Stable homotopy theory of dendroidal sets

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Chapter 1

Introduction

The focus of this thesis is on dendroidal sets and their stable homotopy theory. In this introductory chapter we would like to motivate the reader for this topic and present our main results. We will first give a short overview of the basic principles of algebraic topology and, in particular, explain what is understood under the term homotopy theory. Next, we wish to present the theory of dendroidal sets as a generalization of the theory of simplicial sets. Therefore, we will shortly review certain aspects of simplicial sets (that are probably familiar to the reader) with an aim of making the passage from simplicial sets to dendroidal sets natural. We will review the main results from the theory of dendroidal sets that have been known before writing this thesis and on which our work is based. In the final part of this chapter we will present our new results and explain how the rest of the thesis is organized.

1.1 Algebraic structures in topology

1.1.1 Homotopy invariants

The subject of this thesis falls under the area of mathematics called algebraic topology. One of the main objectives of topology is to classify all topological spaces up to various equivalence relations. Two spaces are homeomorphic (or topologically equivalent) if there is a continuous bijection between them such that the inverse is also a continuous map. Although classifying spaces up to a homeomorphism is the most important question, we often consider coarser equivalence relations.

For two continuous maps $f, g\colon X \to Y$ we say that f is *homotopic* to g if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. A map $f\colon X \to Y$ is called a *homotopy equivalence* if there is a map $g\colon Y \to X$ such that fg is homotopic to the identity 1_Y and gf is homotopic to the identity 1_X. In such cases, we say that g is a homotopy inverse of f and that the spaces X and Y are homotopy equivalent (or of the same homotopy type). With this definition at hand we may consider classification of all topological spaces up to homotopy equivalence. Obviously, every homeomorphism is a homotopy equivalence, but not vice versa.

To show that two spaces are not of the same type we consider invariants shared by all spaces of the same type. An invariant might be a certain property of a space (e.g. compactness or connectedness) or a mathematical object assigned to it (e.g. the number of connected components, the Euler characteristic, the Betti numbers, the cohomology ring, the homotopy groups etc.). Typically, to each topological space we assign an object with a certain algebraic structure and to each continuous map we assign a morphism respecting that algebraic structure. For example, to a topological space X we might assign a group $\pi(X)$ and to a continuous map $f\colon X \to Y$ a group homomorphism $\pi(f)\colon \pi(X) \to \pi(Y)$. The most important invariants in algebraic topology are functorial. This means that they are respecting the composition of maps, i.e. $\pi(fg) = \pi(f)\pi(g)$. We say that $\pi\colon \mathrm{Top} \to \mathrm{Grp}$ is a functor from the category Top of topological spaces and continuous maps to the category Grp of groups and group homomorphisms. If $\pi\colon \mathrm{Top} \to \mathcal{C}$ is a functor that sends homotopic maps to the same morphism in a category $\mathcal{C}$, then π sends homotopy equivalences to isomorphisms. We say that π is a *homotopy invariant*.

Now, consider the *homotopy category* Ho(Top). The objects of Ho(Top) are topological spaces and morphisms are homotopy equivalence classes of continuous maps. There is a functor $\gamma\colon \mathrm{Top} \to \mathrm{Ho}(\mathrm{Top})$ sending each continuous map to its equivalence class. The pair $(\mathrm{Ho}(\mathrm{Top}), \gamma)$ has the following universal property. Every homotopy invariant $\pi\colon \mathrm{Top} \to \mathcal{C}$ induces a functor $\overline{\pi}\colon \mathrm{Ho}(\mathrm{Top}) \to \mathcal{C}$ such that $\pi = \gamma\overline{\pi}$. This illustrates that by studying homotopy invariants we study the homotopy category and functors defined on it.

Category theory provides an efficient way to compare various invariants and to study various properties as we are passing from a topological to an algebraic context. The language of categories will be used throughout this thesis in an essential way.

1.1.2 Homotopy invariant algebraic structures

Algebraic topology also studies topological spaces with an additional algebraic structure and the way the algebraic and the topological structures interact. Here is a basic example. Let X be a topological monoid and Y a space that is homotopy equivalent to X. One can transfer the multiplicative structure from X to Y via the homotopy equivalence. The transferred structure will not satisfy the strictly associative law $(y_1y_2)y_3 = y_1(y_2y_3)$ for all $y_1, y_2, y_3 \in Y$, but the maps $F_0, F_1\colon Y \times Y \times Y \to Y$ given by $F_0(y_1, y_2, y_3) = (y_1y_2)y_3$ and $F_1(y_1, y_2, y_3) = y_1(y_2y_3)$ will be homotopic. In other words, for any three points $y_1, y_2, y_3 \in Y$ there will be a path $[0, 1] \to Y$ from the point $(y_1y_2)y_3$ to the point $y_1(y_2y_3)$. We say that the multiplication on Y is *associative up to a homotopy*.

Furthermore, for any four points $y_1, y_2, y_3, y_4 \in Y$ we have five paths connecting the five points corresponding to different ways of bracketing four variables as in the following picture:

$$\begin{tikzcd}[column sep=small]
 & & (y_1y_2)(y_3y_4) \arrow[drr, dash] & & \\
((y_1y_2)y_3)y_4 \arrow[urr, dash] \arrow[dr, dash] & & & & y_1(y_2(y_3y_4)) \\
 & (y_1(y_2y_3))y_4 \arrow[rr, dash] & & y_1((y_2y_3)y_4) \arrow[ur, dash] &
\end{tikzcd}$$

These five paths form a map from the boundary of a pentagon to Y. It can be seen that the transferred structure on Y contains enough information to extend this map to the interior of the pentagon. As we can do this for any choice of four points, we obtain a map $Y^{\times 4} \times K_4 \to Y$, where K_4 is the pentagon and $Y^{\times 4} = Y \times Y \times Y \times Y$. Informally, we think of this map as a 2-dimensional homotopy between the homotopies.

This is not all there is to be said. For each positive integer $n \geqslant 3$, there is an $(n-2)$-dimensional polyhedron K_n and a map $Y^{\times n} \times K_n \to Y$, which we think of as a higher dimensional homotopy relating the lower dimensional homotopies. We can build an infinite tower of higher homotopies that all together form a coherent system. We say that Y is *associative up to a coherent homotopy* or that it is an A_∞-space. The A_∞-spaces have been first studied by J. Stasheff in 1963 and the polyhedra K_n are called Stasheff polyhedra.

Moreover, if we start with an A_∞-space X and transfer this structure to a homotopy equivalent space, we again obtain an A_∞-space. So, the structure of an A_∞-space is a homotopy invariant algebraic structure.

1.1.3 Loop spaces and infinite loop spaces

There are plenty of examples of A_∞-spaces that are of great interest. If X is a space and x_0 is a point in X, we may consider the space ΩX of all loops $f\colon [0,1] \to X$ such that $f(0) = f(1) = x_0$. With the concatenation of loops as a multiplication, every loop space ΩX is an A_∞-space.

Loop spaces have an additional property of being *group–like*. Let us explain what that means. If Y is a space with a multiplication that is associative up to a homotopy, then the set of connected components $\pi_0(Y)$ is an associative monoid. If $\pi_0(Y)$ is also a group, we say that Y is group–like. For every loop $f \in \Omega X$, there is a loop $\overline{f}$ given by $\overline{f}(t) = f(1-t)$ (hence $\overline{f}$ is traversing the same path as f only in the reverse direction). The concatenated loops $f\overline{f}$ and $\overline{f}f$ are homotopic to the constant loop at x_0. As the connected components of ΩX are homotopy equivalence classes of loops in X, this implies that the class $[\overline{f}]$ is an inverse for the class $[f]$ in $\pi_0(\Omega X)$. Hence, every loop space is a group–like A_∞-space.

Actually, this algebraic structure characterizes loop spaces (up to homotopy). Stasheff proved the following *recognition principle*: a topological space Y is homotopy equivalent to a loop space ΩX of another space X if and only if Y is a group–like A_∞-space.

We can repeat the whole story starting with a commutative monoid. The relevant algebraic structure that is both associative and commutative up to a coherent homotopy is called the E_∞-structure. So, if Y is an E_∞-space, we have a homotopy between maps $G_0, G_1 \colon Y \times Y \to Y$ given by $G_0(x,y) = xy$ and $G_1(x,y) = yx$ and a whole tower of higher dimensional homotopies expressing the relations between homotopies. Typical examples are infinite loop spaces. An infinite loop space is a space Y such that for every positive integer n there exists a space X such that Y is homotopy equivalent to the n-fold loop space $\Omega^n X$. J. M. Boardman and R. M. Vogt gave a recognition principle saying that group–like E_∞-spaces are (up to homotopy) exactly the infinite loop spaces, [BV73].

There are spaces which are commutative up to homotopy, but are not commutative up to a coherent homotopy. Such spaces are, for example, double loop spaces. If Y is a double loop space, $Y = \Omega^2 X$, for any two elements $f, g \in Y$ there are paths in Y from fg to gf, but there are no higher homotopies relating these paths.

By the recognition principle, group–like E_∞-spaces correspond to infinite loop spaces. Note that an infinite loop space actually consists of a sequence of space $(X_n)_{n\geq 0}$ and weak equivalences $X_0 \to \Omega X_1$, $X_1 \to \Omega X_2$, etc. In stable homotopy theory such a structure is called an Ω-spectrum.

Let us say just a few things about spectra. We will discuss only one of many models for spectra (the easiest one to explain) and we will stay quite informal in the discussion (e.g. we will not specify does the word "space" refer to a CW complex or a simplicial set and we will be sloppy about the basepoints).

For a pointed space (Y, y_0), the reduced suspension ΣY is the space given by

$$\Sigma Y = Y \times [0,1] /_{Y \times \{0,1\} \cup \{y_0\} \times [0,1]}.$$

Note that the suspension and loop space constructions are adjoint, so a map $X_n \to \Omega X_{n+1}$ corresponds to a map $\Sigma X_n \to X_{n+1}$. A *spectrum* X consists of a sequence of pointed spaces $(X_n)_{n\in\mathbb{Z}}$ together with structure maps $\Sigma X_n \to X_{n+1}$. Every infinite loop space gives an example of a spectrum with an additional property of being an Ω-spectrum (i.e. the adjoints $X_n \to \Omega X_{n+1}$ of the structure maps are weak equivalences). Every topological space X gives a spectrum $\Sigma^\infty X$ with $(\Sigma^\infty X)_n = \Sigma^n X$ and the structure maps $\Sigma(\Sigma^n X) \to \Sigma^{n+1}$ being identities.

