
4. *$H(\alpha \vee \beta) \leq H(\alpha) + H(\beta)$.*

5. If $f : X \to X$ is a continuous map, then $H(f^{-1}(\alpha)) \le H(\alpha)$. If f is also surjective, then $H(f^{-1}(\alpha)) = H(\alpha)$.

Proof. (See [41] for proofs of 3., 4., and 5. above.) ■

We will need the following lemma, which is used in the proof of Theorem 5.21, and in our results.

Lemma 5.20 *[41] If $(a_n)_{n\ge 1}$ is a sequence of nonnegative real numbers such that $a_{n+p} \le a_n + a_p$ for each $n, p \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\inf_n \frac{a_n}{n}$.*

Proof. Fix $p > 0$. Each $n > 0$ can be written as $n = kp+i$ for some $0 \le i < p$. Then we have

$$\frac{a_n}{n} = \frac{a_{i+kp}}{i+kp} \le \frac{a_i}{kp} + \frac{a_{kp}}{kp} \le \frac{a_i}{kp} + \frac{ka_p}{kp} \le \frac{a_i}{kp} + \frac{a_p}{p}.$$

If $n \to \infty$, then $k \to \infty$, so we have

$$\limsup_n \frac{a_n}{n} \le \frac{a_p}{p},$$

and hence

$$\limsup_n \frac{a_n}{n} \le \inf_p \frac{a_p}{p}.$$

Since

$$\inf_p \frac{a_p}{p} \le \liminf_n \frac{a_n}{n},$$

it follows that $\lim_n \frac{a_n}{n}$ exists and is equal to $\inf_n \frac{a_n}{n}$. ■

Theorem 5.21 *[41] If α is an open cover of X and $f : X \to X$ is continuous, then*

$$\lim_{n\to\infty} \frac{H(\vee_{i=0}^{n} f^{-i}(\alpha))}{n}$$

exists.

Proof. If we set

$$a_n = H\left(\vee_{i=0}^{n-1} f^{-i}(\alpha)\right)$$

then by the Lemma 5.20 it suffices to show that $a_{n+k} \le a_n + a_k$ for $k, n \ge 1$. We have

$$\begin{aligned} a_{n+k} &= H\left(\vee_{i=0}^{n+k-1} f^{-i}(\alpha)\right) \\ &\le H\left(\vee_{i=0}^{n-1} f^{-i}(\alpha)\right) + H\left(f^{-n} \vee_{j=0}^{k-1} f^{-j}(\alpha)\right) \text{ by Remark (4) and } (*) \\ &\le a_n + a_k \text{ by Remark (5)} . \end{aligned}$$

■

Definition 5.22 *If α is an open cover of the compact topological space X, and $f : X \to X$ is continuous, then the **entropy of** f **relative to** α, denoted by $h(f, \alpha)$, is given by*

$$h(f, \alpha) = \lim_{n \to \infty} \frac{H(\vee_{i=0}^{n} f^{-i}(\alpha))}{n}.$$

Remarks 5.23

1. *$h(f, \alpha) \ge 0$.*
2. *If $\alpha < \beta$, then $h(f, \alpha) \le h(f, \beta)$.*
3. *$h(f, \alpha) \le H(\alpha)$.*

Proof. See [41]. ■

Definition 5.24 *If $f : X \to X$ is continuous, the **topological entropy** $h(f)$ of f is given by*

$$h(f) = \sup_{\alpha} h(f, \alpha)$$

where α ranges over all open covers of X.

Remarks 5.25

1. *$\infty \ge h(f) \ge 0$.*
2. *In the definition of $h(f)$ one can take the supremum over finite open covers of X. This follows from the fact that if $\alpha < \beta$, then $h(f, \alpha) \le h(f, \beta)$.*

3. If id_X denotes the identity map from X to X, then $h(id_X) = 0$.

4. If Y is a closed subset of X and $f(Y) = Y$, then $h(f|Y) \leq h(f)$.

Proof. *See [41].* ■

Theorem 5.26 *[41] If X_1, X_2 are compact spaces and $f_i : X_i \to X_i$ are continuous for $i = 1, 2$, and if $\phi : X_1 \to X_2$ is a continuous map with $\phi(X_1) = X_2$ and $\phi \circ f_1 = f_2 \circ \phi$, then $h(f_1) \geq h(f_2)$. If ϕ is a homeomorphism, then $h(f_1) = h(f_2)$.*

Proof. Let α be an open cover of X_2. Then

$$\begin{aligned} h(f_2, \alpha) &= \lim_{n\to\infty} \frac{1}{n} H\left(\vee_{i=0}^{n-1} f_2^{-i}(\alpha)\right) \\ &= \lim_{n} \frac{1}{n} H\left(\phi^{-1} \vee_{i=0}^{n-1} f_2^{-i}(\alpha)\right) \text{ by (5)} \\ &= \lim_{n} \frac{1}{n} H\left(\vee_{i=0}^{n-1} \phi^{-1} f_2^{-i}(\alpha)\right) \text{ by } (*) \\ &= \lim_{n} \frac{1}{n} H\left(\vee_{i=0}^{n-1} f_1^{-i} \phi^{-1}(\alpha)\right) \\ &= h\left(f_1, \phi^{-1}(\alpha)\right). \end{aligned}$$

Hence, $h(f_2) \leq h(f_1)$. If ϕ is a homeomorphism then $\phi^{-1} f_2 = f_1 \phi^{-1}$ so, by the above, $h(f_1) \leq h(f_2)$. ■

Theorem 5.27 *[41] If $f : X \to X$ is a homeomorphism of a compact space X, then $h(f) = h(f^{-1})$.*

Proof.

$$\begin{aligned} h(f, \alpha) &= \lim_{n\to\infty} \frac{1}{n} H\left(\vee_{i=0}^{n-1} f^{-i}(\alpha)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} H\left(f^{n-1}\left(\vee_{i=0}^{n-1} f^{-i}(\alpha)\right)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} H\left(\vee_{i=0}^{n-1} f^{i}(\alpha)\right) \\ &= H(f^{-1}, \alpha). \end{aligned}$$

■

Theorem 5.28 *If $f : X \to X$ is a continuous map of a compact metric space X, then $h(f^n) = n\, h(f)$.*

Proof. See [41], Theorem 7.10. ∎

5.3 Topological entropy of closed subsets of $[0,1]^2$

5.3.1 Background and notation

Sometimes it is convenient to index our factor spaces, sometimes not. Suppose for each integer $i \geq 0$, $I_i = [0,1]$. The Hilbert cube is $I^\infty = [0,1]^\infty = \prod_0^\infty I_i$.

We often need to talk about various projections from a subset of I^∞ into an interval or a product of intervals. Unless it leads to confusion, for a subset X of I^∞, and a point $x = (x_0, x_1, ...)$ in X, $\pi_i(x) = x_i$. (That is, we do not specify the momentary domain of π_i.) Likewise, if N is a positive integer, $x = (x_0, x_1, ..x_N)$, $x \in X \subset I^{N+1}$, then $\pi_i(x) = x_i$ for $0 \leq i \leq N$. Also, we make following definitions.

- We use both $\mathbb{N}$ and $\mathbb{Z}^+$ to denote the positive integers.
- Let m, n be integers and $0 \leq m < n$. Put $\langle m, n\rangle = \{m, m+1, ..., n\}$, and we call $\langle m, n\rangle$ the **integer interval from** m **to** n. Then $\pi_{\langle m,n\rangle}(x) = (x_m, x_{m+1}, ..., x_n)$. We define $\langle m, \infty\rangle$ to be the set $\{m, m+1, ...\}$.
- Let $A = \{n_1, n_2, ...\}$ denote a subset of the nonnegative integers (not necessarily listed in order, and either finite or infinite). Then $\pi_A(x) = (x_{n_1}, x_{n_2}, ...)$.
- If A is a subset of the space X, then A° denotes the interior of A in X, and $\overline{A}$ denotes the closure of A in X.
- Suppose $x = (x_0, x_1, ..., x_n)$ is a point in I^{n+1} and $(y_0, y_1, ...)$ is a point in I^∞. Then we define $x \oplus y$ to be the point $(x_0, ..., x_n, y_0, y_1, ...)$ in I^∞.

- The metric we use on I^∞ is $d(x,y) = \sum_{i=0}^{\infty} \frac{|\pi_i(x)-\pi_i(y)|}{2^i}$, where x and y are points in I^∞.

- The **shift map** $\sigma : I^\infty \to I^\infty$ is defined by $\sigma((x_0, x_1, x_2, ...)) = (x_1, x_2, ...)$. The shift map takes I^∞ continuously onto itself. Also, if $f : I_i \to 2^{I_{i-1}}$ for each $i > 0$, and $M = \varprojlim \mathbf{f}$, then $\sigma(M) = M$ (for proof see [22, p. 62]). Hence, M is invariant under the action of σ.

- For $A \subset I_0 \times I_1$, define $A^{-1} = \{(x,y) : (y,x) \in A\}$. More generally, if N is a positive integer and $A \subset \prod_{i=0}^{N} I_i$, then we define

  $A^{-1} = \left\{(x_N, x_{N-1}, \ldots, x_1, x_0) \in \prod_{i=0}^{N} I_i : (x_0, x_1, \ldots, x_{N-1}, x_N) \in A\right\}.$

- Suppose X, Y are topological spaces, and α is a collection of sets that covers X. Then $\alpha \times Y$ denotes the collection $\{A \times Y : A \in \alpha\}$, which covers $X \times Y$.

- Suppose α is a collection of (open) sets in the space X, and $H \subset X$. Put $\alpha \cap H := \{A \cap H : A \in \alpha\}$.

5.3.2 Preliminary results

Before we define topological entropy we need some background information on the closed sets and open covers we are using.

The following examples demonstrate that if G is a closed subset of $[0,1]^2$, then

1. it may be the case that $\mathbf{G} := \star_{i=1}^{\infty} G = \emptyset$ (and that G is of limited interest), and

2. even if $\mathbf{G} := \star_{i=1}^{\infty} G \neq \emptyset$, it may be the case that $\sigma(\mathbf{G}) \neq \mathbf{G}$.

Example 5.29 *Suppose G is the closed subset $[\frac{2}{3}, 1] \times [0, \frac{1}{3}]$ of $I_i \times I_{i+1}$. Then $\mathbf{G} := \star_{i=1}^{\infty} G = \emptyset$. In fact, $G \star G = \emptyset$.*

Example 5.30 *Let $L_0 = I_0 \times \{p\}$ and $L_1 = I_0 \times \{q\}$, where $0 \leq p < q \leq 1$.Suppose G is the closed subset $L_0 \cup L_1$ of $I_0 \times I_1$. Then $\mathbf{G} := \star_{i=1}^{\infty} G$ is a Cantor set of arcs, and $\sigma(\mathbf{G})$ is a Cantor set and is a proper subset of $\mathbf{G}$.*

Proof. Let $C = \{s = (s_1, s_2, \ldots) : s_i \in \{p, q\} \text{ for each } i > 0\}$. Then C is a Cantor set contained in $\mathbf{G}$. Moreover, for each $s \in C$, $I_0 \times \{s\}$ is an arc contained in $\mathbf{G}$, and $\mathbf{G} = \cup\{I_0 \times \{s\} : s \in C\}$. Hence, $\mathbf{G}$ is a Cantor set of arcs. Since $\sigma(\mathbf{G}) = C$, C is a proper subset of $\mathbf{G}$. ■

Proposition 5.31 *If G is a nonempty closed subset of $I_0 \times I_1$, then $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$ if and only if for every integer $m \geq 2$, $\star_{i=1}^{m} G \neq \emptyset$.*

Proof. If $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$ it follows from the definition that $\star_{i=1}^{m} G \neq \emptyset, \forall m \in \mathbb{N}$. Now, suppose $\star_{i=1}^{m} G \neq \emptyset$ for every integer $m \geq 2$. We will inductively define point in $\mathbf{G} = \star_{i=1}^{\infty} G$. First observe the following: If $(x_0, \ldots, x_{m-1}, x_m) \in \star_{i=1}^{m} G$ for some integer $m \geq 2$ then $(x_0, \ldots, x_{m-1}) \in \star_{i=1}^{m-1} G$. $(*)$

For $m = 2$ we have $G \star G \neq \emptyset$ so from the above it follows that there is a point $(x, y) \in G$ and $z \in [0, 1]$ such that $(x, y, z) \in G \star G$.

Now, for given $m = k$ we have that $\star_{i=1}^{k} G \neq \emptyset$. So, there are points $(x_0, x_1, \ldots, x_k) \in \star_{i=1}^{k} G$ and $x_{k+1} \in [0, 1]$ such that $(x_0, x_1, \ldots, x_k, x_{k+1}) \in \star_{i=1}^{k+1} G$. This follows from $(*)$ and assumption $\star_{i=1}^{k+1} G \neq \emptyset$.

Therefore we constructed a sequence $x_0, x_1, \ldots, x_k, \ldots$, such that for each positive integer $m \geq 2, (x_0, x_1, \ldots, x_m) \in \star_{i=1}^{m} G$ i.e. $(x_0, x_1, \ldots, x_k, \ldots) \in \star_{i=1}^{\infty} G$. Therefore we have $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$. ■

Proposition 5.32 *Let G be a nonempty closed subset of $I_0 \times I_1$. If there is some point $(x, y) \in G$ such that (y, x) is also in G, then $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$.*

Proof. Let $(x, y) \in G$ such that $(y, x) \in G$. Then, for each integer $m \geq 2$ we have that $(x, y, x, y, \ldots, x) \in \star_{i=1}^{m} G$ if m is even or $(x, y, x, \ldots, y) \in \star_{i=1}^{m} G$ if m is odd. In both cases $\star_{i=1}^{m} G \neq \emptyset$ and therefore, from Proposition 5.31 it follows that $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$. ■

Corollary 5.33 *If $G = G^{-1}$, where $G^{-1} = \{(y, x) : (x, y) \in G\}$ is a nonempty closed subset of $I_0 \times I_1$, then $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$.*

Proof. Since $G \neq \emptyset$, there is some point $(x, y) \in G$. Since $G = G^{-1}$, the point $(y, x) \in G$. From previous proposition it follows that $\mathbf{G} \neq \emptyset$. ■

Proposition 5.34 *Suppose n is a positive integer. If G is a nonempty closed subset of $I_0 \times I_1$ that contains a finite set of points $\{(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)\}$, then $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$. Furthermore, $\mathbf{G}$ contains a point x of period n under the action of σ.*

Proof. The point $(x_0, x_1, \ldots, x_n, x_0, \ldots, x_n, \ldots) \in \mathbf{G}$, so $\mathbf{G} \neq \emptyset$. Let $y_0 = (x_0, \ldots, x_n)$ and $y_n = (x_n, x_0, x_1 \ldots x_{n-1})$. For each $0 < i < n$, let $y_i = (x_i, \ldots, x_n, x_0, \ldots x_{i-1})$. For $0 \leq i \leq n$, let $z_i = y_i \oplus y_i \oplus y_i \oplus \ldots$. Then each $z_i \in \mathbf{G}$, and $\sigma(z_i) = z_{i+1}$ for $0 \leq i < n$, and $\sigma(z_n) = z_0$. Hence, $\sigma^n(z_0) = z_0$. ■

Proposition 5.35 *If G is a nonempty closed subset of $I_0 \times I_1$ and $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$, then $\sigma(\mathbf{G}) \subset \mathbf{G}$.*

Proof. Suppose $x = (x_0, x_1, \ldots) \in \sigma(\mathbf{G})$. Then there is $y = (y_0, y_1, \ldots) \in \mathbf{G}$ such that $\sigma(y) = x$. Now $\sigma(y) = (y_1, y_2, \ldots) = x$, so $x_{i-1} = y_i$ for each $i > 0$. Since $y \in \mathbf{G}$, for each $i > 0$, $(y_{i-1}, y_i) \in G$. Then for each $i > 1$, $(y_{i-1}, y_i) = (x_{i-2}, x_{i-1}) \in G$. Then $x \in \mathbf{G}$. ■

Proposition 5.36 *Let G be a nonempty closed subset of $I_0 \times I_1$, $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$ and $\cap_{i=0}^{\infty} \sigma^i(\mathbf{G}) = \mathbf{G}^*$. Then $\mathbf{G}^* \neq \emptyset$ and $\mathbf{G}^* \subset \mathbf{G}$. Furthermore, $\sigma(\mathbf{G}^*) = \mathbf{G}^*$.*

Proof. Since $\sigma^n(\mathbf{G}) \subset \sigma^{n-1}(\mathbf{G})$ for $n > 0$ and $\sigma^n(\mathbf{G})$ is closed for each $n \in \mathbb{N}$, we have $\mathbf{G}^* \neq \emptyset$ and $\mathbf{G}^* \subset \mathbf{G}$. It remains to prove that $\sigma(\mathbf{G}^*) = \mathbf{G}^*$. Let $x \in \sigma(\mathbf{G}^*)$. Then $x \in \sigma\left(\cap_{i=0}^{\infty} \sigma^i(\mathbf{G})\right) \subseteq \cap_{i=0}^{\infty} \sigma\left(\sigma^i(\mathbf{G})\right) = \cap_{i=1}^{\infty} \sigma^i(\mathbf{G}) \overset{5.35}{\subseteq} \cap_{i=0}^{\infty} \sigma^i(\mathbf{G}) = \mathbf{G}^*$.

Now, let $x = (x_1, x_2, \ldots) \in \mathbf{G}^* = \cap_{i=0}^{\infty} \sigma^i(\mathbf{G})$. In particular, it follows that $x = (x_1, x_2, \ldots) \in \sigma(\mathbf{G})$ and similar as in the proof of the previous proposition we get that there is $x_0 \in I$ such that $(x_0, x_1, x_2, \ldots) \in \mathbf{G}$. Continuing in the same way we get that $(x_0, x_1, x_2, \ldots) \in \sigma^n(\mathbf{G}), \forall n \in \mathbb{N}$. Therefore,

$(x_0, x_1, x_2, \ldots) \in \mathbf{G}^*$. Finally, $x = \sigma((x_0, x_1, x_2, \ldots)) \in \sigma(\mathbf{G}^*)$. ■

For G is a nonempty closed subset of $I_0 \times I_1$ and $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$, we will call the set $\mathbf{G}^* = \cap_{i=0}^{\infty} \sigma^i(\mathbf{G})$ the **kernel** of $\mathbf{G}$.

5.3.3 Grid covers

Suppose G is a closed subset of I^∞. Let $\tau = \{T_1, \ldots, T_n\}$ be a minimal open cover of $[0, 1]$ by open intervals. Let N be a positive integer. The **grid generated by** τ **for** N is the collection $\mathcal{T}$ of basic open sets in I^∞

$$\mathcal{T} = \{T_{i_0} \times T_{i_1} \times \ldots \times T_{i_N} \times I^\infty : i_j \in \langle 1, n \rangle\}.$$

Since $\mathcal{T}$ is an open cover of I^∞ by basic open sets, it is therefore also a cover of G by basic open sets. We will say that T is a **grid cover** of G. Likewise,

$$\mathcal{S} = \{T_{i_0} \times T_{i_1} \times \ldots \times T_{i_N} : i_j \in \langle 1, n \rangle\}.$$

is a **grid cover** of I^{N+1} by basic open sets, and is also therefore a cover of any closed subset K of I^{N+1}.

Proposition 5.37 *Suppose G is a closed subset of I^∞. If α is an open cover of G by open sets in I^∞, then there is a grid cover $\mathcal{T}$ of I^∞ such that $\mathcal{T}' = \{T \in \mathcal{T} : T \cap G \neq \emptyset\}$ refines α and covers G. If α' is an open cover of G by open sets in the subspace G, then there is a grid cover $\mathcal{T}$ of I^∞ such that $\mathcal{T}^* = \{T \cap G : T \in \mathcal{T}\}$ refines α' and covers G.*

Proof. Suppose α is an open cover of G by open sets in I^∞. Then there is a collection β of basic open sets in I^∞ that refines α and covers G. We also assume that for each $B \in \beta$, each projection $\pi_k(B)$ is an open interval (relative to $[0, 1]$). Since G is compact, there is a finite subcover β' of β. Let $\beta' = \{B_1, \ldots, B_m\}$. G is compact metric space with inherited metric that on I^∞, so there is a collection δ of basic open sets such that $\delta = \{D_1, \ldots, D_m\}$, $\overline{D_i} \subset B_i$ for $1 \leq i \leq m$, and δ covers G. Again, we can choose the collection δ so that for each $D \in \delta$, each projection $\pi_k(D)$ is an open interval relative to $[0,1]$. Since each D_i is a basic open set, there is some positive integer N such that $\pi_j(D_i) = I$ for each $j > N$.

Let $\pi_k(\overline{D_i}) = [a_{i,k}, b_{i,k}]$ for each $0 \le k \le N$, $1 \le i \le m$, and let

$$\mathcal{E} = \{x : x \in \{a_{i,k}, b_{i,k}\}, 0 \le k \le N, 1 \le i \le m\} \cup \{0,1\}.$$

Since $\mathcal{E}$ is a finite subset of I, we can list the members of $\mathcal{E}$ in increasing order as $\mathcal{E} = \{0 = t_0, t_1, \ldots, t_l = 1\}$. Then each $\pi_k(\overline{D_j})$ is a unique union of consecutive intervals of the form $[t_{i-1}, t_i]$. If $x = (x_0, x_1, \ldots) \in G$, there is some D_i such that $x \in D_i$, which implies that for each $k \le N$, $x_k \in \pi_k(\overline{D_i})$, and there is some t_{j_k} such that $x_k \in [t_{j_k}, t_{j_k+1}]$. Thus, $x \in \prod_{k=0}^{N}[t_{j_k}, t_{j_k+1}] \times I^\infty$.

Suppose $\epsilon > 0$. Let $\pi_k(\overline{D_i})^+ = (a_{i,k} - \epsilon, b_{i,k} + \epsilon) \cap [0,1]$ for each $0 \le k \le N$, $1 \le i \le m$. Let $D_i^+ = \prod_{k=0}^{N} \pi_k(\overline{D_i})^+ \times I^\infty$. We can choose $\epsilon > 0$ so small that

(1) $\epsilon < \min \frac{\{|t_{i+1} - t_i| \,:\, i=0,\ldots,l-1\}}{16}$, and

(2) $\overline{D_i} \subset D_i^+ \subset B_i$.

Then each D_i^+ is a union of members of

$$\mathcal{T} = \left\{ \prod_{i=0}^{N}((t_{j_i} - \epsilon, t_{j_i+1} + \epsilon) \cap [0,1]) \times I^\infty : j_i \in \langle 0, l \rangle \right\}.$$

Hence, if $\mathcal{T}' = \{T \in \mathcal{T} : T \cap G \neq \emptyset\}$, then $\mathcal{T}' > \delta > \beta > \alpha$ and $\mathcal{T}'$ covers G.

Now let us prove the second part.

Suppose α' is an open cover of G by open sets in the subspace G. Then there is an open cover α of G by open sets in I^∞ such that $\alpha \cap G = \alpha'$.

Now we apply the statement proved above and get $\mathcal{T}^*$ as $\mathcal{T}' \cap G$. ■

Proposition 5.38 *Suppose m is a positive integer and G is a closed subset of I^{m+1}. If α is an open cover of G by open sets in I^{m+1}, then there is a grid cover $\mathcal{T}$ of I^{m+1} such that $\mathcal{T}' = \{T \in \mathcal{T} : T \cap G \neq \emptyset\}$ refines α and covers G. If α' is an open cover of G by open sets in the subspace G, then there is a* *grid cover $\mathcal{T}$ of I^{m+1} such that $\mathcal{T}^* = \{T \cap G : T \in \mathcal{T}\}$ refines α' and covers G.*

Proof.

The proof is similar to the proof of Proposition 5.37 but for the completeness of the thesis we present it.

Suppose α is an open cover of G by open sets in I^∞. Then there is a collection β of basic open sets in I^∞ that refines α and covers G. We also assume that for each $B \in \beta$, each projection $\pi_k(B)$ is an open interval (relative to $[0,1]$). Since G is compact, there is a finite subcover β' of β. Let $\beta' = \{B_1, \ldots, B_p\}$. G is compact metric space with inherited metric that on I^∞, so there is a collection δ of basic open sets such that $\delta = \{D_1, \ldots, D_p\}$, $\overline{D_i} \subset B_i$ for $1 \le i \le p$, and δ covers G. Again, we can choose the collection δ so that for each $D \in \delta$, each projection $\pi_k(D)$ is an open interval relative to $[0,1]$.

Let $\pi_k(\overline{D_i}) = [a_{i,k}, b_{i,k}]$ for each $0 \le k \le m$, $1 \le i \le p$, and let $\mathcal{E} = \{x : x \in \{a_{i,k}, b_{i,k}\}, 0 \le k \le m, 1 \le i \le p\} \cup \{0,1\}$.

Since $\mathcal{E}$ is a finite subset of I, we can list the members of $\mathcal{E}$ in increasing order as $\mathcal{E} = \{0 = t_0, t_1, \ldots, t_l = 1\}$. Then each $\pi_k(\overline{D_j})$ is a unique union of consecutive intervals of the form $[t_{i-1}, t_i]$. If $x = (x_0, x_1, \ldots, x_m) \in G$, there is some D_i such that $x \in D_i$, which implies that for each $k \le m$, $x_k \in \pi_k(\overline{D_i})$, and there is some t_{j_k} such that $x_k \in [t_{j_k}, t_{j_k+1}]$. Thus, $x \in \prod_{k=0}^{m}[t_{j_k}, t_{j_k+1}]$.

Suppose $\epsilon > 0$. Let $\pi_k(\overline{D_i})^+ = (a_{i,k} - \epsilon, b_{i,k} + \epsilon) \cap [0,1]$ for each $0 \le k \le m$, $1 \le i \le p$. Let $D_i^+ = \prod_{k=0}^{m} \pi_k(\overline{D_i})^+$. We can choose $\epsilon > 0$ so small that $\epsilon < \min \frac{\{|t_{i+1} - t_i| \ : \ i = 0, \ldots, l-1\}}{16}$ and $\overline{D_i} \subset D_i^+ \subset B_i$. Then each D_i^+ is a union of members of

$$\mathcal{T} = \left\{ \prod_{i=0}^{N} ((t_{j_i} - \epsilon, t_{j_i+1} + \epsilon) \cap [0,1]) : j_i \in \langle 0, l \rangle \right\}.$$

Hence, if $\mathcal{T}' = \{T \in \mathcal{T} : T \cap G \ne \emptyset\}$, then $\mathcal{T}' > \delta > \beta > \alpha$ and $\mathcal{T}'$ covers G.

Proving the second part now follows easy.

Suppose α' is an open cover of G by open sets in the subspace G. Then there is an open cover α of G by open sets in I^∞ such that $\alpha \cap G = \alpha'$.

Now we apply the statement proved above and get $\mathcal{T}^*$ as $\mathcal{T}' \cap G$.

■

For a grid cover $\mathcal{T}$ of I^{m+1} or I^{∞}, we refer to the members of $\mathcal{T}$ as **boxes**. Setting up the machinery for a definition of topological entropy of G, a closed subset of $[0,1]^2$ (and later for G a closed subset of I^{m+1}), takes some doing, but once in place, we will be able to compute topological entropy by "counting" the boxes our relevant sets intersect.

5.3.4 Main results

We index our intervals for bookkeeping purposes. For convenience, we also write I^{∞} for $\prod_{i=m}^{\infty} I_i$ (for m a positive integer). Suppose G is a closed subset of $I_0 \times I_1$. We can define the **topological entropy** of G as follows:

1. First, let $\alpha = \{A_1, \ldots, A_n\}$ be a minimal open cover of I_0 by intervals. Then $N^*(\alpha) = n$. For each positive integer $m > 1$, let

$$\alpha^m = \left\{ \prod_{j=0}^{m-1} A_{k_j} : k_j \in \langle 1, n \rangle, 0 \leq j \leq m-1 \right\}.$$

2. If K is a closed subset of $\prod_{i=0}^{m-1} I_i$ ($m > 1$ a positive integer or $m = \infty$), and β is a collection of open sets in $\prod_{i=0}^{m-1} I_i$ that covers K, let $N(K, \beta)$ denote the least cardinality of a subcover of K in β.

3. Then $\alpha^2 = \{A_i \times A_j : i, j \in \langle 1, n \rangle\}$ is a cover of G by open subsets of $I_0 \times I_1$, and $N(G, \alpha^2) \leq n^2$.

4. Now

$$\alpha^3 = \{A_{i_0} \times A_{i_1} \times A_{i_2} : i_k \in \langle 1, n \rangle, 0 \leq k \leq 2\},$$

is a cover of $G \star G$ by open sets in $\prod_{j=0}^{2} I_j$ and $N(G \star G, \alpha^3) \leq n^3$.

5. Note that $\alpha^2 \star \alpha^2$ contains more sets than does α^3 since it contains sets of the form $(A_i \times A_j) \star (A_k \times A_l)$ for $i, j, k, l \leq n$, and $(A_i \times A_j) \star (A_k \times A_l) = A_i \times (A_j \cap A_k) \times A_l$, which is nonempty as long as $A_j \cap A_k \neq \emptyset$. However, a minimal subcover of $G \star G$ in $\alpha^2 \star \alpha^2$ has the same number of elements

as a minimal subcover of α^3, since each set $A_i \times (A_j \cap A_k) \times A_l$ is contained in at least one member of α^3.