We can replace every spectrum X with an equivalent Ω-spectrum Y given by

$$Y_n = \operatorname*{colim}_k \Omega^k X_{k+n}.$$

The notion of equivalence of spectra is related to the stable homotopy groups. We define stable homotopy groups in the following way. The (standard) homotopy group $\pi_m(X_n)$ can be viewed as the group of homotopy classes $\pi_m(X_n) = [S^m, X_n]$. Applying the suspension functor and then composing with the structure map $\Sigma X_n \to X_{n+1}$ gives us a map

$$\pi_m(X_n) = [S^m, X_n] \to [\Sigma S^m = S^{m+1}, \Sigma X_n] \to [S^{m+1}, X_{n+1}] = \pi_{m+1}(X_{n+1}).$$

For each integer k, we can define the stable homotopy group $\pi_k^s(X)$ of a spectrum X by

$$\pi_k^s(X) = \operatorname*{colim}_n \pi_{k+n}(X_n).$$

If X is a space, Freudenthal's suspension theorem implies that the sequence $\pi_{k+n}(\Sigma^n X)$ *stabilizes*, i.e. there is an m such that for each $k \geq m$ the map $\pi_k(\Sigma^n X) \to \pi_{k+1}(\Sigma^{n+1} X)$ is an isomorphism. Hence, studying spectra means that we are studying stable phenomena in the homotopy theory of spaces.

A spectrum is called *connective* if all stable homotopy groups $\pi_k^s(X)$ are trivial for $k < 0$. Every infinite loop space gives a connective spectrum as we now explain. A loop space ΩX depends only on the connected component of the basepoint with respect to which we consider the loops in X. Hence given an infinite loop space $(X_n)_{n \geq 0}$, we may assume that all the spaces $X_1, X_2, \dots$ are connected. Using that $\pi_n(\Omega X) = \pi_{n+1}(X)$, we see that $\pi_k(X_n) = 0$ for $k < n$. This implies that $\pi_k^s(X) = 0$ for all $k < 0$. Moreover, if X is a connective spectrum, then its associated Ω-spectrum gives an infinite loop space. Hence, connective spectra are equivalent to infinite loop spaces, i.e. to grouplike E_∞-spaces.

Spectra are important because they represent generalized cohomology theories. If E is a spectrum, then a cohomology theory H_E represented by E is given by

$$H_E^n(X) = [X, E_n].$$

By Brown's representability theorem, every generalized cohomology theory is represented by a spectrum. The standard example is the Eilenberg-MacLane spectrum HA for a commutative group A. The spectrum HA represents the singular cohomology $H^*(X; A)$ and consists of Eilenberg-MacLane spaces $HA_n = K(A, n)$ having the property that

$$\pi_k(K(A,n), *) = \begin{cases} A, & k = n; \\ 0, & k \neq n. \end{cases}$$

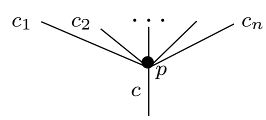
So, the stable homotopy groups of HA are all trivial, except $\pi_0^s(HA)$ which is A.

1.1.4 Operads

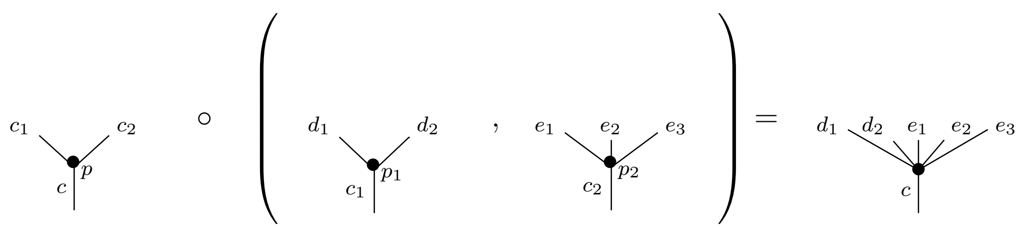
We discussed the recognition principle for (infinite) loop spaces. A recognition principle has been proven also for n-fold loop spaces for all positive integers n. The fundamental work on this subject is the book [May72] by J. P. May. As the relevant algebraic structure is quite complicated, May described it in terms of an action of an operad on an n-fold loop space. The term operad has been coined by May and used in loc. cit. for the first time.

The structure of an operad is very rich. Hence, in this introduction we will only illustrate the main idea how to think of an operad and give a more precise definition in Chapter 2. Instead of starting with May's definition, we will first consider a notion of a coloured operad, which is a common natural generalization of the notion introduced by May and of a notion of a category. After that we will discuss other relevant variants.

In a category, every morphism has one source (the domain) and one target (the codomain). We wish to consider a generalization of the notion of a category where we allow morphisms with more than one source. A *coloured operad* P has a class of colours (or objects) $\mathrm{col}(P)$ and for each sequence of colours $c_1, c_2, \ldots, c_n, c \in \mathrm{col}(P)$ there is a set $P(c_1, c_2, \ldots, c_n; c)$. We think of an element $p \in P(c_1, c_2, \ldots, c_n; c)$ as an n-ary operation from the n-tuple $(c_1, \ldots, c_n)$ to the colour c and depict it as a tree



Further part of the structure is a composition law for all composable operations. For example, if we have operations $p_1 \in P(d_1, d_2; c_1)$, $p_2 \in P(e_1, e_2, e_3; c_2)$ and $p \in P(c_1, c_2; c)$ then there is a composite $p \circ (p_1, p_2) \in P(d_1, d_2, e_1, e_2, e_3; c)$. Pictorially:



We require that the composition is associative in the obvious sense. Also, every set $P(c; c)$ contains an element 1_c called the identity on colour c and we require that the identities act as neutral elements for the composition. The last part of the structure is an action of the symmetric group Σ_n, which is given in the sense that for $\sigma \in \Sigma_n$ and $p \in P(c_1, \ldots, c_n; c)$ there is an operation $\sigma^* p \in P(c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)}; c)$ and for $\sigma, \tau \in \Sigma_n$ we have $(\tau\sigma)^* p = \sigma^*(\tau^* p)$. The last requirement is that the composition is compatible with this action.

A *morphism of operads* $f \colon P \to Q$ is given by a map $f \colon \mathrm{col}(P) \to \mathrm{col}(Q)$ and maps

$$P(c_1, \ldots, c_n; c) \to Q(f(c_1), \ldots, f(c_n); f(c))$$

respecting the compositions, identities and the actions of the symmetric groups. We denote by Oper the category of coloured operads and morphisms of operads.

We can consider any category as a coloured operad which has only unary operations. In fact, if Cat is the category of small categories and functors, then there is an inclusion of categories $j_!\colon \mathrm{Cat} \to \mathrm{Oper}$. The functor has a right adjoint functor $j^*\colon \mathrm{Oper} \to \mathrm{Cat}$. The functor j^* takes an operad and associates to it a category by forgetting all non-unary operations.

Some authors use the name symmetric multicategories for coloured operads. The word symmetric emphasizes that a part of the structure is the action of the symmetric groups. We can also consider a variant of the notion of an operad in which there is no action of the symmetry groups. Such operads are called *nonsymmetric operads*.

A big class of examples of coloured operads is given by the symmetric monoidal categories. If a category $\mathcal{E}$ has a symmetric tensor product $\otimes$, we obtain an operad $O_{\mathcal{E}}$ in the following way. The colours of $O_{\mathcal{E}}$ are the objects of $\mathcal{E}$ and for each sequence of objects $x_1, \ldots, x_n, x$ in $\mathcal{E}$ the set of n-operations is given by

$$O_{\mathcal{E}}(x_1, \ldots x_n; x) = \mathrm{Hom}_{\mathcal{E}}(x_1 \otimes \cdots \otimes x_n, x).$$

The composition, identities and symmetries are given in the obvious way.

Operads provide an extremely efficient way to encode complicated algebraic structures in different contexts. This is because we can consider *enriched coloured operads*. For example, we say that a coloured operad is enriched in topological spaces if each operation set $P(c_1, \ldots, c_n; c)$ has a structure of a topological space such that the composition and the action of the symmetric groups are given by continuous maps. In general, we can consider coloured operads enriched in any cocomplete closed symmetric monoidal category (such as simplicial sets or chain complexes).

We have already discussed one of the earliest examples of an operad. The sequence $(K_n)_n$ of Stasheff polyhedra can be given the structure of a nonsymmetric topological operad with one colour. We think of each point in the space K_n as an abstract n-ary operation. It would take time to describe the composition precisely, so we will not attempt to do that here. Roughly speaking, the composition comes from embeddings of lower dimensional polyhedra as faces of higher dimensional polyhedra in a similar way that one can embed the interval $[0, 1] = K_3$ into each of the five sides of the pentagon K_4.

For an A_∞-space X, the maps $X^{\times n} \times K_n \to X$ show that each point of K_n represents one n-operation on X. We say that the Stasheff polyhedra form a nonsymmetric A_∞-operad and that an A_∞-space is an algebra over that operad. Informally, an operad captures an algebraic structure that is realized in its algebras (in a similar way as a group captures an algebraic structure realized in its representations).

Here is a very compact definition. An algebra over a coloured operad P enriched in a closed symmetric monoidal category $\mathcal{E}$ is a morphism of enriched operads $P \to O_{\mathcal{E}}$. Note that if P is an operad with only one colour c, this boils down to choosing one object X in $\mathcal{E}$ and giving the action of the operad P on X in terms of the maps

$$P(n) \to \mathrm{Hom}_{\mathcal{E}}(X^{\otimes n}, X), \quad \text{or equivalenty,} \quad X^{\otimes n} \otimes P(n) \to X$$

where $P(n) = P(c, \ldots, c; c)$ is the object of n-ary operations.

To give a recognition principle for iterated loop spaces, May considered one-coloured symmetric topological operads called the little cubes operads. In Chapter 2 we will give all the necessary definitions and show that every n-fold loop space is an algebra over the operad of little n-dimensional cubes. The important thing to remember is that the iterated loop spaces have a structure that is commutative up to a homotopy and we can not capture such a structure using nonsymmetric operads.

Let us end this section with just a few remarks about the homotopy theory of operads. We have seen that there is an operad A_∞ whose algebras are the A_∞-algebras, i.e. algebras that are associative up to a coherent homotopy. On the other hand, it is easy to describe an operad whose algebras are associative algebras (i.e. monoids). We call this operad Ass. In 1973, Boardman and Vogt have described a construction which more generally takes a topological operad P (with one colour) and gives a topological operad P_∞ (or $W(P)$) such that the P_∞-algebras are the "P-algebras up to a coherent homotopy". We call this construction the *Boardman-Vogt resolution.* Considering the operad P_∞ as a resolution of the operad P fits into the general formalism of homotopy theory, but there has gone 30 years until making this statement precise.

In the 1990's there has been a revival of the theory of operads led by discovery of many applications in different parts of mathematics and mathematical physics. Motivated by this, various authors considered homotopy theory of operads in different contexts (e.g. Kontsevich, Hinich etc.). This led to the development of an axiomatic approach to homotopy theory of one-coloured operads and algebras over an operad, [BM03]. The approach of C. Berger and I. Moerdijk extends to the case of the operads in various symmetric monoidal categories with a suitable interval object. In [BM06], they develop the notion of the Boardman-Vogt construction for operads enriched in such a monoidal category. The case of algebras over coloured operads was discussed in [BM07]. In the next sections we will discuss what is meant by the axiomatic approach to homotopy theory in general, and we will return to some of these results after that.