6. We can continue this process for each $m \in \mathbb{N}$:

$$\alpha^{m+1} = \left\{ \prod_{j=0}^{m} A_{k_j} : k_j \in \langle 1, n \rangle, 0 \le j \le m \right\} \qquad (*)$$

is an open cover of $\star_{i=1}^m G$ and $N(\star_{i=1}^m G, \alpha^{m+1}) \le n^{m+1}$. Again, a minimal subcover of $\star_{i=1}^m G$ by elements of $\star_{i=1}^m \alpha^2$ has the same number of elements as a minimal subcover of $\star_{i=1}^m G$ by elements of α^{m+1}. Since using the cover $\star_{i=1}^m \alpha^2$ is sometimes more convenient, we continue to use both covers. Without loss of generality, we assume that a minimal subcover (in both α^{m+1} and $\star_{i=1}^m \alpha^2$) consists of sets of the form $\prod_{j=0}^m A_{k_j}$.

In several following statements, to simplify notation, we will denote elements of $\alpha^{m+1}, m \in \mathbb{N}$, with A_i^{m+1} for some positive integer i instead as in $(*)$ if particular sets A_{k_j} are not important for proofs. Likewise, if $\beta^{m+1} = \left\{ \prod_{j=0}^m B_{k_j} : k_j \in \langle 1, n \rangle, 0 \le j \le m \right\}$, we denote its elements with B_i^{m+1}.

Proposition 5.39 *Suppose G is a closed subset of $I_0 \times I_1$. Let $\alpha = \{A_1, A_2, \ldots, A_n\}$ denote a minimal open cover of I_0 by open intervals and let $\mathbf{G} = \star_{i=1}^\infty G$.*

(i) For each positive integer m, $0 \le N(\star_{i=1}^m G, \alpha^{m+1}) \le n^{m+1}$. If $\mathbf{G} \ne \emptyset$, $0 < N(\star_{i=1}^m G, \alpha^{m+1})$.

(ii) If $\mathbf{G} \ne \emptyset$, $1 = N(\star_{i=1}^m G, \alpha^{m+1})$ if and only if there is a finite sequence $A_{j_0}, A_{j_1}, \ldots A_{j_m}$ (with each $1 \le j_i \le n$) such that $\mathbf{G} \subset (A_{j_0} \times \ldots \times A_{j_m}) \times I^\infty$.

(iii) If α, β are both minimal open covers of I_0 by open intervals and $\alpha < \beta$, then for each $m > 0$, $N(\star_{i=1}^m G, \alpha^{m+1}) \le N(\star_{i=1}^m G, \beta^{m+1})$.

Proof. First two statements are obvious so let us prove the third one. Let $k = N(\star_{i=1}^m G, \beta^{m+1})$ and $\{B_1^{m+1}, B_2^{m+1}, \ldots, B_k^{m+1}\}$ be a subcover of $\star_{i=1}^m G$ in β^{m+1} of minimal cardinality. For each $1 \le i \le k$, there is some $A_i^{m+1} \in \alpha^{m+1}$ such that $B_i^{m+1} \subset A_i^{m+1}$. Then $\{A_1^{m+1}, A_2^{m+1}, \ldots, A_k^{m+1}\}$ is a subcover of $\star_{i=1}^m G$ in α^{m+1} of cardinality k. Hence, $N(\star_{i=1}^m G, \alpha^{m+1}) \le N(\star_{i=1}^m G, \beta^{m+1})$. ■

Proposition 5.40 *If α is a minimal open cover of I_0 by open intervals, k, l are positive integers, and $K \subset \star_{i=1}^k G$, $L \subset \star_{i=1}^l G$, K, L are closed, then $\alpha^{k+1} \star \alpha^l$ is a cover of $\prod_{i=0}^{k+l} I_i$ and of $K \star L$ by open sets in $\prod_{i=0}^{k+l} I_i$, as is $(\star_{i=1}^k \alpha^2) \star (\star_{i=1}^l \alpha^2) = \star_{i=1}^{k+l} \alpha^2$. Furthermore, $N(\star_{i=1}^{k+l} G, \alpha^{k+l+1}) = N((\star_{i=1}^{k+l} G, \star_{i=1}^{k+l} \alpha^2) \le N(\star_{i=1}^k G, \alpha^{k+1}) N(\star_{i=1}^l G, \alpha^{l+1}) = N(\star_{i=1}^k G, \star_{i=1}^k \alpha^2) N(\star_{i=1}^l G, \star_{i=1}^l \alpha^2)$.*

Proof. Showing that $\alpha^{k+1} \star \alpha^l$ and $(\star_{i=1}^k \alpha^2) \star (\star_{i=1}^l \alpha^2) = \star_{i=1}^{k+l} \alpha^2$ are open covers of $\prod_{i=0}^{k+l} I_i$ and of $K \star L$ by open sets in $\prod_{i=0}^{k+l} I_i$ is straightforward. Let $\{A_1^{k+1}, A_2^{k+1}, \ldots, A_p^{k+1}\}$ be a subcover of $\star_{i=1}^k G$ in α^{k+1} of minimal cardinality and let $\{B_1^{l+1}, B_2^{l+1}, \ldots, B_q^{l+1}\}$ be a subcover of $\star_{i=1}^l G$ in α^{l+1} of minimal cardinality. Then $\{A_i^{k+1} \star B_j^{l+1} : 1 \le i \le p, 1 \le j \le q\}$ is a subcover of $\star_{i=1}^{k+l} G$ in α^{k+l+1}, and

$$\begin{aligned} N(\star_{i=1}^{k+l} G, \alpha^{k+l+1}) =& N(\star_{i=1}^{k+l} G, \star_{i=1}^{k+l} \alpha^2) \\ \overset{5.39}{\le} & N(\star_{i=1}^k G, \alpha^{k+1}) N(\star_{i=1}^l G, \alpha^{l+1}) \\ =& N(\star_{i=1}^k G, \star_{i=1}^k \alpha^2) N(\star_{i=1}^l G, \star_{i=1}^l \alpha^2). \end{aligned}$$

Equalities above follow from the 6th step at the beginning of this section. ■

Proposition 5.41 *If α, β are both minimal open covers of I_0 by open intervals, then for each $m > 0$, $\alpha^{m+1} \vee \beta^{m+1} = (\alpha \vee \beta)^{m+1}$, and $N(\star_{i=1}^m G, \alpha^{m+1} \vee \beta^{m+1}) \le N(\star_{i=1}^m G, \alpha^{m+1}) N(\star_{i=1}^m G, \beta^{m+1})$.*

Proof. Showing that $(\alpha \vee \beta)^{m+1} = \alpha^{m+1} \vee \beta^{m+1}$ is straightforward and we omit it. Let $\{A_1^{m+1}, A_2^{m+1}, \ldots, A_k^{m+1}\}$ be a subcover of $\star_{i=1}^m G$ in α^{m+1} of minimal cardinality and let $\{B_1^{m+1}, B_2^{m+1}, \ldots, B_l^{m+1}\}$ be a subcover of $\star_{i=1}^m G$ in β^{m+1} of minimal cardinality. Then $\{A_i^{m+1} \cap B_j^{m+1} : 1 \le i \le k, 1 \le j \le l\}$

is a subcover of $\star_{i=1}^{m} G$ in $\alpha^{m+1} \vee \beta^{m+1}$, and $N(\star_{i=1}^{m} G, \alpha^{m+1} \vee \beta^{m+1}) \leq N(\star_{i=1}^{m} G, \alpha^{m+1}) N(\star_{i=1}^{m} G, \beta^{m+1})$. ■

Proposition 5.42 *If K is a closed subset of $G \subset I_0 \times I_1$, m is a positive integer, and $\alpha = \{A_1, \ldots, A_n\}$ is a minimal open cover of I_0 by open intervals, then $N(\star_{i=1}^{m} K, \alpha^{m+1}) \leq N(\star_{i=1}^{m} G, \alpha^{m+1})$.*

Proof. Suppose $\{A_1^{m+1}, A_2^{m+1}, \ldots, A_k^{m+1}\}$ is an open subcover of minimum cardinality of $\star_{i=1}^{m} G$ in α^{m+1}. Since $\star_{i=1}^{m} K \subset \star_{i=1}^{m} G$, $\{A_1^{m+1}, A_2^{m+1}, \ldots, A_k^{m+1}\}$ is also an open subcover of K. Hence, $N(\star_{i=1}^{m} K, \alpha^{m+1}) \leq N(\star_{i=1}^{m} G, \alpha^{m+1})$. ■

Proposition 5.43 *Suppose l and m are positive integers and $\alpha = \{A_1, \ldots, A_n\}$ is a minimal open cover of I_0 by open intervals,. Then α^{l+1} is a grid cover of $\star_{i=1}^{l} G$, α^{l+m+1} is a grid cover of $\star_{i=1}^{l+m} G$, and $\alpha^{l+1} \times \prod_{i=l+1}^{m+1} I_i$ is an open cover of $\star_{i=1}^{l+m} G$. Then $N(\star_{i=1}^{l} G, \alpha^{l+1}) \leq N(\star_{i=1}^{l+m} G, \alpha^{l+1} \times \prod_{i=l+1}^{l+m} I_i) \leq N(\star_{i=1}^{l+m} G, \alpha^{l+m+1})$.*

Proof. Suppose $\left\{A_j^{l+1} \times \prod_{i=l+1}^{l+m} I_i\right\}_{j=1}^{k}$ is a subcover of $\star_{i=1}^{l+m} G$ in $\alpha^{l+1} \times \prod_{i=l+1}^{l+m} I_i$ of least cardinality. Then $\{A_j^{l+1}\}_{j=1}^{k}$ is a subcover of $\star_{i=1}^{l} G$ in α^{l+1}. Hence, $N(\star_{i=1}^{l} G, \alpha^{l+1}) \leq N(\star_{i=1}^{l+m} G, \alpha^{l+1} \times \prod_{i=l+1}^{l+m} I_i)$. Since α^{l+m+1} refines $\alpha^{l+1} \times \prod_{i=l+1}^{l+m} I_i$, we have $N(\star_{i=1}^{l+m} G, \alpha^{l+1} \times \prod_{i=l+1}^{l+m} I_i) \leq N(\star_{i=1}^{l+m} G, \alpha^{l+m+1})$. The result follows. ■

Theorem 5.44 *If $\alpha = \{A_1, \ldots, A_n\}$ is a minimum open cover of I_0 by intervals, G is a closed subset of $I_0 \times I_1$ and $\mathbf{G} \neq \emptyset$, then*

$$\lim_{m \to \infty} \frac{\log N(\star_{i=1}^{m} G, \alpha^{m+1})}{m} = \lim_{m \to \infty} \frac{\log N(\star_{i=1}^{m} G, \star_{i=1}^{m} \alpha^{2})}{m}$$

exists.

Proof.

Let $a_m = \log N(\star_{i=1}^{m} G, \alpha^{m+1}) = \log N(\star_{i=1}^{m} G, \star_{i=1}^{m} \alpha^{2})$ for each $m \in \mathbb{N}$. Then $1 \leq N(\star_{i=1}^{m} G, \alpha^{m+1}) \leq n^{m+1}$, so

$$0 \leq a_m = \log N(\star_{i=1}^{m} G, \alpha^{m+1}) \leq (m+1) \log n.$$

By Lemma 5.20, it suffices to show that $a_{m+k} \leq a_m + a_k$. We have

$$\alpha^{m+k+1} \subset (\star_{i=1}^{m} \alpha^2) \star (\star_{i=m+1}^{m+k} \alpha^2),$$

because $(\star_{i=1}^{m} \alpha^2) \star (\star_{i=m+1}^{m+k} \alpha^2)$ contains more sets than does α^{m+k+1} since it contains sets of the form $(A_1 \times \cdots \times A_m \times A_{m+1}) \star (A'_{m+1} \times A_{m+2} \times \cdots \times A_{m+k+1}) = (A_1 \times \cdots \times A_{m-1} \times (A_{m+1} \cap A'_{m+1}) \times A_{m+2} \times \cdots \times A_{m+k+1})$, which is nonempty as long as $A_{m+1} \cap A'_{m+1} \neq \emptyset$. However, a minimal subcover of $\star_{i=1}^{m} G$ in α^{m+k+1} and in $(\star_{i=1}^{m} \alpha^2) \star (\star_{i=m+1}^{m+k} \alpha^2)$ has the same number so we have

$$N(\star_{i=1}^{m+k} G, \alpha^{m+k+1}) = N(\star_{i=1}^{m+k} G, (\star_{i=1}^{m} \alpha^2) \star (\star_{i=m+1}^{m+k} \alpha^2)).$$

Since $N(\star_{i=1}^{m} G, \alpha^{m+1})$ is the cardinality of a minimal subcover of $\star_{i=1}^{m} G$ in $\star_{i=1}^{m} \alpha^2$, and $N(\star_{i=1}^{k} G, \alpha^{k+1})$ is the cardinality of a minimal subcover of $\star_{i=1}^{k} G = \star_{i=m+1}^{m+k} G$ in $\star_{i=m+1}^{m+k} \alpha^2$, $(\star_{i=1}^{m} \alpha^2) \star (\star_{i=m+1}^{m+k} \alpha^2)$ is a cover of $\star_{i=1}^{m+k} G$ in $\prod_{i=0}^{m+k} I_i$. Thus,

$$N(\star_{i=1}^{m+k} G, \alpha^{m+k+1}) = N(\star_{i=1}^{m+k} G, (\star_{i=1}^{m} \alpha^2) \star (\star_{i=m+1}^{m+k} \alpha^2))$$
$$= N(\star_{i=1}^{m+k} G, (\star_{i=1}^{m} \alpha^2) \star (\star_{i=1}^{k} \alpha^2)) \leq N(\star_{i=1}^{m} G, \star_{i=1}^{m} \alpha^2) N(\star_{i=1}^{k} G, \star_{i=1}^{k} \alpha^2),$$

and we have

$$a_{m+k} = \log(N(\star_{i=1}^{m+k} G, \alpha^{m+k+1})) \leq \log(N(\star_{i=1}^{m} G, \star_{i=1}^{m} \alpha^2) N(\star_{i=1}^{k} G, \star_{i=1}^{k} \alpha^2)) =$$

$$\log(N(\star_{i=1}^{m} G, \star_{i=1}^{m} \alpha^2) + \log(N(\star_{i=1}^{k} G, \star_{i=1}^{k} \alpha^2)) = a_m + a_k.$$

■

Following the notation in Theorem 5.44, for $\mathbf{G} \neq \emptyset$, define $\text{ent}(G, \alpha)$ to be

$$\text{ent}(G, \alpha) = \lim_{m \to \infty} \frac{\log N(\star_{i=1}^{m} G, \alpha^{m+1})}{m}.$$

If $\mathbf{G} = \emptyset$, define $\text{ent}(G, \alpha) = 0$.

Remarks 5.45

1. $\text{ent}(G, \alpha) \geq 0$.

2. *If $\alpha < \beta$, α, β both minimal covers of I_0 by open intervals, then* $\mathrm{ent}(G, \alpha) \leq \mathrm{ent}(G, \beta)$.

Proof. The first statement is obvious. Let us prove the second one. For each positive integer m, $\alpha^m < \beta^m$. If $\left\{B_1^{m+1}, \ldots, B_k^{m+1}\right\}$ is a minimal subcover of $\star_{i=1}^m G$ in β^{m+1}, then for each $1 \leq i \leq k$, there is some $A_i^{m+1} \in \alpha^{m+1}$ such that $B_i^{m+1} \subset A_i^{m+1}$. Thus, $\left\{A_1^{m+1}, \ldots, A_k^{m+1}\right\}$ is a subcover of $\star_{i=1}^m G$ in α^{m+1}, and $\mathrm{ent}(G, \alpha) \leq \mathrm{ent}(G, \beta)$. ■

Corollary 5.46 *If K is a closed subset of $G \subset I_0 \times I_1$, m is a positive integer, and $\alpha = \{A_1, \ldots, A_n\}$ is a minimal open cover of I_0 by open intervals, then* $\mathrm{ent}(K, \alpha) \leq \mathrm{ent}(G, \alpha)$.

Proof. This follows directly from the fact that $N(\star_{i=1}^m K, \alpha^{m+1}) \leq N(\star_{i=1}^m G, \alpha^{m+1})$ for each positive integer m. ■

Finally, we can define **topological entropy** of G:

Definition 5.47 *We define* $\mathrm{ent}(G) = \sup_\alpha \{\mathrm{ent}(G, \alpha)\}$, *where α ranges over all minimal covers of I_0 by open intervals (in I_0).*

Theorem 5.48 *Let G be a closed subset of $I_0 \times I_1$ and $G^{-1} = \{(x, y) : (y, x) \in G\}$. Then* $\mathrm{ent}(G) = \mathrm{ent}(G^{-1})$.

Proof. For a positive integer m, note that a point $(x_0, x_1, \ldots, x_m) \in \star_{i=1}^m G$ if and only if $(x_m, x_{m-1}, \ldots, x_0) \in \star_{i=1}^m G^{-1}$. Suppose $\alpha = \{A_1, \ldots, A_n\}$ is a minimal cover of I_0 by open intervals. Suppose m is a positive integer. Then $A_{i_0} \times A_{i_1} \times \cdots \times A_{i_m} \in \alpha^{m+1}$ if and only if $A_{i_m} \times A_{i_{m-1}} \times \cdots \times A_{i_0} \in \alpha^{m+1}$, and $(A_{i_0} \times A_{i_1} \times \cdots \times A_{i_m}) \cap \left(\star_{j=1}^m G\right) \neq \emptyset$ if and only if $(A_{i_m} \times A_{i_{m-1}} \times \cdots \times A_{i_0}) \cap \left(\star_{j=1}^m G^{-1}\right) \neq \emptyset$. Then $N(\star_{i=1}^m G, \alpha^{m+1}) = N(\star_{i=1}^m G^{-1}, \alpha^{m+1})$ for each m. Hence, $\mathrm{ent}(G, \alpha) = \mathrm{ent}(G^{-1}, \alpha)$ for each cover α, and the result follows. ■

Proposition 5.49 *If β is an open cover (in I^∞) of $\star_{i=1}^\infty G$, $\sigma : I^\infty \to I^\infty$ is the shift map then $\sigma^{-1}(\beta) := \{\sigma^{-1}(B) : B \in \beta\} = \{I_0 \times B : B \in \beta\}$ is also an open cover (in I^∞) of $\star_{i=1}^\infty G$.*

Proof. Suppose $x = (x_0, x_1, \ldots) \in \star_{i=1}^{\infty} G$. Then $\sigma(x) = (x_1, x_2, \ldots) \in \star_{i=1}^{\infty} G$ by Proposition 5.35, and there is some $B \in \beta$ such that $\sigma(x) \in B$. Since $I_0 \times B = \sigma^{-1}(B)$, $x \in \sigma^{-1}(B) \in \sigma^{-1}(\beta)$. ■

If α is an open cover of I^{∞}, G is a closed subset of $I_0 \times I_1$, and $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$, let $\alpha^* = \{A \cap \mathbf{G} : A \in \alpha\}$ denote the corresponding open cover of $\mathbf{G}$ by open sets in $\mathbf{G}$.

Theorem 5.50 *Suppose G is a closed subset of $I_0 \times I_1$, $\mathbf{G} = \star_{i=1}^{\infty} G \neq \emptyset$, and $\sigma(\mathbf{G}) = \mathbf{G}$. If $\alpha = \{A_1, \ldots, A_n\}$ is a minimal open cover of I_0 by open intervals, then* $\operatorname{ent}(G, \alpha) = h(\sigma, (\alpha^{M+1} \times I^{\infty})^*)$ *for each positive integer M.*

Proof.

Let $\alpha = \{A_1, \ldots, A_n\}$ be a minimal open cover of I_0 by intervals.

Fix the positive integer M. Let

$$\beta = \left\{ \prod_{j=0}^{M} A_{i_j} \times I^{\infty} : A_{i_j} \in \alpha \text{ and } \left(\prod_{j=0}^{M} A_{i_j} \times I^{\infty} \right) \cap \mathbf{G} \neq \emptyset \right\},$$

and let

$$\gamma = \Big\{ A_{k_0} \times A_{k_1} \times \ldots \times A_{k_M} \times A_{k_{M+1}} \times I^{\infty} : (k_j)_{j=0}^{M+1} \text{ is a finite sequence of members of } \{1, \ldots, n\} \text{ of length } M+2 \Big\}.$$

For $B = \prod_{j=0}^{M} A_{i_j} \times I^{\infty} \in \beta$, $\sigma^{-1}(B) = I_0 \times \prod_{j=0}^{M} A_{i_j} \times I^{\infty}$. Then

$$\begin{aligned} \sigma^{-1}(\beta) \vee \beta &= \left\{ \sigma^{-1}(B_1) \cap B_2 : B_1 = \prod_{j=0}^{M} A_{i_j} \times I^{\infty}, B_2 = \prod_{j=0}^{M} A_{k_j} \times I^{\infty} \in \beta \right\} \\ &= \left\{ \left(I_0 \times \prod_{j=0}^{M} A_{i_j} \times I^{\infty} \right) \cap \left(\prod_{j=0}^{M} A_{k_j} \times I^{\infty} \right) : (i_j)_{j=0}^{M} \text{ and } (k_j)_{j=0}^{M} \right. \\ &\qquad \left. \text{are finite sequences of members of } \{1, \ldots, n\} \text{ of length } M+1 \right\}. \end{aligned}$$

If $B_1 = \prod_{j=0}^{M} A_{i_j} \times I^{\infty} \in \beta$ and $B_2 = \prod_{j=0}^{M} A_{k_j} \times I^{\infty} \in \beta$, then

$$\sigma^{-1}(B_1)\cap B_2 = \left(I_0\times\prod_{j=0}^{M}A_{i_j}\times I^\infty\right)\cap\left(\prod_{j=0}^{M}A_{k_j}\times I^\infty\right)$$
$$= A_{k_0}\times(A_{k_1}\cap A_{i_0})\times\ldots\times(A_{k_M}\cap A_{i_{M-1}})\times A_{i_M}\times I^\infty$$
$$\subset A_{k_0}\times A_{k_1}\times\ldots\times A_{k_M}\times A_{i_M}\times I^\infty.$$

Hence, the collection $\sigma^{-1}(\beta)\vee\beta$ refines the collection

$$\gamma = \big\{A_{k_0}\times A_{k_1}\times\ldots\times A_{k_M}\times A_{k_{M+1}}\times I^\infty : (k_j)_{j=0}^{M+1} \text{ is a finite sequence of members of } \{1,\ldots,n\} \text{ of length } M+2\,\big\}.$$

Then $(\sigma^{-1}(\beta)\vee\beta)^*$ refines the collection γ^*, so $\gamma^* < (\sigma^{-1}(\beta)\vee\beta)^*$, and $N(\mathbf{G},\gamma^*)\le N(\mathbf{G},(\sigma^{-1}(\beta)\vee\beta)^*)$.

But γ also refines $\sigma^{-1}(\beta)\vee\beta$, and so γ^* refines $(\sigma^{-1}(\beta)\vee\beta)^*$. Thus, $N(\mathbf{G},\gamma^*)\ge N(\mathbf{G},(\sigma^{-1}(\beta)\vee\beta)^*)$. Then $N(\mathbf{G},\gamma^*) = N(\mathbf{G},(\sigma^{-1}(\beta)\vee\beta)^*)$.

Note that $N(\mathbf{G},\gamma^*) = N(\star_{i=1}^{M+1}G,\alpha^{M+2})$.

We can continue: By similar arguments, for each positive integer l,

$$N(\mathbf{G},(\vee_{i=0}^{l}\sigma^{-i}\beta)^*) = N(\mathbf{G},\alpha^{M+l+1}\times I^\infty) = N(\star_{i=1}^{M+l+1}G,\alpha^{M+l+1}).$$

Now $\beta=\alpha^{M+1}\times I^\infty$, and for l a positive integer, $N(\mathbf{G},(\vee_{i=0}^{l}\sigma^{-i}\beta)^*) = N(\star_{i=1}^{M+l}G,\alpha^{M+l+1})$. Then $\log(N(\mathbf{G},(\vee_{i=0}^{l}\sigma^{-i}\beta)^*)) = \log(N(\star_{i=1}^{M+l}G,\alpha^{M+l+1}))$. It follows that

$$h(\sigma,\alpha^{M+1}\times I^\infty) = \lim_{l\to\infty}\frac{\log(N(\mathbf{G},(\vee_{i=0}^{l}\sigma^{-i}\beta)^*))}{l} = \lim_{l\to\infty}\frac{\log(N(\star_{i=1}^{M+l}G,\alpha^{M+l+1}))}{l},$$

while

$$\operatorname{ent}(G,\alpha) = \lim_{l\to\infty}\frac{\log(N(\star_{i=1}^{l}G,\alpha^{l+1}))}{l}.$$

For each positive integer k, let $\log(N(\star_{i=1}^{k}G,\alpha^{k+1})) = a_k$.

Then $\log(N(\star_{i=1}^{l}G,\alpha^{l+1})) = a_l$ and $\log(N(\star_{i=1}^{M+l}G,\alpha^{M+l+1})) = a_{M+l}$. Furthermore, $a_l \leq a_{M+l} \leq a_M + a_l$. (This is because $N(\star_{i=1}^{M+l}G,\alpha^{M+l+1}) \leq N(\star_{i=1}^{M}G,\alpha^{M+1})N(\star_{i=1}^{l}G,\alpha^{l+1})$.) Then $\frac{a_l}{l} \leq \frac{a_{l+M}}{l} \leq \frac{a_l}{l} + \frac{a_M}{l}$. By Lemma 5.21, $\lim_{l\to\infty}\frac{a_l}{l}$ exists, and

$$\lim_{l\to\infty}\frac{a_l}{l} \leq \lim_{l\to\infty}\frac{a_{l+M}}{l} \leq \lim_{l\to\infty}(\frac{a_l}{l}+\frac{a_M}{l}) = \lim_{l\to\infty}\frac{a_l}{l} + \lim_{l\to\infty}\frac{a_M}{l} = \lim_{l\to\infty}\frac{a_l}{l}.$$

It follows that

$$\lim_{l\to\infty}\frac{\log(N(\star_{i=1}^{l}G,\alpha^{l+1}))}{l} = \lim_{l\to\infty}\frac{\log(N(\star_{i=1}^{M+l}G,\alpha^{M+l+1}))}{l},$$

and thus, $\mathrm{ent}(G,\alpha) = h(\sigma,(\alpha^{M+1}\times I^{\infty})^*)$ for each positive integer M. ■

Theorem 5.51 *Suppose G is a closed subset of $I_0 \times I_1$, $\mathbf{G} = \star_{i=1}^{\infty}G$, and $\sigma(\mathbf{G}) = \mathbf{G}$. If $\alpha = \{A_1,\ldots,A_n\}$ is a minimal open cover of I_0 by open intervals, then* $\mathrm{ent}(G) = h(\sigma)$.

Proof. Since each open cover of $\mathbf{G}$ is refined by the grid cover $\alpha^{M+1}\times I^{\infty}$ for some minimal open cover by intervals α of I_0, the result follows. ■

Theorem 5.52 *If $f : I \to I$ is a continuous function and G is the graph of f^{-1}, i.e. $G = \{(y,x) : (x,y) \in \Gamma(f)\}$ then $h(f) =$* $\mathrm{ent}(G)$.

Proof. This follows from Theorem 5.50, and Ye's result that $h(f) = h(\sigma)$ in [42].

■

5.4 Topological entropy of closed subsets of $[0,1]^N$

Before we can investigate further we need to define and explore the topological entropy of closed subsets of $[0,1]^N$ for N a positive integer. So suppose N is a positive integer and H is a closed subset of $\prod_{i=0}^{N} I_i$. We can define the **topological entropy** of H as follows:

1. Let $\alpha = \{A_1, \dots, A_n\}$ be a minimal open cover of I_0 by intervals. Let

$$\beta = \left\{ \prod_{j=0}^{N} A_{k_j} : k_j \in \langle 1, n \rangle, 0 \le j \le N \right\}.$$

Hence, β is the grid cover of $\prod_{i=0}^{N} I_i$ determined by α, and β therefore covers H. Since $N^*(\beta) = n(N+1) := n_\beta$, we can list the members of $\beta = \{B_1, B_2, \dots, B_{n_\beta}\}$. For each positive integer $m > 1$, let

$$\beta^m = \left\{ \prod_{j=0}^{m-1} B_{k_j} : k_j \in \langle 1, n_\beta \rangle, 0 \le j \le m-1 \right\}.$$

2. Then $\beta \star \beta = \{B_i \star B_j : 1 \le i, j \le n_\beta\}$ is a cover of $H \star H$ by open subsets of $\prod_{i=0}^{2N} I_i$, and $N(H \star H, \beta \star \beta) \le n_\beta^2$. Note that $\beta \star \beta$ refines α^{2N+1} and $\beta \star \beta$ is refined by α^{2N+1}, so $N(H \star H, \beta \star \beta) = N(H \star H, \alpha^{2N+1})$.

3. We can continue this process for each $m \in \mathbb{N}$:

$$\star_{i=1}^{m} \beta = \left\{ \star_{j=1}^{m} \beta_{k_j} : k_j \in \langle 1, n_\beta \rangle, 1 \le j \le m \right\}$$

is an open cover of $\star_{i=1}^{m} H$ and $N(\star_{i=1}^{m} H, \star_{i=1}^{m} \beta) \le n_\beta^m$. Again, a minimal subcover of $\star_{i=1}^{m} H$ by elements of $\star_{i=1}^{m} \beta$ has the same number of elements as a minimal subcover of $\star_{i=1}^{m} H$ by elements of α^{mN+1}. Since using the cover $\star_{i=1}^{m} \beta$ is sometimes more convenient, we continue to use both covers. Without loss of generality, we may assume that a minimal subcover (in both α^{mN+1} and $\star_{i=1}^{m} \beta$) consists of sets of the form $\prod_{j=0}^{mN} A_{k_j}$, where each $k_j \in \langle 1, n \rangle$.