1.2 Homotopy theory

1.2.1 Homotopy theory in various categories

One of the most studied invariants of a pointed space (X, x_0) are its homotopy groups $\pi_n(X, x_0)$. A *weak homotopy equivalence* is a map $f\colon X \to Y$ such that for any choice of a basepoint x_0 in X the induced maps $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ are group isomorphisms. In general, for a weak homotopy equivalence $f\colon X \to Y$ there does not have to exist a map $g\colon Y \to X$ that is also a weak equivalence and which might be considered an inverse to f. So, the existence of a weak homotopy equivalence between spaces is a reflexive and a transitive relation, but it is not symmetric. This relation generates an equivalence relation $\sim$ and we say that spaces X and Y belong to the same weak homotopy type if $X \sim Y$. To study weak homotopy types, we would like to construct a category where all weak homotopy equivalences become isomorphisms.

A similar situation appears in homological algebra. There we study the category of (bounded) chain complexes and consider the class of quasi-isomorphisms, i.e. the maps that induce isomorphisms on the homology groups. If $f\colon C_\bullet \to D_\bullet$ is a quasi-isomorphism, there does not have to exist a map $g\colon D_\bullet \to C_\bullet$ inverse to f (in any relevant sense).

There is a common framework to deal with both situations and it is given by the notion of a localization of categories, [GZ67]. The idea is that given a category $\mathcal{C}$ and a class W of morphisms in $\mathcal{C}$, we want to add inverses to all the morphisms in W and obtain a category $\mathcal{C}[W^{-1}]$. Also, we want this new category to be equipped with a functor $\gamma\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ that is universal among all functors that send morphisms in W to isomorphisms. To obtain such a category $\mathcal{C}[W^{-1}]$ it is necessary to add all the possible compositions of the morphisms in $\mathcal{C}$ with the newly added inverses of morphisms in W. This procedure can lead to a foundational problem as the size of the new class of morphisms might become too large to be allowed by the set theoretical axioms. Such problems do not occur in the two examples we mentioned above.

In homological algebra, the localization of the category of bounded chain complexes in an abelian category $\mathcal{A}$ at the class of quasi-isomorphisms is called the derived category of $\mathcal{A}$. The existence of the derived category is a basic result that is obtained by considering the homotopy category of the projective resolutions, see [GM10]. In the case of topological spaces, the CW approximation theorem implies that weak homotopy types can be modelled by CW-complexes. The set-theoretical size issue does not appear because the universal property of the localization of the category Top with respect to weak equivalences is satisfied by an actual category - the homotopy category on CW-complexes, [Hat02].

Let us mention some other constructions and principles that have shown very useful in homotopy theory. One guiding principle of homotopy theory is to approximate objects with nicer ones - we have already mentioned projective resolutions of a chain complex and CW approximations of topological spaces. CW-complexes are built by glueing in cells of higher and higher dimension along their boundaries. The obvious advantage of such spaces is that we can work with them by induction on the dimension of the cells. We also consider relative cell complexes consisting of a pair (X, A) such that X is build out of A by attaching cells. In the categorical language, relative cell complexes are obtained by a sequence of pushouts and transfinite compositions. Each inclusion $A \to X$ which is a relative cell complex has the *homotopy extension property*. This property says that each homotopy $A \times [0,1] \to Y$ between maps $f, g\colon A \to Y$ can be extended to a homotopy $X \times [0,1] \to Y$ between maps $F, G\colon X \to Y$ whenever we are given a map F extending f. The maps having the homotopy extension properties are called *cofibrations*.

There is also a dual notion of fibrations, which are important as they give long exact sequences - a basic tool for calculating homotopy groups. Let D^n denote the n-dimensional disk. A *(Serre) fibration* $E \to Y$ has the *homotopy lifting property*, i.e. every homotopy $D^n \times [0,1] \to Y$ between maps $f, g\colon D^n \to Y$ can be lifted to a homotopy $D^n \times [0,1] \to E$ between maps $F, G\colon D^n \to E$ whenever we have a map F lifting f. Moreover, let $i\colon A \to X$

be a cofibration, $p\colon E \to Y$ a Serre fibration and let $u\colon A \to E$ and $v\colon X \to Y$ be maps such that $pu = vi$. If i or p is also a weak equivalence, then there exists a map $f\colon X \to E$ such that $pf = v$ and $fi = u$. We express this diagrammatically like this

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "v"'] \arrow[ur, dotted, "f"] & Y \end{tikzcd}$$

We will see that generalizing the notion of fibrations and cofibrations is an important part of defining a "homotopy theory" in a non-topological context.

1.2.2 Axiomatic homotopy theory

In 1967, D. Quillen united these considerations from topology and homological algebra under the name *homotopical algebra* by introducing the formalism of *(closed) model structures*, [Qui67] and [Qui69]. A model structure on a category consists of three classes of maps: weak equivalences, fibrations and cofibrations. These classes have to satisfy five axioms which we will discuss in detail in Chapter 2.

The fundamental theorem of homotopical algebra states that a category endowed with a model structure admits a localization with respect to the class of weak equivalences. It is important to note that the localization depends only on the class of weak equivalences, but the additional structure (in terms of fibrations and cofibrations) ensures that the localization exists. Let us give slightly more details. The axioms of a model structure allow one to introduce the notion of homotopic morphisms. In general, this notion does not give an equivalence relation on morphisms between arbitrary two object. Nonetheless, the axioms allow us to identify a subclass of cofibrant-fibrant objects for which these problems disappear. The existence of the localization with respect to weak equivalences then follows as one can show that it is equivalent to the homotopy category on cofibrant-fibrant objects. So, the localization is usually called the associated *homotopy category*.

The axioms for model categories are very powerful, but checking them can be tedious. So, constructing a model structure is a non-trivial job, but worthy of the effort as once it is done we can obtain various results from the general theory of model categories. Note that we will often use that a model structure is uniquely determined by cofibrations and fibrant objects (cf. Proposition 2.3.20).

Quillen showed that the category of topological spaces admits a model structure with the weak equivalences being the weak homotopy equivalences, the cofibrations being the retracts of relative cell complexes and the fibrations being Serre fibrations. For this reason, we say a model category gives a presentation (or a model) for a particular homotopy theory.

Homotopy theories can be compared. We say that two model categories are Quillen equivalent if there is a pair of adjoint functors between them inducing an equivalence of the associated homotopy categories. In [Qui67], Quillen also showed that there is a model structure on the category of simplicial sets and that it is equivalent to the model structure on topological spaces. We will devote more time to simplicial sets in the next few sections.

Before we proceed, let us mention one more example that motivates studying homotopy in an axiomatic way. The setting of differential graded algebras is not well suited for homotopical considerations. One issue is that the free algebra functor from chain complexes to commutative differential graded algebras over a field of positive characteristic does not preserve weak equivalences. The solution is to extend the Dold-Kan correspondence between bounded chain complexes and simplicial abelian groups to commutative rings. The free commutative algebra functor preserves weak equivalences between cofibrant simplicial abelian groups, so we see that from the homotopical point of view one should work with simplicial commutative algebras.

1.2.3 A categorical definition of simplicial sets

Simplicial methods were introduced in terms of triangulations of topological spaces in order to calculate the homotopy invariants using combinatorial models. A simplicial complex is build out of vertices (the 0-simplices), edges (the 1-simplices), triangles (the 2-simplices), tetrahedra (the 3-simplices) and higher dimensional simplices of every dimension.

Each geometrical n-simplex x has $n+1$ distinct faces $d_0x, \ldots, d_nx$ which are glued together along their faces. If we label the vertices of x as $v_0, \ldots, v_n$, then we think of d_kx as the face opposite to (or not containing) vertex v_k.

Since there are spaces that do not admit simplicial approximations, the notion of a geometrical simplicial complex has been generalized to the notion of a simplicial set. A *simplicial set* consists of a sequence of sets X_n whose elements are thought of as abstract n-simplices. For every positive integer n, there are face maps

$$d_i^n \colon X_n \to X_{n-1}, \quad i = 0, 1, \ldots, n.$$

An element $d_i^n(x)$ is called the i-th face of a simplex x. Contrary to the geometrical simplicial complex, simplices of a simplicial set do not need to have distinct faces. Also, for each nonnegative n there are degeneracy maps

$$s_j^n \colon X_n \to X_{n+1}, \quad j = 0, 1, \ldots, n.$$

If x is a 1-simplex of X, then we think of the 2-simplices $s_0^2(x)$ and $s_1^2(x)$ as triangles that were degenerated to a segment (so two vertices are in one endpoint of x and the third point is in the other endpoint of x). Allowing the faces to coincide and considering simplices as degeneracies of lower-dimensional simplices makes it possible to approximate more spaces by simplicial sets than by simplicial complexes.

There is a more compact definition of a simplicial set using the language of categories. Let Δ denote the category with exactly one object $[n]$ for each nonnegative integer n, the finite linear order $[n] = \{0 < 1 < \ldots < n\}$. The morphisms in Δ are the nondecreasing functions. Note that among the morphisms, we have the injections

$$\partial_n^i \colon [n-1] \to [n], \quad i = 0, 1, \ldots, n$$

which are uniquely determined by not having i in the image. We call them elementary face maps. Also, there are elementary degeneracy maps

$$\sigma_n^j \colon [n] \to [n+1], \quad j = 0, 1, \ldots, n$$

which are the unique surjections sending j and $j+1$ to j.

It is easy, but essential to see that every morphism in the category Δ is a composition of the elementary face and degeneracy maps. Hence, a functor $X \colon \Delta^{op} \to \mathrm{Set}$ is determined by its value on the objects and on the elementary face and degeneracy maps. If we denote

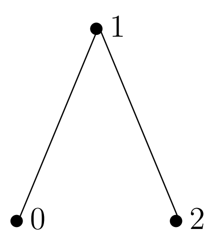
$$X([n]) = X_n, \quad X(\partial_n^i) = d_i^n, \quad X(\sigma_n^i) = s_j^n,$$

we see that such a functor is exactly a simplicial set. The morphisms between simplicial sets are natural transformations between functors and we denote the category of simplicial sets by sSet. Because of their nice categorical properties and the combinatorial flavour, simplicial sets form a convenient context for doing homotopy theory.

1.2.4 Homotopy theory of simplicial sets

To give a combinatorial definition of a homotopy group one restricts to a special kind of simplicial sets, as was done by D. Kan in [Kan57]. First, we must introduce some terminology. For each n there is a simplicial set $\Delta[n] = \mathrm{Hom}_\Delta(-, [n])$ called a representable simplicial set. The Yoneda lemma implies that there is a bijection $X_n \cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta[n], X)$, i.e. any simplex $x \in X_n$ can be thought of as map $x \colon \Delta[n] \to X$. In particular, each elementary face map $[n-1] \to [n]$ is an $(n-1)$-simplex of $\Delta[n]$, so it corresponds to a map $\Delta[n-1] \to \Delta[n]$. The union of the images of all elementary face maps form a subobject $\partial\Delta[n]$ of $\Delta[n]$ which we call the *boundary*. If we omit the image of the k-th face map from the boundary we get the *horn* $\Lambda^k[n]$. Here is an example:

$\Lambda^1[2] =$



A map $\Lambda^k[n] \to X$ corresponds to the union of simplices $x_i \in X_{n-1}, i \neq k$ satisfying the compatibility condition $d_j x_i = d_{i-1} x_j$ for all $j < i$ and $j, i \neq k$.

A *Kan complex* is a simplicial set X such that for every map $\Lambda^k[n] \to X$ there is an extension $\Delta[n] \to X$. We say that X admits fillers for all horns and write the diagram

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & X \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

Quillen showed that the category of simplicial sets admits a model structure which is equivalent to the model structure on topological spaces. We will call this model structure on simplicial sets the *Kan-Quillen model structure*. The adjoint functors exhibiting that equivalence are the geometric realization functor which assigns to each simplicial set X its realization $|X|$ as a topological space, and the singular complex functor which assigns to each topological space Y the simplicial set $\mathrm{Sing}(Y)$ given by

$$\mathrm{Sing}(Y)_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta[n]|, Y).$$

In the Kan-Quillen model structure the weak equivalences are those maps which induce weak homotopy equivalences under the geometric realization functor. The cofibrations are exactly the monomorphisms, and this class is the smallest class closed under pushouts and transfinite compositions which contains boundary inclusions $\partial\Delta[n] \to \Delta[n]$. All simplicial sets are cofibrant in this model structure because every simplicial set can be obtained by inductively glueing in simplices along their boundary. The fibrant objects are exactly the Kan complexes.

We have mentioned that every topological space is weakly equivalent to a CW-complex. In fact, a geometric realization of every simplicial set is a CW-complex and Quillen showed that for a topological space Y one CW-approximation is given by $|\mathrm{Sing}(Y)| \to Y$. On the other hand, every singular complex of a topological space is a Kan complex and any simplicial set X is weakly equivalent to the Kan complex $\mathrm{Sing}(|X|)$. In Chapter 2 we will mention Quillen's small object argument which gives a procedure how to replace a simplicial set with a weakly equivalent Kan complex "combinatorially", i.e. without referring to topological spaces.

Simplicial approximations make some problems of algebraic topology more approachable, but straightforward calculations can contain very complicated combinatorial arguments. Arguments involving the combinatorics of horns become more conceptual if we use anodyne extensions as introduced in [GZ67]. By definition, anodyne extensions are the elements of the smallest class closed under pushouts, transfinite compositions and retracts which contains all horn inclusions $\Lambda^k[n] \to \Delta[n]$. In the Kan-Quillen model structure, anodyne extensions are exactly those cofibrations that are also weak equivalences.

The category sSet admits a tensor product defined by $(X \times Y)_n = X_n \times Y_n$. With this tensor product, sSet is a closed symmetric monoidal category. More precisely, for simplicial sets X and Y there is a simplicial set $\underline{\mathrm{hom}}(X, Y)$ given by $\underline{\mathrm{hom}}(X, Y)_n = \mathrm{Hom}(X \times \Delta[n], Y)$

such that for all simplicial sets X, Y and Z there is a natural bijection

$$\mathrm{Hom}_{\mathrm{sSet}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathrm{sSet}}(X, \underline{\mathrm{hom}}(Y, Z)).$$

The *pushout-product property* relates the model structure and the tensor product: for a monomorphism $A \to B$ and an anodyne extension $K \to L$, the canonical map

$$A \times L \sqcup_{A \times K} B \times K \to B \times L$$

is an anodyne extension, too. The pushout-product property follows formally from the special case for the generating maps, i.e. the boundary and horn inclusions. For the special case one needs to provide an explicit combinatorial proof. Important consequence of this is that the simplicial sets $\underline{\mathrm{hom}}(X, Y)$ are compatible with the model structure. For example, $\underline{\mathrm{hom}}(X, Y)$ is a Kan complex whenever Y is a Kan complex.

1.2.5 Higher categorical point of view

The tools of homotopy theory also apply to the study of categories through simplicial methods. Any linear order $[n]$ can be thought of as a category. The objects of that category are the numbers $0, 1, \dots, n$ and there is a unique map from i to j if and only if $i \leq j$. The nerve of a category $\mathcal{C}$ is a simplicial set $N(\mathcal{C})$ given by $N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$. In other words, an n-simplex of the nerve is a sequence

$$c_0 \to c_1 \to \ldots \to c_n$$

of n composable morphisms in $\mathcal{C}$. For $0 < k < n$, the k-th face is obtained by composing the two maps $c_{k-1} \to c_k \to c_{k+1}$, while for $k = 0$ or $k = n$, we delete one map on the corresponding end of the sequence.

A horn $\Lambda^1[2] \to N(\mathcal{C})$ is given by two morphisms $f\colon c_0 \to c_1$ and $g\colon c_1 \to c_2$. Such a pair is at the same time a 2-simplex of $N(\mathcal{C})$. This means that for any horn $\Lambda^1[2] \to N(\mathcal{C})$ there is a unique filler $\Delta[2] \to N(\mathcal{C})$. One can think of this as saying that the three faces of a 2-simplex are the maps

$$f\colon c_0 \to c_1, \quad g\colon c_1 \to c_2 \quad \text{and} \quad gf\colon c_0 \to c_2.$$

Given f and g, the map gf is uniquely determined and the 2-simplex is witnessing that gf is the composition of f and g. Moreover, by associativity of the composition it follows that the nerve of a category admits unique fillers for all horns $\Lambda^k[n] \to N(\mathcal{C})$ with $0 < k < n$ and $n \geq 1$. We say that $N(\mathcal{C})$ is a strict inner Kan complex. The word strict emphasizes that the existing fillers are unique, while the word inner refers to the fact that the fillers do not necessarily exist for $k = 0$ or $k = n$. Indeed, the nerve $N(\mathcal{C})$ has fillers for outer horns if and only if all morphisms in $\mathcal{C}$ are invertible, i.e. if $\mathcal{C}$ is a groupoid.

This suggests an approach to homotopy coherent categories, i.e. structures similar to categories in which the composition is not strictly associative, but only up to a coherent homotopy. We say that a simplicial set X is an *inner Kan complex* if it admits fillers of all inner horns $\Lambda^k[n] \to X$, $0 < k < n$ and $n \geq 1$.

Historically, the first such example has been studied by Boardman and Vogt. They noticed that the weak A_∞-maps between A_∞-spaces can not be organized into a category because there is no way to compose weak A_∞-maps so that the composition is strictly associative. Instead, there is an inner Kan complex whose 0-simplices are A_∞-spaces, 1-simplices are weak A_∞-maps and higher simplices correspond to higher homotopy coherences for the composition of such maps.

In the last two decades inner Kan complexes have been intensively studied by A. Joyal under the name quasi-categories ([Joy08]) and J. Lurie under the name ∞-categories ([Lur09]). In particular, Joyal showed that the category of simplicial sets admits a model structure for which the cofibrations are monomorphisms, while the fibrant objects are exactly the quasi-category. The Kan-Quillen and the Joyal model structure on simplicial sets are models for two different homotopy theories. They have the same cofibrations, but there are more weak equivalences in the Kan-Quillen structure.

Quasi-categories provide just one of many models for homotopy coherent categories. Here, under the notion of a homotopy coherent category we refer to an abstract idea or a philosophy which is realized in the specific models. It depends on the context which model is used, but all models should share the following basic principles. First of all, a homotopy coherent category is a structure which consists of 0-cells (the objects), 1-cells (the morphisms), 2-cells (the morphisms between morphisms) and so on to infinity. Secondly, for any two composable 1-cells $f\colon x \to y$ and $g\colon y \to z$ there is a 2-cell witnessing that there is a candidate $h\colon x \to z$ for a composition of f and g. The uniqueness of the composition for ordinary categories is replaced by the requirement that the space of all these candidates is contractible, i.e. homotopy equivalent to a point. The third principle states that all n-cells for $n > 1$ are invertible. In the model given by the inner Kan complexes this follows from the lifting property with respect to horns. Because of these principles, the term $(\infty, 1)$-category is also in use for a homotopy coherent category.

1.2.6 Simplicially enriched categories

Another model for homotopy coherent categories is given by the theory of simplicially enriched categories. A *simplicially enriched category* is a category $\mathcal{C}$ such that for any two objects X and Y in $\mathcal{C}$ there is a simplicial set $\underline{\hom}(X, Y)$ of morphisms. We will denote by $\mathrm{sCat} := \mathrm{Cat}(\mathrm{sSet})$ the category of simplicially enriched categories. We already mentioned one example; at the end of section 1.2.4 we have seen a simplicial enrichment for the category sSet.

For the simplicial set $\underline{\hom}(X, Y)$, the 0-simplices are maps between X and Y, the 1-simplices are homotopies between maps and the higher simplices correspond to higher homotopy coherences. So if we think of n-simplices of $\underline{\mathrm{hom}}(X,Y)$ as the $(n+1)$-cells of $\mathcal{C}$, we see that the first principle of homotopy coherent categories is satisfied. On the other hand, the third principle is satisfied only if all $\underline{\mathrm{hom}}(X,Y)$ are Kan complexes. In that case we say that $\mathcal{C}$ is enriched in Kan complexes. So, the categories enriched in Kan complexes give a model for homotopy coherent categories. Concerning the second principle, note that the composition in this model is strictly associative. This might be too restrictive for some applications, but it might be an advantage in other.

Nevertheless, this model is equivalent to Joyal's model as we now explain. In [Ber04], J. Bergner showed that the category sCat admits a model structure in which the fibrant objects are exactly the categories enriched in Kan complexes. The first proof that the Joyal model structure on sSet and the Bergner model structure on sCat are equivalent was based on comparing both models to other intermediate models (namely, the Segal categories of C. Simpson, [HS98], and the complete Segal spaces of C. Rezk, [Rez91]).

Simplicially enriched categories can be directly related to simplicial sets by another nerve construction (defined by Cordier and Porter, [CP86]). To describe this functor, we first define simplicially enriched categories $W([n])$. The objects of the category $W([n])$ are numbers $0, 1, \ldots, n$. For $0 \leq i < j \leq n$, we define a poset

$$P_{i,j} = \{I \subset \{i, i+1, \ldots, j\} : i, j \in I\}$$

ordered by the inclusions of subsets. The simplicial enrichment in $W([n])$ is given by the nerves of these posets, i.e. $\underline{\mathrm{hom}}(i,j) = NP_{i,j}$. The composition is induce by taking the unions of subsets. The *homotopy coherent nerve* functor $hcN\colon \mathrm{sCat} \to \mathrm{sSet}$ is defined by

$$hcN(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{sCat}}(W([n]), \mathcal{C}).$$

The functor hcN admits a left adjoint functor $hc\tau\colon \mathrm{sSet} \to \mathrm{sCat}$. The pair $(hc\tau, hcN)$ forms a Quillen equivalence as was proven by Joyal, Tierney and Bergner. A more direct approach is given by J. Lurie in [Lur09].