4. **Remarks** Using the notation from before, suppose H is a closed subset of $\prod_{i=0}^{N} I_i$. Let $\mathbf{H} = \star_{i=1}^{\infty} H$.

   - For each positive integer m, $0 \le N(\star_{i=1}^{m} H, \alpha^{mN+1}) = N(\star_{i=1}^{m} H, \star_{i=1}^{m} \beta) \le n^{mN+1}$, and $0 \le N(\star_{i=1}^{m} H, \alpha^{mN+1}) = N(\star_{i=1}^{m} H, \star_{i=1}^{m} \beta) \le n_\beta^m$. If $\mathbf{H} \neq \emptyset$, $0 < N(\star_{i=1}^{m} H, \star_{i=1}^{m} \beta)$.

- If $\mathbf{H} \neq \emptyset$, $1 = N(\star_{i=1}^m H, \star_{i=1}^m \beta)$ if and only if there is a sequence $A_{j_0}, A_{j_1}, \ldots A_{j_{mN}}$ (with each $1 \leq j_i \leq n$) such that $\mathbf{H} \subset (A_{j_0} \times \ldots \times A_{j_{mN}}) \times I^\infty$.
- As before, if α, γ are both minimal open covers of I_0 by open intervals and $\alpha < \gamma$, then for each $m > 0$, $N(\star_{i=1}^m H, \alpha^{mN+1}) \leq N(\star_{i=1}^m H, \gamma^{mN+1})$.
- As before, if α, γ are both minimal open covers of I_0 by open intervals, then for each $m > 0$, $\alpha^{mN+1} \vee \gamma^{mN+1} = (\alpha \vee \gamma)^{mN+1}$, and $N(\star_{i=1}^m H, \alpha^{mN+1} \vee \gamma^{mN+1}) \leq N(\star_{i=1}^m H, \alpha^{mN+1}) N(\star_{i=1}^m H, \gamma^{mN+1})$.
- If K is a closed subset of $H \subset \prod_{i=0}^N I_i$, m is a positive integer, and $\alpha = \{A_1, \ldots, A_n\}$ is a minimal open cover of I_0 by open intervals, then $N(\star_{i=1}^m K, \alpha^{mN+1}) \leq (\star_{i=1}^m H, \alpha^{mN+1})$.
- Suppose l and m are positive integers. Then α^{lN+1} is a grid cover of $\star_{i=1}^l H$, $\alpha^{lN+mN+1}$ is a grid cover of $\star_{i=1}^{l+m} H$, and $\alpha^{lN+1} \times \prod_{i=lN+1}^{mN+1} I_i$ is an open cover of $\star_{i=1}^{l+m} G$. Then $N(\star_{i=1}^l H, \alpha^{lN+1}) \leq N(\star_{i=1}^{l+m} H, \alpha^{lN+1} \times \prod_{i=lN+1}^{lN+mN} I_i) \leq N(\star_{i=1}^{l+m} H, \alpha^{lN+mN+1})$.

5. If $\alpha = \{A_1, \ldots, A_n\}$ is a minimal open cover of I_0 by intervals, H is a closed subset of $\prod_{i=0}^N I_i$ and $\mathbf{H} \neq \emptyset$, then

$$\lim_{m \to \infty} \frac{\log N(\star_{i=1}^m H, \alpha^{mN+1})}{m} = \lim_{m \to \infty} \frac{\log N(\star_{i=1}^m H, \star_{i=1}^m \beta)}{m}$$

exists.

Proof.

Let $a_m = \log N(\star_{i=1}^m H, \alpha^{mN+1}) = \log N(\star_{i=1}^m H, \star_{i=1}^m \beta)$ for each $m \in \mathbb{N}$. Then $1 \leq N(\star_{i=1}^m H, \alpha^{mN+1}) \leq n^{mN+1}$, so

$$0 \leq a_m = \log N(\star_{i=1}^m H, \alpha^{mN+1}) \leq (mN+1) \log n.$$

By Lemma 5.20, it suffices to show that $a_{m+k} \leq a_m + a_k$. We have

$$\alpha^{m+k+1} \subset (\star_{i=1}^m \beta) \star (\star_{i=1}^k \beta) = \star_{i=1}^{m+k} \beta,$$

and also

$$\alpha^{m+k+1} \text{ refines } (\star_{i=1}^{m}\beta) \star (\star_{i=1}^{k}\beta) = \star_{i=1}^{m+k}\beta.$$

Then

$$N(\star_{i=1}^{m+k}H, \alpha^{mN+kN+1}) = N(\star_{i=1}^{m+k}H, (\star_{i=1}^{m}\beta) \star (\star_{i=1}^{k}\beta)).$$

Since $N(\star_{i=1}^{m}H, \alpha^{mN+1})$ is the cardinality of a minimal subcover of $\star_{i=1}^{m}H$ in $\star_{i=1}^{m}\beta$, and $N(\star_{i=1}^{k}H, \alpha^{kN+1})$ is the cardinality of a minimal subcover of $\star_{i=1}^{k}H$ in $\star_{i=1}^{k}\beta$, $(\star_{i=1}^{m}\beta)\star(\star_{i=m+1}^{m+k}\beta)$ is a cover of $\star_{i=1}^{m+k}H$ in $\prod_{i=0}^{mN+kN} I_i$. Thus,

$$N(\star_{i=1}^{m+k}H, \alpha^{mN+kN+1}) = N(\star_{i=1}^{m+k}H, (\star_{i=1}^{m}\beta) \star (\star_{i=1}^{k}\beta))$$

$$= N(\star_{i=1}^{m+k}H, (\star_{i=1}^{m}\beta) \star (\star_{i=1}^{k}\beta)) \leq N(\star_{i=1}^{m}H, \star_{i=1}^{m}\beta)N(\star_{i=1}^{k}H, \star_{i=1}^{k}\beta),$$

and we have

$$a_{m+k} = \log(N(\star_{i=1}^{m+k}H, \alpha^{mN+kN+1})) \leq \log(N(\star_{i=1}^{m}H, \star_{i=1}^{m}\beta)N(\star_{i=1}^{k}H, \star_{i=1}^{k}\beta)) =$$

$$\log(N(\star_{i=1}^{m}H, \star_{i=1}^{m}\beta) + \log(N(\star_{i=1}^{k}H, \star_{i=1}^{k}\beta)) = a_m + a_k.$$

■

6. If $\mathbf{H} \neq \emptyset$ and α is minimal open cover of I_0, define $\text{ent}(H, \alpha)$ to be

$$\text{ent}(H, \alpha) = \lim_{m\to\infty} \frac{\log N(\star_{i=1}^{m}H, \alpha^{mN+1})}{m}.$$

If $\mathbf{H} = \emptyset$, define $\text{ent}(H, \alpha) = 0$.

7. **Remarks**

   (a) $\text{ent}(H, \alpha) \geq 0$.

   (b) If $\alpha < \beta$, α, β both minimal covers of I_0 by open intervals, then $\text{ent}(H, \alpha) \leq \text{ent}(H, \beta)$.

(c) If K is a closed subset of $H \subset \prod_{i=0}^{N} I_i$, m is a positive integer, and $\alpha = \{A_1, \ldots, A_n\}$ is a minimal open cover of I_0 by open intervals, then $\text{ent}(K, \alpha) \leq \text{ent}(H, \alpha)$.

8. Finally, we define $\text{ent}(H) = \sup_{\alpha} \{\text{ent}(H, \alpha)\}$, where α ranges over all minimal covers of I_0 by open intervals (in I_0).

Theorem 5.53 *Let H be a closed subset of $\prod_{i=0}^{N} I_i$ and*

$$H^{-1} = \{(x_N, x_{N-1}, \ldots, x_1, x_0) : (x_0, x_1, \ldots, x_{N-1}, x_N) \in H\}.$$

Then $\text{ent}(H) = \text{ent}(H^{-1})$.

Proof. Let $\alpha = \{A_1, \ldots, A_n\}$ be a minimal open cover of I_0 by open intervals and m a positive integer. First note that $(x_0, x_1, \ldots, x_{mN+1}) \in \star_{i=1}^{m} H$ if and only if $(x_{mN+1}, x_{mN}, \ldots, x_1, x_0) \in \star_{i=1}^{m} H^{-1}$. Now, let β be defined as at the beginning of this section. We have that $B_{i_0} \times B_{i_1} \times \cdots B_{i_m} \in \beta^{m+1}$ if and only if $B_{i_m} \times B_{i_{m-1}} \times \cdots B_{i_0} \in \beta^{m+1}$ and $B_{i_0} \times B_{i_1} \times \cdots B_{i_m} \cap \star_{i=1}^{m} H \neq \emptyset$ if and only if $B_{i_m} \times B_{i_{m-1}} \times \cdots B_{i_0} \cap \star_{i=1}^{m} H^{-1} \neq \emptyset$. Therefore, $N(\star_{i=1}^{m} H, \alpha^{mN+1}) = N(\star_{i=1}^{m} H, \beta^{m+1}) = N(\star_{i=1}^{m} H^{-1}, \beta^{m+1}) = N(\star_{i=1}^{m} H^{-1}, \alpha^{mN+1})$. Hence, $\text{ent}(H, \alpha) = \text{ent}(H^{-1}, \alpha)$ for each α, and the result follows. ■

Proposition 5.54 *Let H be a closed subset of $\prod_{i=0}^{N} I_i$, β is an open cover (in I^∞) of $\star_{i=1}^{\infty} H$, $\sigma : I^\infty \to I^\infty$ is a shift map, then $\sigma^{-N}(\beta) := \left\{\sigma^{-N}(B) : B \in \beta\right\} = \left\{\prod_{i=0}^{N-1} I_i \times B : B \in \beta\right\}$ is also an open cover (in I^∞) of $\star_{i=1}^{\infty} H$.*

Proof. Suppose $x = (x_0, x_1, \ldots) \in \star_{i=1}^{\infty}$. Then, by Proposition 5.35 it follows $\sigma^N(x) = (x_N, x_{N+1}, \ldots) \in \star_{i=1}^{\infty} H$ and there is some $B \in \beta$ such that $\sigma^N(x) \in B$. Since $\prod_{i=0}^{N-1} I_i \times B = \sigma^{-N}(B), x \in \sigma^{-N}(B) \in \sigma^{-N}(\beta)$. ■

Suppose H is a closed subset of $\prod_{i=0}^{N} I_i$ and $\mathbf{H} = \star_{i=1}^{\infty} H$. If α is an open cover of I^∞, let $\alpha^* = \{A \cap \mathbf{H} : A \in \alpha\}$ denote the corresponding open cover of $\mathbf{H}$ by open sets in $\mathbf{H}$.

Theorem 5.55 *Suppose H is a closed subset of $\prod_{i=0}^{N} I_i$, $\mathbf{H} = \star_{i=1}^{\infty} H \neq \emptyset$, and $\sigma^N(\mathbf{H}) = \mathbf{H}$. Suppose M is a positive integer. If $\alpha = \{A_1, \ldots, A_n\}$ is a* *minimal open cover of* I_0 *by open intervals, then* $\operatorname{ent}(H,\alpha) = h(\sigma^N, (\alpha^{MN+1} \times I^\infty)^*)$.

Proof. This proof is similar to the proof of Theorem 5.50, but a little more difficult technically so we present it.

Let $\alpha = \{A_1, \ldots, A_n\}$ be a minimal open cover of I_0 by intervals.

Fix the positive integer M. Let

$$\beta = \left\{ \prod_{j=0}^{MN} A_{i_j} \times I^\infty : A_{i_j} \in \alpha \text{ and } \left(\prod_{j=0}^{MN} A_{i_j} \times I^\infty \right) \cap \mathbf{H} \neq \emptyset \right\},$$

and let

$$\gamma = \Big\{ A_{k_0} \times A_{k_1} \times \ldots \times A_{k_{(M+1)N}} \times I^\infty : (k_j)_{j=0}^{(M+1)N} \text{ is a finite sequence of members of } \{1, \ldots, n\} \text{ of length } (M+1)N+1 \Big\}.$$

For $B = \prod_{j=0}^{MN} A_{i_j} \times I^\infty \in \beta$, $\sigma^{-N}(B) = \prod_{i=0}^{N-1} I_i \times \prod_{j=0}^{MN} A_{i_j} \times I^\infty$. Then

$$\begin{aligned}\sigma^{-N}(\beta) \vee \beta &= \left\{ \sigma^{-N}(B_1) \cap B_2 : B_1 = \prod_{j=0}^{MN} A_{i_j} \times I^\infty, B_2 = \prod_{j=0}^{MN} A_{k_j} \times I^\infty \in \beta \right\} \\ &= \Bigg\{ \left(\prod_{i=0}^{N-1} I_i \times \prod_{j=0}^{MN} A_{i_j} \times I^\infty \right) \cap \left(\prod_{j=0}^{MN} A_{k_j} \times I^\infty \right) : (i_j)_{j=0}^{MN} \text{ and } \\ &\qquad (k_j)_{j=0}^{MN} \text{ are finite sequences of members of } \{1, \ldots, n\} \text{ of length } (M+1)N+1 \Bigg\}.\end{aligned}$$

If $B_1 = \prod_{j=0}^{MN} A_{i_j} \times I^\infty \in \beta$ and $B_2 = \prod_{j=0}^{MN} A_{k_j} \times I^\infty \in \beta$, then

$$\sigma^{-N}(B_1) \cap B_2 = \left(\prod_{i=0}^{N-1} I_i \times \prod_{j=0}^{MN} A_{i_j} \times I^\infty \right) \cap \left(\prod_{j=0}^{MN} A_{k_j} \times I^\infty \right)$$
$$= \prod_{i=0}^{N-1} A_{k_i} \times (A_{k_N} \cap A_{i_0}) \times \ldots \times (A_{k_{MN}} \cap A_{i_{(M-1)N}}) \times \prod_{l=(M-1)+1}^{MN} A_{i_l} \times I^\infty$$
$$\subset \ A_{k_0} \times A_{k_1} \times \ldots \times A_{k_{MN}} \times A_{i_{(M-1)N+1}} \times \ldots \times A_{i_{MN}} \times I^\infty.$$

Hence, the collection $\sigma^{-1}(\beta) \vee \beta$ refines the collection

$$\gamma = \Big\{ A_{k_0} \times A_{k_1} \times \ldots \times A_{k_{(M+1)N}} \times I^\infty : (k_j)_{j=0}^{(M+1)N} \text{ is a finite sequence of members of } \{1, \ldots, n\} \text{ of length } (M+1)N+1 \Big\}.$$

Then $(\sigma^{-N}(\beta) \vee \beta)^*$ refines the collection γ^*, so $\gamma^* < (\sigma^{-N}(\beta) \vee \beta)^*$, and $N(\mathbf{H}, \gamma^*) \leq N(\mathbf{H}, (\sigma^{-N}(\beta) \vee \beta)^*)$.