Our notation comes from the fact that we can, in a somewhat unorthodox way, view the enriched categories as the enriched operads with only unary operations. The simplicial categories $W([n])$ are given by the Boardman-Vogt resolution (as studied by Berger and Moerdijk in [BM07]) of the category $[n]$ thought of as a (discrete) simplicially enriched operad.

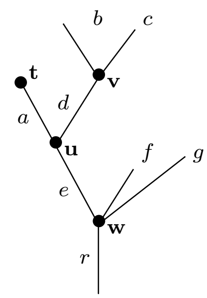
Building on the work of Joyal, Lurie developed ∞-categorical analogues of many aspects of category theory. Moreover, Lurie showed that the approach based on quasi-categories makes the tools of homotopy theory applicable to algebraic geometry (*brave new algebra*, [Lur11]). All of this made quasi-categories ubiquitous in modern homotopy theory and a subject of research in its own right.

1.3 Dendroidal sets

1.3.1 The category Ω of trees

Dendroidal sets have been introduced by I. Moerdijk and I. Weiss ([MW07]) in order to develop a theory of homotopy coherent operads analogously to Joyal's approach to homotopy coherent categories.

The main idea is to extend the category Δ of linear orders to a larger category Ω of finite rooted trees. We will give a precise definition in Chapter 3. Here is an example of one such tree:

$T =$ 

This tree has eight edges: a root r, inner edges a, d, e and leaves b, c, f, g. It has four vertices t, u, v, w. To each such tree T we associate a coloured operad $\Omega(T)$ whose colours are the edges of T and whose operations are generated by the vertices of T. For example, there is an operation $u \in \Omega(T)(a, d; e)$. Since $\Omega(T)$ is an operad it has identities and operations generated by composition and the action of the symmetry group. So, among many others, there is an operation

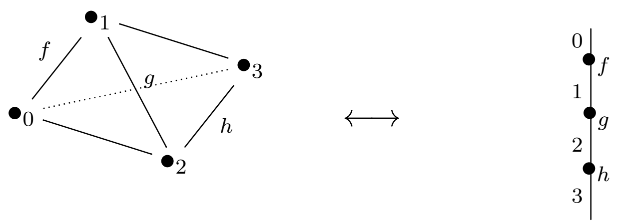
$$w \circ (u, 1_f, 1_g) \in \Omega(T)(a, d, f, g; r).$$

Also, there is an operation $\tau^* v \in \Omega(c, b; d)$ which is the image of the operation $v \in \Omega(b, c; d)$ under the action of the transposition $\tau \in \Sigma_2$.

The morphisms between trees T and S in Ω are the morphisms of operads between $\Omega(T)$ and $\Omega(S)$. Hence we consider the category Ω as a subcategory of the category Oper of coloured operads.

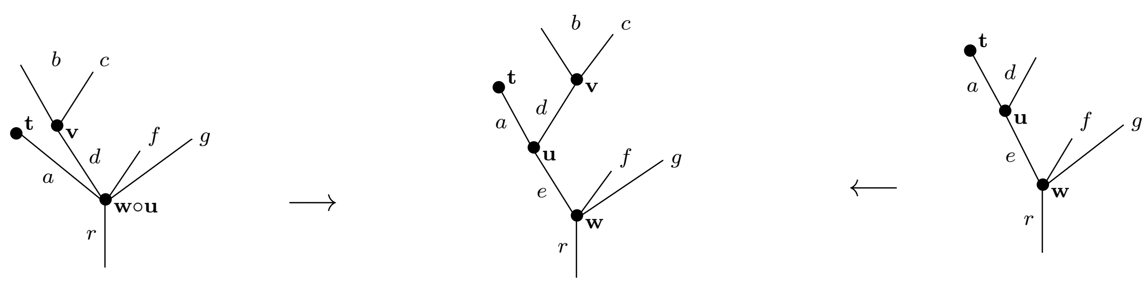
There is an inclusion $i \colon \Delta \to \Omega$ given by sending a linear order $[n]$ to the linear tree L_n with n vertices. Let us elaborate on this point as it demands a change of perspective. One usually thinks about the simplices geometrically as generalized tetrahedra. A geometrical n-simplex has $n+1$ vertices, but it correspond to a linear tree with $n+1$ edges. So, the vertices of the simplex become edges of the tree and the vertices of the trees correspond only to those edges of the simplex connecting consecutive vertices.

We give a picture for the case $n = 3$ in which the reader should notice the duality between the vertices and the edges:



The theory of dendroidal sets is similar to the theory of simplicial sets in many respects. As in Δ, in the category Ω we can also consider elementary face and degeneracy maps. If we look at the above picture carefully we can see that the inner faces of $\Delta[n]$ correspond to contracting an edge of the linear tree, while the outer faces correspond to chopping off the top or the bottom vertex. One can notice the resemblance to the nerve construction.

This is generalized to trees in the way that by now might be obvious. We will describe the elementary face maps precisely in Section 3.1 and discuss them in more details in Section 4.2. Let us just give an example of one inner face map (the arrow on the left) and one top face map (the arrow on the right).



Note that the tree in the middle also has a bottom face which is obtained by chopping off the bottom vertex w and all edges attached to it except the inner edge e. The inner edge e is the root of this face. On the other hand the tree on the left does not have a bottom face because the bottom vertex $w \circ u$ is attached to two inner edges (a and d) and chopping this vertex off would yield two disconnected trees.

One very important difference to Δ is that in Ω objects can have non-trivial automorphisms. For example, if a tree has one vertex with n inputs, then the group of automorphisms of that tree is the symmetric group Σ_n. More generally, the automorphisms are generated by the permutations of inputs of vertices respecting the structure of the subtrees above these inputs. One can show that every morphism in Ω can be written as a composition of elementary face maps, elementary degeneracy maps and isomorphisms. Dealing with the isomorphisms adds an additional layer to the complexity of the theory.

1.3.2 Dendroidal sets and operads

A *dendroidal set* is a functor $X\colon \Omega^{op} \to \mathrm{Set}$. For a tree T, elements of the set $X_T = X(T)$ are called dendrices of X of shape T. The morphisms of dendroidal sets are the natural transformations and we let dSet denote the category of dendroidal sets. We can also consider the representable dendroidal sets, which we denote by $\Omega[T] = \mathrm{Hom}_\Omega(-, T)$. The Yoneda lemma implies that the dendrices $x \in X_T$ correspond to maps $x\colon \Omega[T] \to X$.

Now let us relate the following four categories: Cat, Oper, sSet and dSet. The inclusion $i\colon \Delta \to \Omega$ induces a pair of adjoint functors $(i_!\colon \mathrm{sSet} \to \mathrm{dSet}, i^*\colon \mathrm{dSet} \to \mathrm{sSet})$. Moreover, by the functor $i_!$ the category of simplicial sets is embedded in the category of dendroidal sets. If X is a dendroidal set, we say that i^*X is its underlying simplicial set.

For simplicial sets we have considered the nerve functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$. For coloured operads, there is an analogously defined functor $N_d\colon \mathrm{Oper} \to \mathrm{dSet}$ which we call the *dendroidal nerve functor*. For an operad P it is given by

$$N_d(P)_T = \mathrm{Hom}_{\mathrm{Oper}}(\Omega(T), P).$$

Both functors N and N_d admit left adjoint functors. We denote the left adjoint of N by τ and the left adjoint of N_d by τ_d. These adjunctions fit in the following diagram

$$\begin{tikzcd}
\mathrm{sSet} \arrow[r, shift left, "\tau"] \arrow[d, shift left, "i_!"] & \mathrm{Cat} \arrow[l, shift left, "N"] \arrow[d, shift left, "j_!"] \\
\mathrm{dSet} \arrow[r, shift left, "\tau_d"] \arrow[u, shift left, "i^*"] & \mathrm{Oper} \arrow[l, shift left, "N_d"] \arrow[u, shift left, "j^*"]
\end{tikzcd}$$

where we have that $i^*N_d = Nj^*$, $i_!N = N_d j_!$ and $\tau_d i_! = j_!\tau$, but $j^*\tau_d \neq \tau i^*$ (for a counterexample, see 3.1.5. in [MT10]).

Another important aspect is the tensor product and the simplicial enrichment coming from it. The category of simplicial sets is simplicially enriched and the enrichment came from the Cartesian product $X \times Y$. As any functor category, the category of dendroidal sets also admits a monoidal structure coming from the Cartesian product, but we will consider another tensor product which is closely related to operads. In Chapter 2, we will give the definition of the Boardman-Vogt tensor product $\otimes_{BV}$ on the category of coloured operads. This tensor product is designed so that a $P \otimes_{BV} Q$-algebra is equipped with a structure of a P-algebra and a structure of a Q-algebra in a compatible way (in the sense that they have a common unit and distribute over each other).

For two representable dendroidal sets, the tensor product is defined by

$$\Omega[T] \otimes \Omega[S] = N_d(\Omega(T) \otimes_{BV} \Omega(S)).$$

One can apply the usual arguments of category theory to define the binary operation $\otimes$ for arbitrary dendroidal sets (using that every dendroidal set is a colimit of representables).

This binary tensor product is not associative, but it can be extended to an unbiased symmetric tensor product (for details, see Section 6.3 of [HHM13]). Moreover, the category of dendroidal sets is weakly enriched in simplicial sets (see Section 3.5 loc. cit.). For dendroidal sets X and Y, there is a simplicial set given by

$$\underline{\mathrm{hom}}(X, Y)_n = \mathrm{Hom}_{\mathrm{dSet}}(X \otimes i_!\Delta[n], Y)$$

and there are bijections

$$\mathrm{Hom}_{\mathrm{dSet}}(X \otimes i_!K, Y) \cong \mathrm{Hom}_{\mathrm{sSet}}(K, \underline{\mathrm{hom}}(X, Y))$$

which are natural in the dendroidal sets X and Y and the simplicial set K. This simplicial enrichment is of great use in building different model structures on dendroidal sets and it will be used in many parts of this thesis. We discuss model categories that are weakly enriched in simplicial sets in Section 2.3 and the tensor product on dendroidal sets in Section 3.1.

The theory of dendroidal sets is set up so that many concepts from the simplicial setting can be directly translated to the dendroidal setting. Nonetheless, the theory of dendroidal sets has a much richer structure so proving properties usually comes down to combinatorial statements for which new ideas must be used. In most cases, the results we know for simplicial sets might be used as a checking point for new ideas or a source of inspiration, but often a more general argument devised for the study of dendroidal sets has the case of simplicial sets as a direct consequence.

1.3.3 The operadic model structure

The category of dendroidal sets has a model structure that is related to operads as the Joyal model structure is related to categories. We first describe the cofibrations.

Since there is a notion of a face of a representable dendroidal set $\Omega[T]$, we can also define the boundary $\partial\Omega[T]$ and the horns $\Lambda^f[T]$ in the same way as we did for simplicial sets. We consider the smallest class of morphisms containing the boundary inclusions $\partial\Omega[T] \to \Omega[T]$ and closed under the pushouts, retracts and transfinite compositions. Because trees have non-trivial automorphisms this class is not the class of all monomorphisms, but a smaller class of *normal monomorphisms*. The objects which can be obtained by glueing in cells along their boundaries are exactly those objects for which the action of the automorphisms on dendrices is free. These objects are called normal dendroidal sets. In the category of simplicial sets we have considered model structures in which cofibrations are monomorphisms and hence all objects are cofibrant. In the case of dendroidal sets we are interested in the model structures for which the cofibrations are normal monomorphisms and so only the normal objects will be cofibrant.

Next, we turn to discuss fibrant objects. An inner horn $\Lambda^f[T]$ is obtained from the boundary $\partial\Omega[T]$ by removing a face corresponding to an inner edge f of T. In [MW07], Moerdijk and Weiss showed that a dendroidal set X is the dendroidal nerve of an operad if and only if there are unique fillers for all inner horns $\Lambda^f[T] \to X$. In analogy with ∞-categories, they have defined *inner Kan dendroidal set* (or ∞-operads) as dendroidal sets having fillers for all inner horns.

Inner Kan dendroidal sets are models for homotopy coherent operads. In [CM13a], D.-C. Cisinski and I. Moerdijk have shown that the category of dendroidal sets is endowed with a model structure such that the fibrant objects are exactly the inner Kan dendroidal sets. We will refer to this model structure as the *operadic model structure*. The "restriction" of this model structure to simplicial sets gives exactly the Joyal model structure.

Another model for homotopy coherent operads is given by simplicially enriched operads. The two models can be compared by a homotopy coherent version of the dendroidal nerve functor. Let us denote the category of simplicially enriched operads by sOper. As we noticed already in the case of categories, the idea is to apply the Boardman-Vogt resolution to the operads $\Omega(T)$ (considered as discrete simplicially enriched operads) and obtain simplicially enriched operads $W(\Omega(T))$. Extending this to dSet gives a pair of adjoint functors ($hc\tau_d\colon \mathrm{dSet} \to \mathrm{sOper}, hcN_d\colon \mathrm{sOper} \to \mathrm{dSet}$). The *dendroidal homotopy coherent nerve* functor hcN_d is given on a simplicially enriched operad P by

$$hcN_d(P)_T = \mathrm{Hom}_{\mathrm{sOper}}(W(\Omega(T)), P).$$

In subsequent papers, Moerdijk and Cisinski introduced two more models for homotopy coherent operads and showed that all of these models are equivalent. In particular, they have constructed a model structure on simplicially enriched coloured operads and compared the mentioned structures on dSet and sOper to intermediate models of Segal operads and complete dendroidal Segal spaces. The proofs that all four model structures are Quillen equivalent appear as Corollary 6.7 and Theorem 8.15. in [CM13b] and Theorem 8.15. in [CM11].

In [Lur11], Lurie developed a model for homotopy coherent operads in terms of certain fibrations of simplicial sets. The price of working with simplicial sets is paid by an extra layer of complexity. Advantages of that approach can be seen from the applications given in loc. cit., e.g. the proof of the "additivity" theorem for E_n-operads. In [HHM13], Heuts, Hinich and Moerdijk proved that the dendroidal approach is equivalent to Lurie's. This indicates that one could look for different proofs of Lurie's results using the dendroidal setting that might shed a different light on these results.

1.3.4 Homotopy theory of homotopy coherent algebras

Our next goal is to consider the homotopy theory of algebras over an operad from the perspective of dendroidal sets. First, we will take a step back and say something about the corresponding situation in the simplicial context.

If we consider a category C enriched in $\mathcal{E}$ as an operad, then algebras over C are just functors $A\colon C \to \mathcal{E}$ (also called *diagrams* in $\mathcal{E}$). The study of (limits and colimits of) diagrams is one of the basic topics in category theory. Also, the study of homotopy coherent diagrams is an important topic in homotopy theory.

In [Lur09], Lurie considered (homotopy coherent) colimits and limits of diagrams of ∞-categories and of Kan complexes. His approach is based on generalizing the Grothendieck construction to ∞-categories. For an ordinary category $\mathcal{C}$, the classical Grothendieck construction gives a correspondence between (pseudo)functors $\mathcal{C} \to \mathrm{Cat}$ and functors $\mathcal{D} \to \mathcal{C}$ of particular type being "fibrations of categories".

Let us give slightly more details about the case of homotopy coherent diagrams in Kan complexes. For an ∞-category S, we can consider the simplicially enriched category $hc\tau(S)$. We think of a functor $hc\tau(S) \to \mathrm{sSet}$ as a homotopy coherent diagram over S with values in simplicial sets. The category of all functors $hc\tau(S) \to \mathrm{sSet}$ admits a model structure based on the Kan-Quillen model structure on sSet. In that model structure, fibrant diagrams are exactly those having values in Kan complexes.

Lurie showed that this model structure is Quillen equivalent to a model structure on the category $\mathrm{sSet}/_S$. Here, the category $\mathrm{sSet}/_S$ is the slice category whose objects are maps $X \to S$ in sSet and a morphism from $f\colon X \to S$ to $g\colon Y \to S$ is given by a map $h\colon X \to Y$ such that $gh = f$. In this model structure on $\mathrm{sSet}/_S$ the cofibrations are monomorphisms and the fibrant objects are the maps $X \to S$ called left fibrations. A *left fibration* is a map which admits lifts for all horns $\Lambda^k[n] \to \Delta[n]$ with $0 \leq k < n$. This shows that the homotopy theory of homotopy coherent diagrams in Kan complexes can be studied on a smaller model realized by left fibrations in simplicial sets. We wish to comment the analog of this result in the operadic context.

Results of Berger and Moerdijk (in [BM07]) imply that there is a model structure on $\mathrm{Alg}_{\mathrm{sSet}}(hc\tau_d(S))$ for any normal dendroidal set S. Generalizing Lurie's approach to dendroidal sets, Heuts considered a *straightening functor*

$$St_S\colon \mathrm{dSet}/_S \to \mathrm{Alg}_{\mathrm{sSet}}(hc\tau_d(S)).$$

In [Heu11a], it was proven that $\mathrm{dSet}/_S$ admits a model structure, which is called the *covariant model structure*. The cofibrations are the normal monomorphisms and the fibrant objects are left fibrations $X \to S$. Analogously to simplicial sets, a left fibration of dendroidal sets is a map that admits lifts for all horns corresponding to an inner or a top face. Moreover, Heuts showed that St_S admits a right adjoint Un_S and that this pair of functors forms a Quillen equivalence.

Let us comment on the case when S is the nerve of an operad P, i.e. $S = N_dP$. The result above says that if we wish to study the homotopy theory of homotopy coherent P-algebras, we can consider another model for the same theory in which we do not need to resolve the operad P. That other model is given by the covariant model structure on the category $\mathrm{dSet}/_{N_dP}$. The instance where $P = \mathrm{Comm}$ (the operad for commutative monoids) is particularly interesting. As $N_d\mathrm{Comm}$ is the terminal dendroidal set, the category $\mathrm{dSet}/_{N_d\mathrm{Comm}}$ is isomorphic to dSet. We have already discussed that the homotopy coherent commutative algebras are the E_∞-algebras. We write E_∞ for the simplicially enriched operad obtained as the Boardman-Vogt resolution of Comm, i.e.

$$E_\infty = hc\tau_d(N_d\mathrm{Comm}) = W(\mathrm{Comm}).$$

To show that dendroidal sets model E_∞-algebras, let us recall one more notion that Heuts introduced. A dendroidal set X is called a *dendroidal Kan complex* if it admits fillers for all inner horns and all top horns. In [Heu11b], Heuts emphasizes the following results. Under the identification of dSet with $\mathrm{dSet}/_{N_d\mathrm{Comm}}$, the covariant model structure on the category of dendroidal sets has normal monomorphisms as cofibrations and dendroidal Kan complexes as fibrant objects. This model structure is Quillen equivalent to the model structure for E_∞-algebras in simplicial sets.

Hence, dendroidal sets provide many examples of infinite loop spaces. We can take a dendroidal set and “straighten” it to a simplicial E_∞-algebra. Next, we may apply the procedure called the group–like completion to obtain a group–like E_∞-algebra and then apply the recognition principle for infinite loop spaces.

If an E_∞-algebra is not group–like, then the group completion will change its homotopy type, i.e. the resulting E_∞-algebra will not be homotopy equivalent to the starting one. For this reason it is natural to ask if it is possible to perform the group–like completion on the dendroidal side, i.e. to get a dendroidal model for the homotopy theory of group–like E_∞-algebras. One of the main results of this thesis is identifying such a model structure on dendroidal sets.

1.4 Main results and the organization of the thesis

1.4.1 The dendroidal group–like completion

So far we have seen that there are two model structures on dendroidal sets. The operadic model structure has inner Kan dendroidal sets as fibrant objects and it is Quillen equivalent to simplicially enriched operads. The covariant structure has dendroidal Kan complexes as fibrant objects and it is Quillen equivalent to E_∞-algebras in simplicial sets. The main results of this thesis are related to the construction of a third model structure which we call the stable model structure.

In simplicial sets, a Kan complex admits fillers for all horns. By definition, a dendroidal Kan complex admits fillers for inner horns and top horns. This implies, as shown in [Heu11b], that a dendroidal Kan complex also admits fillers for horns with respect to a bottom face when a bottom vertex is unary. Nonetheless, a bottom vertex might have more than one input (see the example on p. 17) and dendroidal Kan complexes in general do not admit fillers for all horns. Hence a new definition is in order.

A *fully Kan dendroidal set* is a dendroidal set which admits fillers for all horns. So, every fully Kan dendroidal set is a dendroidal Kan complex, but not vice versa. One of the main results of this thesis is the following theorem.

Theorem 1.4.1. *The category of dendroidal sets admits a simplicial cofibrantly generated model structure, called the stable model structure, in which the cofibrations are normal monomorphisms and the fibrant objects are the fully Kan dendroidal sets. The weak equivalences between fibrant objects are exactly those maps that induce weak homotopy equivalences on the underlying simplicial sets.*

This theorem has been motivated by the discussion about a potential geometric realization of dendroidal sets. The idea to search for a geometric realization by looking at fully Kan dendroidal sets is due to Urs Schreiber.