But γ also refines $\sigma^{-N}(\beta) \vee \beta$, and so γ^* refines $(\sigma^{-N}(\beta) \vee \beta)^*$. Thus, $N(\mathbf{H}, \gamma^*) \geq N(\mathbf{H}, (\sigma^{-N}(\beta) \vee \beta)^*)$. Then $N(\mathbf{H}, \gamma^*) = N(\mathbf{H}, (\sigma^{-N}(\beta) \vee \beta)^*)$.

Note that $N(\mathbf{H}, \gamma^*) = N(\star_{i=1}^{(M+1)N} H, \alpha^{MN+1})$.

We can continue: By similar arguments, for each positive integer l, $N(\mathbf{H}, (\vee_{i=0}^{l} \sigma^{-iN}(\beta)^*)) = N(\mathbf{H}, \alpha^{(M+l)N+1} \times I^\infty) = N(\star_{i=1}^{(M+l)N} H, \alpha^{(M+l)N+1})$. Now $\beta = \alpha^{MN+1} \times I^\infty$, and for l a positive integer, $N(\mathbf{H}, (\vee_{i=0}^{l} \sigma^{-iN}(\beta)^*)) = N(\star_{i=1}^{M+l} G, \alpha^{(M+l)N+1})$. Then $\log(N(\mathbf{G}, (\vee_{i=0}^{l} \sigma^{-iN}(\beta)^*)) =$ $\log(N(\star_{i=1}^{(M+l)N} H, \alpha^{(M+l)N+1}))$. It follows that

$$h(\sigma^N, \alpha^{MN+1} \times I^\infty) = \lim_{l \to \infty} \frac{\log(N(\mathbf{H}, (\vee_{i=0}^{l} \sigma^{-iN}(\beta)^*))}{l}$$
$$= \lim_{l \to \infty} \frac{\log(N(\star_{i=1}^{(M+l)N} H, \alpha^{(M+l)N+1}))}{l},$$

while

$$\mathrm{ent}(H, \alpha) = \lim_{l \to \infty} \frac{\log(N(\star_{i=1}^{l} H, \alpha^{lN+1}))}{l}.$$

For each positive integer k, let $\log(N(\star_{i=1}^{k}H, \alpha^{kN+1})) = a_k$. Then $\log(N(\star_{i=1}^{l}H, \alpha^{lN+1})) = a_l$ and $\log(N(\star_{i=1}^{M+l}H, \alpha^{(M+l)N+1})) = a_{M+l}$. Furthermore, $a_l \leq a_{M+l} \leq a_M + a_l$. Then $\frac{a_l}{l} \leq \frac{a_{l+M}}{l} \leq \frac{a_l}{l} + \frac{a_M}{l}$. By Lemma 5.21, $\lim_{l\to\infty} \frac{a_l}{l}$ exists, and

$$\lim_{l\to\infty} \frac{a_l}{l} \leq \lim_{l\to\infty} \frac{a_{l+M}}{l} \leq \lim_{l\to\infty} (\frac{a_l}{l} + \frac{a_M}{l}) = \lim_{l\to\infty} \frac{a_l}{l} + \lim_{l\to\infty} \frac{a_M}{l} = \lim_{l\to\infty} \frac{a_l}{l}.$$

It follows that

$$\lim_{l\to\infty} \frac{\log(N(\star_{i=1}^{l}H, \alpha^{lN+1}))}{l} = \lim_{l\to\infty} \frac{\log(N(\star_{i=1}^{M+l}H, \alpha^{(M+l)N+1}))}{l},$$

and thus, $\text{ent}(G, \alpha) = h(\sigma^N, (\alpha^{MN+1} \times I^\infty)^*)$. ■

Theorem 5.56 *Suppose H is a closed subset of $\prod_{i=0}^{N} I_i$, $\mathbf{H} = \star_{i=1}^{\infty}H$, and $\sigma^N(\mathbf{H}) = \mathbf{H}$. If $\alpha = \{A_1, \ldots, A_n\}$ is a minimal open cover of I_0 by open intervals, then* $\text{ent}(H) = h(\sigma^N)$.

Proof. Since each open cover of $\mathbf{H}$ is refined by the grid cover $\alpha^{MN+1} \times I^\infty$ for some minimal open cover by intervals α of I_0, the result follows. ■

Theorem 5.57 *Suppose G is a closed subset of $I_0 \times I_1$, $\mathbf{G} = \star_{i=1}^{\infty}G \neq \emptyset$, and $\sigma(\mathbf{G}) = \mathbf{G}$. Then* $\text{ent}(\star_{i=1}^{k}G) = k\ \text{ent}(G)$, *for each integer $k \geq 2$.*

Proof. Supppose k is a positive integer. Let G be a closed subset of $I_0 \times I_1$ such that $\mathbf{G} = \star_{i=1}^{\infty}G \neq \emptyset$, and $\sigma(\mathbf{G}) = \mathbf{G}$, and let $H = \star_{i=1}^{k}G \subset \prod_{i=0}^{k} I_i$. Let $\alpha = \{A_1, \ldots, A_n\}$ be a minimal open cover of I_0 by open intervals. Then for each positive integer m, $\star_{i=1}^{m}H = \star_{i=1}^{m}(\star_{i=1}^{k}G) = \star_{i=1}^{mk}G$. Hence, $N(\star_{i=1}^{mk}G, \alpha^{mk+1}) = N(\star_{i=1}^{m}H, \alpha^{mk+1})$. Then

$$\begin{aligned}\text{ent}(G, \alpha) &= \lim_{m\to\infty} \frac{1}{m} \log N(\star_{i+1}^{m}G, \alpha^{m+1}) = \lim_{m\to\infty} \frac{1}{mk} \log N(\star_{i+1}^{mk}G, \alpha^{mk+1}) \\ &= \frac{1}{k} \lim_{m\to\infty} \frac{1}{m} \log N(\star_{i+1}^{mk}G, \alpha^{mk+1}) = \frac{1}{k} \text{ent}(\star_{i=1}^{mk}G, \alpha^{mk+1}).\end{aligned}$$

Thus, for every minimal cover α by open intervals of I_0,

$$k\ \text{ent}(G, \alpha) = \text{ent}(\star_{i=1}^{mk}G, \alpha^{mk+1}).$$

The result follows. ■

5.5 Computation and application of topological entropy

In this section we compute the topological entropy for some closed subsets G of I^2. We also suggest an application.

Example 5.58 *Suppose* $G = I^2$. *Then* $\text{ent}(G) = \infty$.

Proof. Suppose $\alpha = \{A_1, \ldots, A_n\}$ is a minimal open cover of I_0 by open intervals. Then for each positive integer m, $N(\star_{i=1}^m G, \alpha^{m+1}) = n^{m+1}$. Thus,

$$\begin{aligned}\text{ent}(G, \alpha^2) &= \lim_{m\to\infty} \frac{1}{m} \log N(\star_{i+1}^m G, \alpha^{m+1}) = \lim_{m\to\infty} \frac{1}{m} \log n^{m+1} \\ &= \lim_{m\to\infty} \frac{m+1}{m} \log n = \log n.\end{aligned}$$

Then

$$\sup_\alpha \text{ent}(G, \alpha^2) = \sup_\alpha \log n = \infty.$$

■

Example 5.59 *Let* $G = \{(x, x)\, |x \in I\}$. *Then* $\text{ent}\,(G) = 0$.

Proof. G is graph of the identity map f^{-1} and the inverse f is a homeomorphism, hence $\text{ent}\,(G) = \text{ent}\,(f) = 0$. ■

Example 5.60 *Let G denote the union of the diagonal from* $(0, 0)$ *to* $(1, 1)$ *and one point* (x, y) *where* (x, y) *is an arbitrary point in* $I \times I$ *such that* $x \neq y$. *Then,* $\text{ent}\,(G) = 0$.

Proof. Set G is the union of the diagonal and one point, $G \star G$ is the union of the diagonal from $(0, 0, 0)$ to $(1, 1, 1)$ and two points, $\star_{i=1}^m G$ is union of diagonal and m different points. Therefore, we have that

$$N\left(G, \alpha^2\right) \leq n + 1, N\left(G \star G, \alpha^3\right) \leq n + 2, \ldots, N\left(\star_{i=1}^m G, \alpha^{m+1}\right) \leq n + m.$$

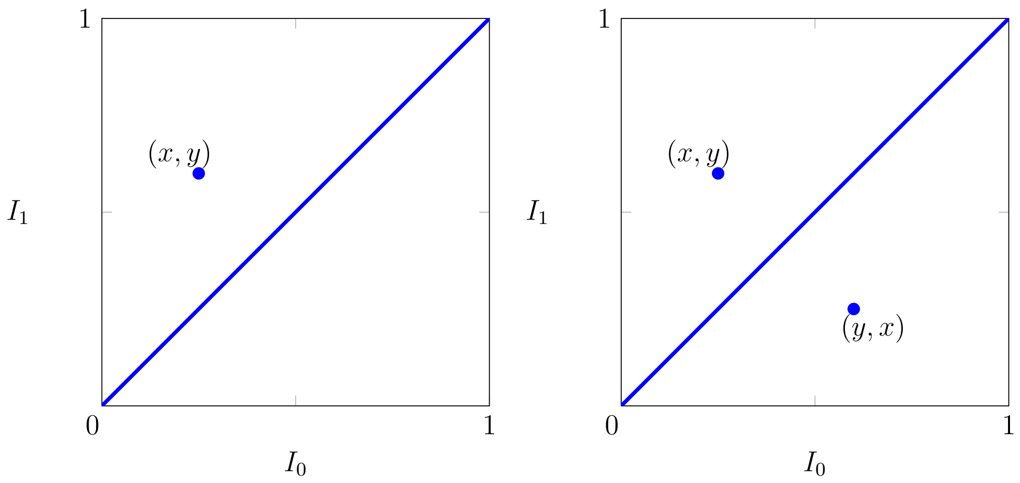


Figure 5.1: Set G from the Example 5.60 (left) and Example 5.61 (right)

Hence,

$$\operatorname{ent}(G,\alpha) = \lim_{m\to\infty} \frac{1}{m} \log N\left(\star_{i=1}^{m} G, \alpha^{m+1}\right) \leq \lim_{m\to\infty} \frac{1}{m} \log(n+m) = 0.$$

So, $\operatorname{ent}(G,\alpha) = 0$ and then $\operatorname{ent}(G) = 0$. ■

If we add finitely many points (x,y) to the G from the last example, such that $x \neq y$, and there is no pair of symmetric points (relative to the diagonal), we get zero entropy in the same way.

Example 5.61 *Let G denote the union of the diagonal from $(0,0)$ to $(0,1)$ and two points (x,y) and (y,x), where (x,y) is an arbitrary point in $I \times I$ such that $x \neq y$. Then,* $\operatorname{ent}(G) = \log 2$.

Proof. Set G is the union of the diagonal and two points, $G \star G$ is the union of the diagonal from $(0,0,0)$ to $(1,1,1)$ and six points, $\star_{i=1}^{m} G$ is union of diagonal and $2^{m+1}-2$ different points that don't lie on the diagonal. Namely, points in $\star_{i=1}^{m} G$ have $m+1$ coordinates and every coordinate is either x or y so there are $2^{m+1}-2$ different points which don't lie on diagonal. So, we have

$$N\left(G,\alpha^2\right) \leq n+2, N\left(G \star G, \alpha^3\right) \leq n+6, \ldots, N\left(\star_{i=1}^{m} G, \alpha^{m+1}\right) \leq n+2^{m+1}-2.$$

Hence, it is easy to see

$$\text{ent}\,(G,\alpha) = \lim_{m\to\infty} \frac{1}{m} \log\left(n + 2^{m+1} - 2\right) = \log 2.$$

Therefore, ent $(G) = \log 2$. ■

Theorem 5.62 *For each $n \in \mathbb{N}$ there exists a closed set $G \subseteq I \times I$ such that* ent $(G) = \log n$.

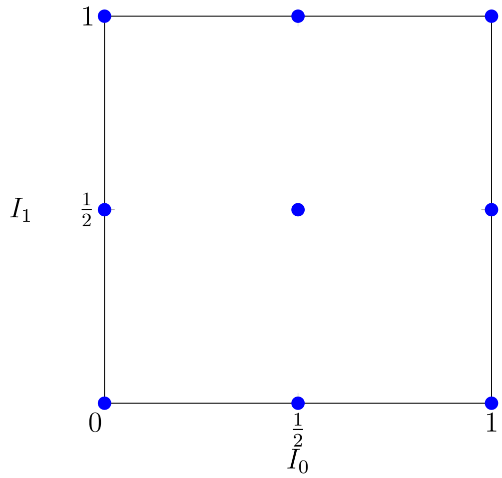


Figure 5.2: Set G from the Example 5.62 for $n = 3$

Proof. Let $n \in \mathbb{N}$ be arbitrary. We define G in following way:

$$G = \left\{ \left(\frac{k}{n-1}, \frac{l}{n-1} \right) : k, l \in \{0, 1, \ldots, n-1\} \right\}.$$

Set G is union of n^2 points, $G \star G$ is union of n^3, $\star_{i=1}^{m} G$ is union of n^{m+1} different points so we have the following:

$$N\left(G, \alpha^2\right) \le n^2, N\left(G \star G, \alpha^3\right) \le n^3, \ldots, N\left(\star_{i=1}^{m} G, \alpha^{m+1}\right) \le n^{m+1}.$$

We have

$$\text{ent}\,(G,\alpha) = \lim_{m\to\infty} \frac{1}{m} \log\left(n^{m+1}\right) = \log n.$$

Therefore, ent $(G) = \log n$ ■

For the next example, G^{-1} is Example 2.14 in [22].

Example 5.63 *Let* $G = \{(x, x\} : x \in I\} \cup (\{1\} \times I)$ *. Then* $\mathrm{ent}\,(G) = 0$.

Proof. Let us denote the following: $L_1 = \{(x, x)\,|x \in I\}$ and $L_2 = \{1\} \times I$. For arbitrary $m \in \mathbb{N}$ we observe $\star_{i=1}^{m} L_i, i \in \{1, 2\}$. We have $m + 1$ combinations. Namely, if $i = 2$ on $j - th$ position (for all preceeding position we put $i = 1$), so $(x_j, x_{j+1}) \in L_2$, on all coordinates of the point x before j we must have 1. Therefore, the product is determined by the last appearance of the L_2 (that is,others don't matter), so we have $m + 1$ possibilities:

$$\star_{i=1}^{m} L_1, \left(\star_{i=1}^{m-1} L_1\right) \star L_2, \ldots, L_2 \star \left(\star_{i=1}^{m-1} L_1\right).$$

All $m + 1$ sets are arcs in I^{m+1} intersecting in point $(1, 1, \ldots, 1) \in I^{m+1}$. Therefore, we have

$$N\left(\star_{i=1}^{m} G, \alpha^{m+1}\right) = n + m\,(n - 1)$$

and

$$\mathrm{ent}\,(G, \alpha) = \lim_{m\to\infty} \frac{1}{m} \log\left(n + m\,(n - 1)\right) \le \lim_{m\to\infty} \frac{1}{m} \log\left(n\,(m + 1)\right) = 0.$$

Hence, $\mathrm{ent}\,(G) = 0$. ■

Example 5.64 *Let* $G = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{1\})$ *. Then,* $\mathrm{ent}\,(G) = 0$.