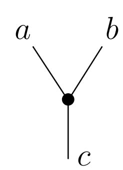
We will prove this theorem in two different ways as each way has its own advantages. One approach is more elementary in the sense that we build a model structure "from scratch" using standard homotopy theoretical arguments. We will discuss this approach in the next section. The other approach was chronologically considered earlier. It is based on the fact that the stable model structure is Quillen equivalent to a model structure on E_∞-algebras for which the fibrant objects are exactly the group–like E_∞-algebras. That result is a product of the joint work with Thomas Nikolaus and is published in [BN14].

An E_∞-algebra X is group–like if and only if the shear map

$$Sh\colon X \times X \to X \times X, \quad (x, y) \mapsto (x, xy)$$

is a weak equivalence (cf. the proof of Lemma 6.3.2). The main idea is that turning the shear map into a weak equivalence corresponds to turning a certain horn inclusion into a weak equivalence in the covariant model structure on dendroidal sets.

The tree with only one vertex which has 2 inputs is called a 2-corolla C_2. It has three faces corresponding to the inclusions of the trivial tree (the tree with no vertices) as each of the three edges. We denote the three edges a, b and c as in the picture



So, this tree has three horns

$$\Lambda^a[C_2] = b \sqcup c, \quad \Lambda^b[C_2] = a \sqcup c \quad \text{and} \quad \Lambda^c[C_2] = a \sqcup b.$$

Let us consider a dendroidal Kan complex X. Informally, we think of a horn $\Lambda^c[C_2] \to X$ as corresponding to a choice of two points x, y in X. Filling this horn to $\Omega[C_2] \to X$ gives a third point z and a binary operation p in X such that $z = p(x, y)$. The horn

$\Lambda^c[C_2]$ corresponds to a top face, so this shows that in a dendroidal Kan complex X we can multiply the points. On the other hand the horn $\Lambda^b[C_2] \to X$ gives us two points x and z and filling this horn to $\Omega[C_2] \to X$ corresponds to an operation p and a point y such that $p(x, y) = z$. Intuitively speaking, if a dendroidal Kan complex X has a lift for every horn $\Lambda^b[C_2] \to X$ then we get certain inverses for the multiplication resembling to the inverse of a shear map.

After presenting this very informal idea, here is how we construct a dendroidal model for group–like E_∞-algebras. We start with the covariant model structure of Heuts and turn the horn inclusion $\Lambda^b[C_2] \to \Omega[C_2]$ into a weak equivalence. The procedure of making a set of morphisms into weak equivalences is a standard tool in the theory of model categories and it goes by the name of Bousfield localization (note that in the associated homotopy category we make some morphisms invertible, so the homotopy category is localized in the sense we discussed in Section 1.2). In this way we obtain a new model category on the category of dendroidal sets with the same cofibrations as before but with less fibrant objects.

To formalize the idea above, we use the explicit description of Heuts' straightening functor mentioned earlier. Using an explicit calculation of the straightening and some model-theoretic arguments we establish a Quillen equivalence between the model for group–like E_∞-algebras and a certain model structure on dendroidal sets. Now we come to the most technical part. We claim that the fibrant objects of that model structure are the fully Kan dendroidal sets, i.e. that this model structure is exactly the stable model structure. To show this we will need to give a few non-trivial combinatorial proofs concerning the lifting properties with respect to horns. That will imply the following result.

Theorem 1.4.2. *The stable model structure on dendroidal sets is Quillen equivalent to a model structure on E_∞-algebras in simplicial sets in which the fibrant objects are exactly the group–like E_∞-algebras.*

1.4.2 An elementary construction of the stable model structure

Let us comment on a more direct approach to the construction of the stable model structure. We can do this using the simplicial enrichment of the category of dendroidal sets. The strategy of the proof is well-known and in Chapter 5 we will follow the presentation in [HHM13], where a model structure on forest sets is constructed in a similar way.

One advantage of this approach is that we can characterize fibrations between fibrant objects. We know that the stable model structure is cofibrantly generated, but we do not know whether the horn inclusions generate all trivial cofibrations. On the other hand fibrant objects (the fully Kan dendroidal sets) and fibrations between fibrant objects are characterized by the lifting property with respect to all horns. This makes it easier to prove that certain functors are a part of a Quillen pair (i.e. that they respect the properties of a model structure). We will use this in the last chapter of the thesis.

To perform the construction in this approach we require compatibility of the main ingredients of the model structure (the horn inclusions) with the simplicial enrichment on dendroidal sets. More concretely, we need the generalization of the pushout-product property to the case when one map is a map of dendroidal sets and the other is a map of simplicial sets. Proving the pushout-product property requires a careful consideration of the combinatorics of faces of trees.

One situation occurs many times in the proof - showing that an inclusion of a given subobject of a representable dendroidal set is a dendroidal anodyne extension (i.e. that it is in the closure of the horn inclusions by pushout and transfinite composition). To deal with this efficiently, we have found simple sufficient conditions one needs to check to conclude that. The main idea is very simple. Informally, if A is a subobject of a dendroidal set B and we wish to prove that the inclusion $A \to B$ is a dendroidal anodyne extension, we will try to show that all the "missing dendrices", i.e. dendrices in $B \setminus A$, can be added in a sequence of steps

$$A = A_1 \subseteq A_2 \subseteq \ldots \subseteq B,$$

such that in the n-th step we fill the horns in the subobject A_n. The sufficient conditions that we have identified enable us to consider "critical" pairs of the form

$$(b_f \colon \Omega[\partial_f T] \to B, b \colon \Omega[T] \to B)$$

of dendrices in $B \setminus A$ (where the dendrex b_f is a face of b) and order them so that in each step we can fill some of the horns of shape $\Lambda^f[T]$. We consider this to give a combinatorial technique that simplifies the proofs of the statements about dendroidal anodyne extensions that are already known (cf. Example 4.2.13, Theorem 4.3.1, Remark 4.3.4). More importantly, we apply it to obtain a new result (cf. Theorem 4.3.2).

1.4.3 Homology of dendroidal sets

We consider one more generalization of a well-known concept for simplicial sets to the dendroidal setting. Namely, we introduce the notion of homology of dendroidal sets that extends the singular homology. To consider singular homology for a simplicial set, one first considers the free abelian simplicial group generated by it. Then we can apply the Dold-Kan correspondence to obtain a (bounded) chain complex for which we calculate the homology. Let us be a bit more explicit about the functor from simplicial abelian groups to chain complexes. If X is a simplicial abelian groups, then X_n is the component of the chain complex in degree n and the differential $d \colon X_n \to X_{n-1}$ is given by the alternating sum of face maps

$$d(x) = \sum_{k=0}^{n} (-1)^k d_k^n(x).$$

One way to generalize this procedure was given by J. Gutierrez, A. Lukacz and I. Weiss. To obtain a differential as an alternating sum they use planar trees. A planar tree is a tree with the additional structure given by a linear order on the inputs of each vertex. Planar trees form a subcategory of the category of nonsymmetric operads. In [GLW11], the mentioned authors give a convention for associating a sign to each face of a planar tree. Based on that they develop a Dold-Kan correspondence between planar dendroidal abelian groups and the planar dendroidal chain complexes. A planar dendroidal chain complex is a graded abelian group which is graded by planar trees and this notion was defined in order to obtain an equivalence of categories just like in the case of simplicial abelian groups and genuine bounded chain complexes.

We take a different approach with a different goal. We want to consider all dendroidal sets (not only the planar version) and we want to construct a functor to (genuine) chain complexes. To obtain a grading by nonnegative integers, the idea is to generate the group in degree n of the associated chain complex by all dendrices that have the shape of a tree with n vertices. For the differential, we use the same convention for signs as in [GLW11], but we want to consider trees as symmetric operads. So, for a dendrex of shape of a tree T we consider multiple copies indexed by all possible planar structures of T. To ensure the functoriality of the construction under the action of non-trivial automorphisms of trees, we identify these copies with an additional sign. In that way we have found an approach that works for trees in general, not only for trees with planar structure.

After explaining conventions for signs, we can define a chain complex associated to a dendroidal set (cf. Definition 7.3.1). We will also consider a variant of this construction where we generate the chain complex only by (the isomorphism classes of) nondegenerate dendrices. For normal dendroidal sets, the two variants yield quasi-isomorphic chain complexes and hence the same homology.

Theorem 1.4.2 shows that dendroidal sets model group-like E_∞-spaces. As group-like E_∞-spaces are equivalent to connective spectra (as discussed in Section 1.1.3) it is natural to compare this new homotopy invariant of dendroidal sets to the standard homology of the corresponding spectra. Our main result shows that they coincide.

Theorem 1.4.3. *Homology of a normal dendroidal set is equivalent to the homology of the corresponding connective spectrum.*

This result also implies the following characterization of weak equivalences in the stable model structure.

Corollary 1.4.4. *A map between fully Kan dendroidal sets is a weak equivalence if and only if it induces isomorphisms in homology.*

1.4.4 Overview of the chapters

The thesis is organized into seven chapters. Following this introductory chapter, in Chapter 2 we present an overview of the general theory of model structures and in Chapter 3 we give the preliminaries on dendroidal sets. These two chapters serve to establish the terminology and to remind the reader of the notions and results that we intensively use in the rest of the thesis. There no new results in these chapters, but we present the material with an eye towards the applications in the latter chapters.

The Chapters 4-7 are devoted to the construction and further properties of the stable model structure on dendroidal sets and these chapters form the essential part of the thesis.

Chapter 4 is the most technical part of the thesis as it involves intricate combinatorial proofs about faces of trees. The main result of this chapter is the dendroidal pushout-product property, which is a basic ingredient for the construction of the stable model structure.

In Chapter 5 we construct the stable model structure. We follow the well-known model-theoretical arguments. The definitions that we make are set up in a way that ensures that we can characterize fibrant objects and fibrations between fibrant objects by the lifting property with respect to all horns.

The last two chapters are concerned about the relation of the stable homotopy theory of dendroidal sets to classical stable homotopy theory. In Chapter 6 we give a different construction of the stable model structure and show that it presents a model for group–like E_∞-spaces, i.e. connective spectra.

In Chapter 7 we introduce a notion of homology of a dendroidal set. We discuss two variants of associating a chain complex to a dendroidal set and we give some non-trivial examples. After that, we show that homology of a dendroidal set is the same as the standard homology of the corresponding connective spectrum.

Chapter 2

Background on axiomatic homotopy theory

In this chapter we recall the well-known results about model categories that will be used in the rest of the thesis. First we discuss relevant categorical properties (e.g. lifting and factorization) and give the definition of the Quillen model structure. Next we discuss examples. We give more details in cases which are important to us - the category of simplicial sets and the category of operads. The third part of this chapter is devoted to various properties and constructions related to model structures. In particular, we will discuss homotopy categories, Quillen functors, (weak) simplicial model categories and left Bousfield localizations.