Proof. Let us denote the following: $L_1 = \{0\} \times [0, 1]$ and $L_2 = [0, 1] \times \{1\}$. For arbitrary $m \in \mathbb{N}$ we observe $\star_{i=1}^{m} L_i, i \in \{1, 2\}$. First, let $i = 2, \forall i \in \{1, \ldots, m\}$. All points in L_1 have first coordinate 0 so we have $x \in \star_{i=1}^{m} L_1$ if and only if $x_1 = x_2 = \ldots x_{m-1} = 0$. Otherwise, let i_0 be smallest integer such that $i_0 = 2$. Since all points in L_2 have second coordinate equal 1, after L_2 in product we cannot have L_1. Therefore, given product is determined with first appearance of L_1. Hence, we have $m + 1$. Each product is an arc in I^{m+1} and they form arc from $(0, 0, \ldots, 0)$ to $(1, 1 \ldots, 1)$ in I^{m+1}, in a way that end point of $\star_{i=1}^{m} L_1$ is origin point of $\left(\star_{i=1}^{m-1} L_1\right) \star L_2$ and so on. Therefore, we have

$$N\left(\star_{i=1}^{m} G, \alpha^{m+1}\right) \le (m + 1)\,n$$

and

$$\text{ent}\,(G,\alpha) \leq \lim_{m\to\infty} \frac{1}{m} \log\left(n\left(m+1\right)\right) = 0.$$

Hence, ent $(G) = 0$. ■

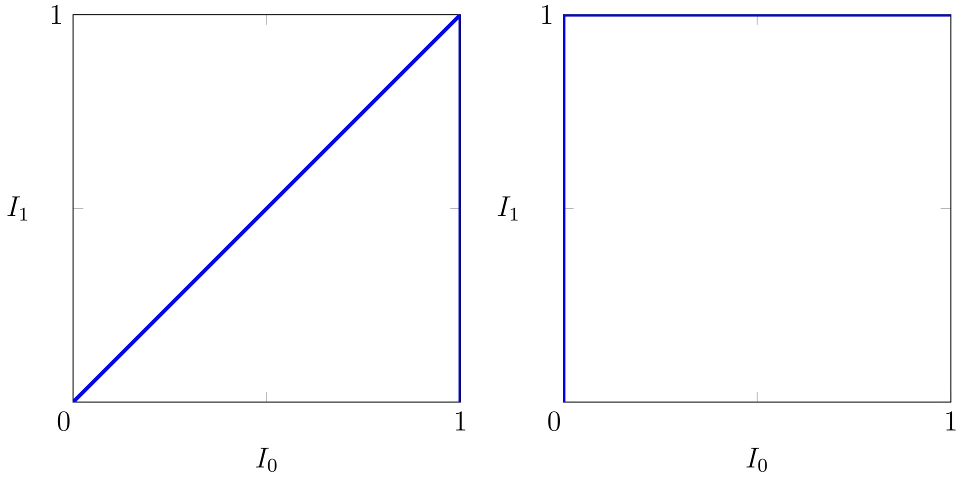


Figure 5.3: Set G from the Example 5.63 (left) and Example 5.64 (right)

Example 5.65 *Let* $G = \{0\} \times [0,1] \cup [0,1] \times \{0\}$ *. Then* ent $(G) = \infty$.

Proof. Let us denote the following: $L_1 = \{0\} \times [0,1]$ and $L_2 = [0,1] \times \{0\}$. For arbitrary $m \in \mathbb{N}, \star_{i=1}^{m} G$ contains $L_1 \star L_2 \star \ldots L_1 \star L_2 = [0,1] \times \{0\} \times \ldots \times \{0\} \times [0,1]$ if m is even and $L_1 \star L_2 \star \ldots L_2 \star L_1 = [0,1] \times \{0\} \times \ldots \times [0,1] \times \{0\}$ if m is odd. Either way, $N\left(\star_{i=1}^{m} G, \alpha^{m+1}\right) > n^{\frac{m}{2}}$. Therefore,

$$\text{ent}\,(G,\alpha) \geq \lim_{m\to\infty} \frac{1}{m} \log\left(n^{\frac{m}{2}}\right) = \log n.$$

and it follows ent $(G) = \infty$. ■

Example 5.66 *Let* $G = I \times \{0\} \cup \{(x, 1-x) : x \in I\}$ *. Then* ent$(G) = \frac{1+\sqrt{5}}{2}$ *i.e. "golden ratio."*

Proof. Let us denote with L_1 line from $(1,0)$ to $(0,0)$ and with L_2 line from $(1,0)$ to $(0,1)$. We have $G = L_1 \cup L_2$. In arbitrary product $\star_{j=1}^{m} L_{i_j}, i_j \in \{1,2\}$, let i_0 be first index such that $L_{i_0} = L_1$. Next coordinate has to be 0 and after that we have only zeros and ones such that we cannot have two neighboring

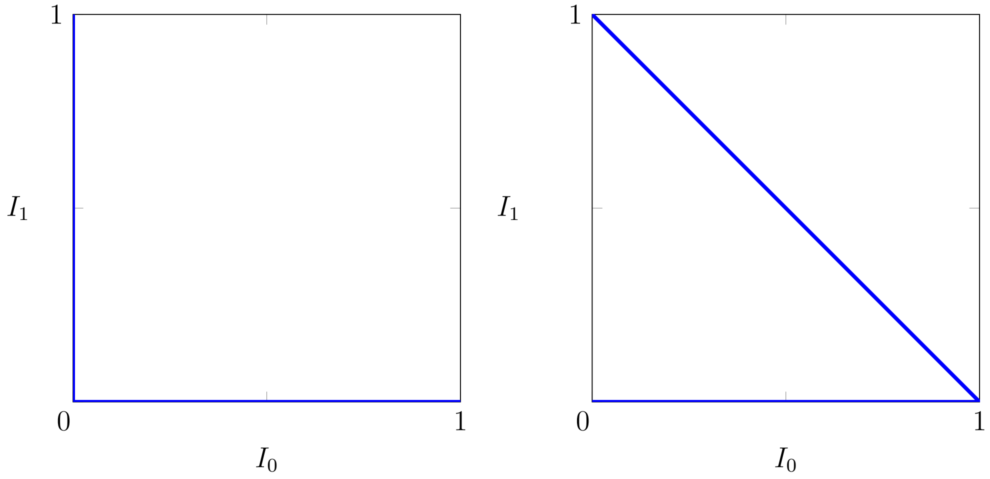


Figure 5.4: Set G from the Example 5.65 (left) and Example 5.66 (right)

ones. If $L_i = L_2, \forall i$ then the product is arc from $(0,1,0,\ldots)$ to $(1,0,1,\ldots)$. So we have

$$\begin{aligned} N\left(\star_{i=1}^{m-1} G, \alpha^m\right) + N\left(\star_{i=1}^{m-2} G, \alpha^{m-1}\right) \leq & N\left(\star_{i=1}^{m} G, \alpha^{m+1}\right) \\ \leq & N\left(\star_{i=1}^{m-1} G, \alpha^m\right) + N\left(\star_{i=1}^{m-2} G, \alpha^{m-1}\right) + n \end{aligned}$$

and hence $nF_{m+2} \leq N\left(\star_{i=1}^{m} G, \alpha^{m+1}\right) \leq n(F_{m+3}-1)$, where F_m is $m-th$ Fibonacci number. Therefore, we have that

$$\operatorname{ent}(G,\alpha) = \lim_{m\to\infty} \frac{1}{m} \log N\left(\star_{i=1}^{m} G, \alpha^{m+1}\right) = \lim_{m\to\infty} \frac{1}{m} \log n(F_{m+3}-1) = \frac{1+\sqrt{5}}{2}.$$

and $\operatorname{ent}(G) = \log \frac{1+\sqrt{5}}{2}$. ■

Example 5.67 *Let $a \in \mathbb{N}$ be arbitrary and let*

$$G_a = \left\{\left(\frac{k}{a}, 0\right) : k \in \{0,\ldots,a\}\right\} \cup \left\{\left(0, \frac{k}{a}\right) : k \in \{0,\ldots,a\}\right\} \subseteq I^2.$$

Then $\operatorname{ent}(G_a) = \log \dfrac{1+\sqrt{1+4a}}{2}$.

Proof. Let us denote number of points in $\star_{i=1}^{m} G_a$ with N_m.
We prove $N_m = N_{m-1} + a\ N_{m-2}$.

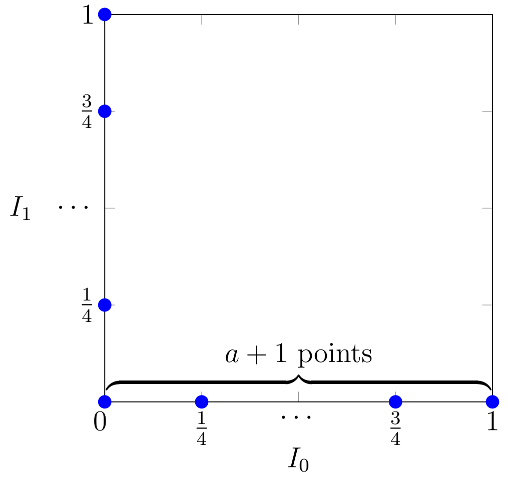


Figure 5.5: Set G_4 in Example 5.67

Proof is combinatorial: let us observe arbitrary $(m+1)-$tuple in $\star_{i=1}^m G_a$. If we have 0 on the first coordinate, second can be any number from $\left\{0, \frac{1}{a}, \frac{2}{a}, \ldots, 1\right\}$ so we can get any $m-$ tuple. If we have non-zero as first coordinate, second coordinate has to be zero, third can be anything (as above) so we can get any $(m-1)-$tuple. Therefore, we get reccurence relation $N_m = N_{m-1} + a\ N_{m-2}$ with initial values $N_1 = 2a+1$ and $N_2 = (a+1)^2 + a$. Solving it using characteristical polynomial $x^2 - x - a = 0$ we get

$$N_m = a_0 \left(\frac{1+\sqrt{1+4a}}{2}\right)^m + b_0 \left(\frac{1-\sqrt{1+4a}}{2}\right)^m$$

where a_0 and b_0 are obtained from initial values.
Therefore

$$N\left(\star_{i=1}^m G_a, \alpha^{m+1}\right) \leq N_m$$

and

$$\lim_{m\to\infty} N\left(\star_{i=1}^m G_a, \alpha^{m+1}\right) = \lim_{m\to\infty} N_m.$$

Now,

$$\operatorname{ent}(G_a, \alpha) = \lim_{m\to\infty} \frac{\log N_m}{m}.$$

By simple calculations we get

$$\operatorname{ent}(G_a, \alpha) = \frac{1 + \sqrt{1 + 4a}}{2}.$$

■

Example 5.68 *Let* $G = \{(x_0, x_1) \in I_0 \times I_1 : x_1 \leq x_0^2\}$ *and let* $bL = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$. *Then* $\operatorname{ent}(G) = \operatorname{ent}(bL) = 0$.

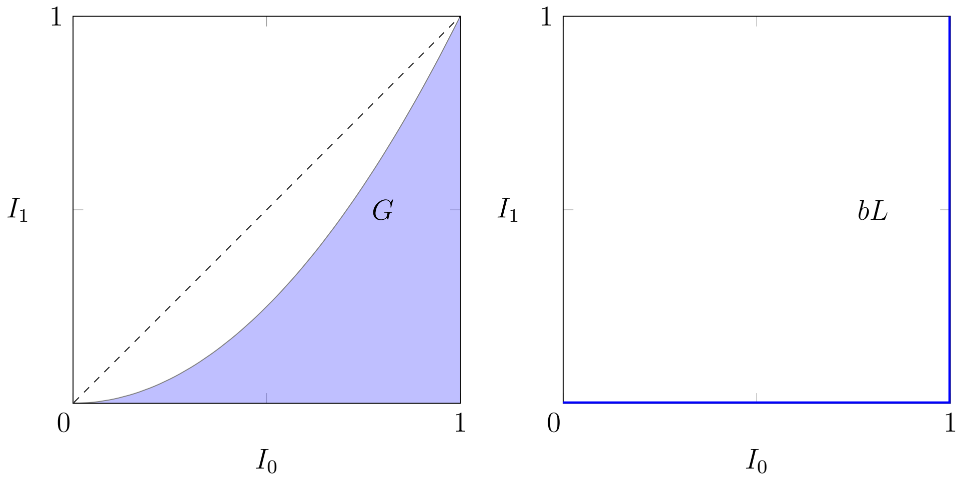


Figure 5.6: Sets G and bL from the Example 5.68

Proof. Mahavier product $\star_{i=1}^{\infty} G$ has very simple dynamics. The orbit of every point $x = (x_0, x_1, \ldots) \neq (1, 1, \ldots)$ under the action of σ, i.e. $x, \sigma(x), \sigma^2(x), \ldots,$ converges to the point $(0, 0, \ldots)$. Points $(0, 0, \ldots)$ and $(1, 1, \ldots)$ are fixed points, where $(0, 0, \ldots)$ is an attracting and $(1, 1, \ldots)$ is a repelling fixed point. So we have that $\operatorname{ent}(\sigma) = 0$ and by Theorem 5.51 therefore $h(G) = 0$. Set $M = \star_{i=1}^{\infty} bL$ indeed is an arc and the entropy is equal 0. Namely, in the Example 5.64 we have that $\operatorname{ent}(bL^{-1}) = 0$ but then by the Theorem 5.48 it follows that $\operatorname{ent}(bL) = 0$. ■

Before stating the proposition we give new notion. If H is closed subset of $[0, 1]^{n+1}$, define $\pi_{\{0,n\}}$ to be the map from H to $[0, 1] \times [0, 1]$ defined by $\pi_{\{0,n\}}(x) = (x_0, x_n)$ for $x = (x_0, x_1, \ldots, x_n) \in H$. For G a closed subset of

$[0,1] \times [0,1]$, and n positive integer, let $G^{0,n} = \pi_{\{0,n\}}(\star_{i=1}^{n} G)$. Therefore, $G^{0,n}$ is the closed subset of $[0,1] \times [0,1]$.

With $\lim_{H_d}$ we denote limit with respect to the Hausdorff metric. Let us recall that Hausdorff metric:

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}.$$

This equation is different from the one in the first chapter but they are actually equivalent.

Proposition 5.69 *Let G be connected and closed subset of $[0,1]^2$ such that $\lim_{H_d} G^{0,n} = bL$ where bL is from the previous example. Then* $\mathrm{ent}(G) = 0$.

Proof. We have $\lim_{H_d} G^{0,n} = bL$ i.e.

$$(\forall \epsilon >)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N})\ n \geq n_0 \to H_d(G^{0,n}, bL) < \epsilon$$

We divide the proof in several steps:

(i) G does not contain any point on diagonal not equal to $(0,0)$ and $(1,1)$. Assume the contrary, i.e. if it does contain point $(x,x), x \neq 0,1$, then $G^{0,n}$ also contains that point, for all $n \in \mathbb{N}$. Then, $d((x,x),bL) = \begin{cases} x, & x \leq \frac{1}{2} \\ 1-x, & x > \frac{1}{2} \end{cases}$, but in either way it is greater than 0.
Therefore, we get $H_d(G^{0,n}, bL) \geq d((x,x),bL) > 0, \forall n \in \mathbb{N}$, hence we cannot have $\lim_{H_d} G^{0,n} = bL$.

(ii) $\pi_0(G) = [0,1]$ and $\pi_1(G) = [0,1]$.
Suppose $\pi_0(G) = J_0$ where J_0 is closed and proper subset of $[0,1]$. Then, $G^{0,n} \subseteq J_0 \times [0,1] \subset [0,1] \times [0,1]$ for all $n \in \mathbb{N}$. Since bL contains $[0,1] \times \{0\}$, there exists point $(x,0) \in bL \setminus (J_0 \times \{0\})$ such that $d((x,0),J_0) = d_0 > 0$ (because J_0 is closed). Now we have that $0 < d_0 = d((x,0), J_0 \times [0,1]) \leq d((x,0), G^{0,n})$ and therefore $H_d(G^{0,n}, bL) \geq d_0 > 0$. So we get $\lim_{H_d} G^{0,n} \neq bL$ which is contradiction.

(iii) G doesn't contain point (x_0, y_0) above diagonal.
Suppose that G contains point (x_0, y_0) above diagonal, i.e. $y_0 > x_0$. Set G is closed, connected and by (ii), $\pi_0(G) = [0,1]$ and $\pi_1(G) = [0,1]$, therefore G intersects diagonal in some point $(x,x), x \in [0,1]$. By (i) it follows that x is equal to 0 or 1.
Suppose that $x = 0$. Since G is connected and closed, it contains an arc connecting points (x_0, y_0) and $(0,0)$ which we denote with K. Since K is an arc, for each positive integer n there exists finite sequence of points $x_{n-1} < x_{n-2} < \ldots < x_0 < y_0$ in $[0,1]$ such that $(x_{n-1}, x_{n-2}, \ldots, x_0, y_0) \in \star_{i=1}^{n} K$. Therefore, since $K \subset G$, we have $H_d(G^{0,n}, bL) \geq H_d(K^{0,n}, bL) \geq d((x_0, y_0), bL) > 0$, hence we cannot have $\lim_{H_d} G^{0,n} = bL$.
If we suppose that $x = 1$, we get the contradiction in the same way.

From (i) and (iii) it follows that set G is under the diagonal, except points $(0,0)$ and $(1,1)$ i.e.

$$(\forall (x,y) \in G \setminus \{(0,0),(1,1)\}, \ y < x.) \ (*)$$

(iv) G contains both $(0,0)$ and $(1,1)$.
We prove that G contains $(1,1)$ in the same way as we proved (i). If we assume contrary, we would get that the distance between G and bL is > 0. Therefore, $\star_{i=1}^{\infty} G$ contains the point $(1,1,1,\ldots)$.
Suppose G doesn't contain $(0,0)$. Since G is closed subset of $[0,1]^2$, then $\pi_0(G)$ is closed subset of $[0,1]$ so there exists minimum of that set and let us denote it with x_{min}. Now, we claim that $\star_{i=1}^{\infty} G$ contains only the point $(1,1,1,\ldots)$. Let assume that there is point $(x_0, x_1, \ldots) \neq (1,1,1,\ldots)$ in $\star_{i=1}^{\infty} G$. But since, by $(*)$ we have $x_0 > x_1 > x_2 > \ldots$, there exists $n \in \mathbb{N}$ large enough such that $x_n < x_{min}$ but that is contradiction. Therefore G contains $(0,0)$.

(v) $\mathrm{ent}(G) = 0$.
Proof is same as for the set G in the previous example so we omit it.

■

Some open problems:

Question 1 *Suppose* L *is a closed subset of* $[0,1] \times [0,1]$, *and* $\mathrm{ent}(L) = 0$. *If* G *is closed subset of* $[0,1] \times [0,1]$, *and* $lim_{H_d}(G^{0,n}) = L$, *is* $\mathrm{ent}(G) = 0$?

Question 2 *Is there for arbitrary* $b \in \mathbb{R}$, *a closed subset* G *of* $[0,1] \times [0,1]$ *that is not graph of a continuous function such that* $\mathrm{ent}(G) = b$?

Question 3 *Can we find a characterisation for closed subsets of* $[0,1] \times [0,1]$, *with zero entropy?*

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Abstract

Generalized inverse limits are generalization of standard inverse limits in a way that in the corresponding inverse system bonding functions are upper semicontinuous (u.s.c.) functions instead of continuous functions. Concept was introduced in 2004 in [31] and later in 2006 in [28] and since then, theory has been developing rapidly.

In the first part we introduce categories $\mathcal{CHU}$ and $\mathcal{CU}$ in which u.s.c. functions are morphisms and compact Hausdorff and compact metric spaces, respectively, are objects. We also introduce the category $\mathcal{ICU}$ of inverse sequences in $\mathcal{CU}$. Then we investigate the induced functions between inverse limits of compact metric spaces with u.s.c. bonding functions. We also show that taking such inverse limits is very close to being a functor (but is not a functor) from $\mathcal{ICU}$ to $\mathcal{CU}$, if morphisms are mapped to induced functions. At the end of the third chapter we give a useful application of the mentioned results.

In the second part new definition of topological entropy is considered, in which is used Mahavier product, introduced in [19]. It is shown that new notion is well defined and that is in line with previous definitions for regular functions [41], using entropy of the shift map. Then, entropy of various examples is calculated, new ones and some well known. Finally, some new results about generalized inverse limits are shown using newly defined objects.

Keywords: category, hyperspace, inverse system, inverse limit, upper semicontinuous function, generalized inverse limit, Mahavier product, topological entropy

Sažetak

Inverzni limesi su imali ključnu ulogu u razvoju teorije kontinuuma u prošlom stoljeću. Također su bili važni i u dinamičkim sustavima. Jedan od razloga za to je njihovo svojstvo da inverzni nizovi s jednostavnim prostorima i jednostavnim veznim preslikavanjima mogu inducirati komplicirane prostore kao njihove inverzne limese. U dinamičkim sustavima, inverzni limesi se koriste za „kodiranje", na neki način, kompliciranih dinamičkih sustava.

Generalizirani inverzni limesi su poopćenje standardnih inverznih limesa na način da u pripadajućem inverznom sustavu vezna preslikavanja nisu neprekidne funkcije nego odozgo poluneprekidne funkcije. Pojam je uveden u [31] 2004. godine, a zatim 2006. godine u [28] razvijen do forme koja se danas koristi. Od tada se teorija intezivno razvija.

U prvom dijelu rada se kategorijski opisuju ti objekti i ispituje se koje se tvrdnje iz standardnog slučaja mogu poopćiti, te se primjenjuje dobivene rezultate na neke konkretne slučaje. Uvodimo dvije kategorije, $\mathcal{CHU}$ i $\mathcal{CU}$. Objekti u $\mathcal{CHU}$ su kompaktni Hausdorffovi prostori s odozgo poluneprekidnim funkcijama (skraćeno u.s.c.) kao morfizmima, a $\mathcal{CU}$ je potkategorija od $\mathcal{CHU}$, s istim morfizmima i kompaktnim metričkim prostorima kao objektima. Pokazuje se da u kategoriji $\mathcal{CHU}$ inverzni limes inverznog sistema (s usmjerenim indeksnim skupom) s u.s.c. višeznačnim funkcijama (kako su ih definirali Ingram i Mahavier) zajedno s projekcijama nije nužno inverzni limes u $\mathcal{CHU}$, ali je takozvani slabi inverzni limes, [10].
Nadalje, razmatraju se inverzni nizovi u kategoriji $\mathcal{CU}$ i dokazuje da s odgovarajućim morfizmima čine kategoriju, označenu s $\mathcal{ICU}$. Promatraju se morfizmi između dvaju inverznih limesa, inducirani odgovarajućim morfizmima u kategoriji $\mathcal{ICU}$. Dokazuju se nužni i dovoljni uvjeti za njihovu egzistenciju

i njihova svojstva. Nadalje, razmatra se standardno pridruživanje između inverznog niza i njegovog inverznog limesa i pokazuje da nije funktor iz kategorije $\mathcal{ICU}$ inverznih nizova u kategoriju $\mathcal{CU}$, ali je jako blizu istom. Naposljetku, na kraju trećeg poglavlja se pokazuje primjena navedenih rezultata.

U drugom dijelu rada se razmatra poopćenje pojma topološke entropije na zatvorene podskupove od $[0,1]^2$ koristeći Mahavierov produkt, uveden u [19]. Pokazuje se da je nova definicije topološke entropije zaista dobra i koristeći entropiju tzv. funkcije pomaka (funkcija u uobičajenom smislu) pokazuje se da je usklađena s prijašnjim definicijama entropije. Naime ako je dana neprekidna funkcija $f : [0,1] \to [0,1]$ i zatvoreni podskup od $[0,1]^2$ kao graf od f^{-1}, nova i tradicionalna topološka entropija su jednake. Zatim se pokazuju razna svojstva za novu definiciju koja se uspješno mogu poopćiti iz teorije s funkcijama te se proširuje definicija na zatvorene podskupove konačnih produkata jediničnog segmenta $[0,1]$. Na kraju primjenjujemo dobivene rezultate za računanje entropije raznih primjera, važnih u teoriji o generaliziranim inverznim limesima i nekih novih.

Ključne riječi: kategorija, hiperprostor, inverzni sustav, inverzni limes, odozgo poluneprekidna funkcija, generalizirani inverzni limes, Mahavierški produkt, topološka entropija

Curriculum vitae

Goran Erceg was born on February 20th, 1985 in Split. He finished Prirodoslovno-matematička gimnazija in Split in 2003. In 2009. he graduated at the University of Split, Faculty of Science with thesis "Hyperspaces and Whitney maps" under supervision of prof. Vlasta Matijević.

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He was on the short visits at the University of Richmond, VA, USA, Lamar University, Texas, USA and several times at the University of Maribor, Slovenia.

List of journal publications

1. Ujević, Nenad; Erceg, Goran; Lekić, Ivan, A family of methods for solving nonlinear equations, *Applied mathematics and computation*, 192 (2007) , 2; 311-318

2. Ujević, Nenad; Erceg, Goran, A generalization of the corrected midpoint-trapezoid rule and error bounds, *Applied mathematics and computation*, 184 (2007) , 2; 216-222

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Participation at conferences and workshops

- Mini conference on continuum theory and dynamical systems, Maribor, Slovenia, June 21-22, 2010.
- Mini workshop on dynamical systems, Zagreb, Croatia, June 23, 2010.
- Generalized inverse limits of inverse sequences of generalized inverse limits, Dubrovnik VII - Geometric Topology, Dubrovnik, Croatia, June 26 - July 3, 2011, Abstracts p. 21
- 5th Croatian Mathematical Congress, Rijeka, Croatia, June 18-21, 2012.
- Induced functions and induced morphisms between generalized inverse limits, 48th Spring Topology and Dynamics Conference, Richmond VA, USA, March 13-15, 2014, Abstracts p. 9.
- Workshop on Inverse Limits with Set-valued Functions, Richmond VA, USA, March 16, 2014.
- Mahavier product and topological entropy, Dubrovnik VIII - Geometric Topology, Geometric Group Theory & Dynamical Systems, Dubrovnik, Croatia, June 22 - 26 2015

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Životopis

Goran Erceg je rođen 20. veljače 1985. u Splitu. gdje je 2003. godine završio Prirodoslovno-matematičku gimnaziju. Godine 2009. diplomirao je na Prirodoslovno-matematičkom fakultetu Sveučilišta Splitu s diplomskim radom *Hiperprostori i Whitneyjeva preslikavanja* pod mentorstvom prof. Vlaste Matijević te stekao zvanje profesor matematike.

Iste godine upisuje Zajednički sveučilišni poslijediplomski doktorski studij matematike Sveučilišta u Osijeku, Sveučilišta u Rijeci, Sveučilišta u Splitu te Sveučilišta u Zagrebu. Član je Topološkog seminara u Splitu čiji je i tajnik od 2015. godine, Splitskog matematičkog društva čiji je i računovođa i član Upravnog odbora od 2015. godine te Hrvatskog matematičkog društva.

Od ožujka 2010. godine zaposlen je kao znanstveni novak-asistent na Prirodoslovno-matematičkom fakultetu Sveučilišta Splitu i projektu Grubi oblik i klasifikacija natkrivanja (177-0372791-0886).

Bio je na kratkim gostovanjima na University of Richmond, VA, USA, Lamar University, Texas, USA te više puta na sveučilištu u Mariboru.

Publikacije

1. Ujević, Nenad; Erceg, Goran; Lekić, Ivan, A family of methods for solving nonlinear equations, *Applied mathematics and computation*, 192 (2007) , 2; 311-318

2. Ujević, Nenad; Erceg, Goran, A generalization of the corrected midpoint-trapezoid rule and error bounds, *Applied mathematics and computation*, 184 (2007) , 2; 216-222

3. Banič, Iztok; Črepnjak, Matevž; Erceg, Goran; Merhar, Matej; Milutinović, Uroš. Inducing functions between inverse limits with upper semicontinuous bonding functions, *Houston journal of mathematics* (0362-1588) 41 (2015), 3; 1021-1037

Sudjelovanja i izlaganja na konferencijama i znanstvenim skupovima

Sudjelovanja

- Mini conference on continuum theory and dynamical systems, Maribor, Slovenia, June 21-22, 2010.
- Mini workshop on dynamical systems, Zagreb, Croatia, June 23, 2010.
- 5th Croatian Mathematical Congress, Rijeka, Croatia, June 18-21, 2012.
- Workshop on Inverse Limits with Set-valued Functions, Richmond VA, USA, March 16, 2014.

Izlaganja

- Generalized inverse limits of inverse sequences of generalized inverse limits, Dubrovnik VII - Geometric Topology, Dubrovnik, Croatia, June 26 - July 3, 2011, Abstracts p. 21
- Induced functions and induced morphisms between generalized inverse limits, 48th Spring Topology and Dynamics Conference, Richmond VA, USA, March 13-15, 2014, Abstracts p. 9.
- Mahavier product and topological entropy, Dubrovnik VIII - Geometric Topology, Geometric Group Theory & Dynamical Systems, Dubrovnik, Croatia, June 22 - 26 2015

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