2.1 Categorical preliminaries and the definition of a model category

2.1.1 Conventions about sets and categories

In this thesis we extensively use the language of category theory. So, we start with the set-theoretical conventions related to foundations of category theory. We will also recall a few essential results on (co)limits and adjunctions, but for more details on category theory we refer the reader to standard textbooks, such as [ML98], [AR94] and [AHS90]. For the theory of monoidal and enriched categories we refer to [Kel82].

Convention 2.1.1. We work with the formalism of Grothendieck universes. Hence we work with the ZFC[1] axiom system and assume the existence of at least one inaccessible cardinal. We fix an inaccessible cardinal α and consider the *universe* V_α. We say that the elements of the universe V_α are *small sets*. Sets which are not in V_α are called *proper classes* (or sometimes *large sets*).

[1]In particular, we assume the axiom of choice, use the well-ordering theorem and transfinite induction.

Convention 2.1.2. We consider *locally small categories*. So, a *category* has a class of objects and for each pair of objects X, Y a small set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms between them. A category is called *small* if its class of objects is a small set, as well. We denote by Set the category of small sets and by Cat the category of small categories.

Convention 2.1.3. For two categories $\mathcal{C}$ and $\mathcal{D}$, we consider functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations between such functors. If $\mathcal{C}$ is small, we call such a functor a *diagram* in $\mathcal{D}$ and there is a category of diagrams which we denote by $[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$.

Remark 2.1.4. If $\mathcal{C}$ is not small, then $\mathcal{D}^{\mathcal{C}}$ is not a category in the universe V_α, but we may assume existence of yet another inaccessible cardinal $\alpha' > \alpha$ such that $\mathcal{D}^{\mathcal{C}}$ is a category in a higher universe $V_{\alpha'}$.

Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a category (not necessarily small). For any object X of $\mathcal{D}$, there is a constant diagram $\Delta_{\mathcal{C}}(X)\colon \mathcal{C} \to \mathcal{D}$ which sends every object of $\mathcal{C}$ to X and every morphism to the identity 1_X. This defines a *constant diagram functor* $\Delta_{\mathcal{C}}\colon \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$.

Definition 2.1.5. A category $\mathcal{D}$ has $\mathcal{C}$*-limits* if there exists a functor $\lim_{\mathcal{C}}\colon \mathcal{C}^{\mathcal{D}} \to \mathcal{D}$ right adjoint to $\Delta_{\mathcal{C}}$. We say that $\mathcal{D}$ is *complete* if it has $\mathcal{C}$-limits for all small categories $\mathcal{C}$.

Analogously, $\mathcal{D}$ has $\mathcal{C}$*-colimits* if the constant diagram functor $\Delta_{\mathcal{C}}$ has a left adjoint $\mathrm{colim}_{\mathcal{C}}$ and it is *cocomplete* if it has $\mathcal{C}$-colimits for all small categories $\mathcal{C}$.

Limits and colimits in any functor category are calculated *pointwise*, hence $\mathcal{D}^{\mathcal{C}}$ is (co)complete if and only $\mathcal{D}$ is. Let $\hat{\mathcal{C}} = [\mathcal{C}^{op}, \mathrm{Set}]$ be the category of presheaves on a small category $\mathcal{C}$. The category Set is complete and cocomplete, so $\hat{\mathcal{C}}$ is, too. For an object X of $\mathcal{C}$ let h_X be the representable functor $h_X = \mathrm{Hom}(-, X)\colon \mathcal{C}^{op} \to \mathrm{Set}$.

Lemma 2.1.6 (Yoneda lemma)**.** *For any presheaf $F\colon \mathcal{C}^{op} \to Set$ and any object X in $\mathcal{C}$ there is a natural bijection* $\mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, F) \cong F(X)$*,* $\alpha \mapsto \alpha_X(1_X)$*.*

Yoneda lemma implies that there is an embedding of categories $h\colon \mathcal{C} \to \hat{\mathcal{C}}$. We will often construct adjunctions $\hat{\mathcal{C}} \leftrightarrows \mathcal{D}$ as Kan extensions along the Yoneda embedding h as implied by the following result.

Proposition 2.1.7. *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a cocomplete category. There is a bijection between the set of functors $F\colon \mathcal{C} \to \mathcal{D}$ and the set of colimit preserving functors $\hat{F}\colon \hat{\mathcal{C}} \to \mathcal{D}$. The functor $\hat{F}$ is the left Kan extension of F along the Yoneda embedding and has a right adjoint $\hat{G}\colon \mathcal{D} \to \hat{\mathcal{C}}$ acting on objects by*

$$\hat{G}(D)(C) = \mathrm{Hom}_{\mathcal{D}}(F(C), D).$$

We will also use the following simple observation.

Lemma 2.1.8. *Let $f\colon A \to B$ be a morphism in a category $\mathcal{C}$. The following statements are equivalent:*

i) Morphism f is an isomorphism.

ii) Map $f^*\colon \hom(B,X) \to \hom(A,X)$ is a bijection for every object X.

iii) Map $f_*\colon \hom(X,A) \to \hom(X,B)$ is a bijection for every object X.

Let λ be a small ordinal. A λ-*sequence* is a colimit preserving functor $Y\colon \lambda \to \mathcal{C}$, i.e. a diagram such that for every limit ordinal $\gamma < \lambda$ the canonical map

$$\operatorname*{colim}_{\beta<\gamma} Y_\beta \to Y_\gamma$$

is an isomorphism. The *composition* of Y is the morphism $Y_0 \to \operatorname{colim}_{\beta<\lambda} Y_\beta$.

Let κ be a small regular cardinal. An ordinal λ is κ-*filtered* if every κ-small subset (i.e. a subset with cardinality strictly smaller than κ) of λ has an upper bound.

Definition 2.1.9. Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. For a small regular cardinal κ, an object X of $\mathcal{C}$ is κ-*small relative to* $\mathcal{D}$ if for any κ-filtered ordinal λ and any λ-sequence $Y\colon \lambda \to \mathcal{C}$ such that each map $Y_\beta \to Y_{\beta+1}$ is in $\mathcal{D}$, the canonical map

$$\operatorname*{colim}_{\beta} \hom(X, Y_\beta) \to \hom(X, \operatorname*{colim}_{\beta} Y_\beta)$$

is a bijection. An object X is *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some small regular cardinal κ. An object X is *small* if it is small relative to the set of all morphisms in $\mathcal{C}$.

Example 2.1.10. Every small set X is $|X|$-small.

Definition 2.1.11. A category $\mathcal{C}$ is *locally presentable* if it is cocomplete, every object in $\mathcal{C}$ is small and there exists a small set S of objects of $\mathcal{C}$ which generates $\mathcal{C}$ under colimits.

Example 2.1.12. a) The category Set is locally presentable.

b) For a small category $\mathcal{C}$ and a locally presentable category $\mathcal{D}$, the functor category $\mathcal{D}^{\mathcal{C}}$ is locally presentable.

Theorem 2.1.13 (Adjoint Functor Theorem, [AR94], 1.66)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be locally presentable categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then*

i) F is left adjoint if and only if it preserves small colimits.

ii) F is right adjoint if and only if it preserves small limits and κ-filtered colimits for some regular cardinal κ.

Let $[n]$ be the linear order $\{0 < 1 < 2 < \cdots < n\}$ considered as a category with a unique morphism $i \to j$ if $i \leq j$. For a category $\mathcal{C}$, $\mathcal{C}^{[1]}$ is the category of arrows of $\mathcal{C}$ and $\mathcal{C}^{[2]}$ is the category of composable pairs of morphisms. Composing morphisms in $\mathcal{C}$ induces the functor $S\colon \mathcal{C}^{[2]} \to \mathcal{C}^{[1]}$, $(f,g) \mapsto g \circ f$.

Definition 2.1.14. Let $\mathcal{C}$ be a category. A *functorial factorization* in $\mathcal{C}$ is a functor $T\colon \mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$ right inverse to S.

2.1.2 Localization of categories

Equivalent categories have the same isomorphism classes of objects. The idea of homotopy theory is to consider a coarser equivalence relation on objects in the following way. We consider a category $\mathcal{M}$ with a class W of morphisms in $\mathcal{M}$ and we wish to construct a category that would have the same objects as $\mathcal{M}$, but in which elements of the class W are isomorphisms.

Definition 2.1.15. Let $\mathcal{M}$ be a category and W a class of morphisms in $\mathcal{M}$. A *localization* of $\mathcal{M}$ with respect to W is a category $M[W^{-1}]$ and a functor $\gamma\colon \mathcal{M} \to \mathcal{M}[W^{-1}]$ such that

i) if $f \in W$, then $\gamma(f)$ is an isomorphism;

ii) if $\mathcal{N}$ is a category and $F\colon \mathcal{M} \to \mathcal{N}$ such that for every $f \in W$, $F(f)$ is an isomorphism in $\mathcal{N}$, then there is a unique functor $\tilde{F}\colon M[W^{-1}] \to \mathcal{N}$ such that $F = \tilde{F}\gamma$.

Remark 2.1.16. A localization $\mathcal{M}[W^{-1}]$ is constructed by formally inverting morphisms in W. In general, this procedure gives a category in a universe which is possibly larger than the universe in which $\mathcal{M}$ is defined (i.e. it might be that the hom-sets are not small sets but proper classes). We say that a localization *exists* if $\mathcal{M}[W^{-1}]$ is a locally small category in the same universe in which $\mathcal{M}$ is.

Remark 2.1.17. The category $\mathcal{M}[W^{-1}]$ is defined by a universal property, so if it exists, it is unique up to an isomorphism.

Remark 2.1.18. We say that $(\mathcal{M}, W)$ is a *category with weak equivalences* if the class W contains all isomorphisms and satisfies the *two-out-of-three* property: if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms in C such that two of the three morphisms f, g and gf are in W, then so is the third.
In this setting the localization $\mathcal{M}[W^{-1}]$ is called the *homotopy category* of $\mathcal{M}$.

Example 2.1.19. Here are the motivating examples of categories with weak equivalences.

a) In homological algebra, we consider the class W of quasi-isomorphisms (i.e. maps inducing isomorphisms on homology) in the category of chain complexes. The localization of the category of chain complexes in an abelian category $\mathcal{A}$ is the *derived category* $D(\mathcal{A})$.

b) In topology, we consider the class W of weak homotopy equivalences (i.e. maps inducing isomorphisms on homotopy groups) between topological spaces. The localization is called the *homotopy category*. Note that this example motivates the terminology.

In Homotopical Algebra ([Qui67]), Daniel Quillen axiomatized homotopy theory by unifying the principles of homological algebra, homotopy theory of topological spaces and homotopy theory of simplicial sets. Quillen introduced the notion of a *model structure*. A model structure consists of a category with weak equivalences equipped with two additional classes of morphisms (called *fibrations* and *cofibrations*) satisfying certain factorization and lifting properties. We will first discuss these properties and then, in the latter parts of this chapter, we will outline Quillen's proof that this additional structure provides the existence of the homotopy category.

2.1.3 Factorization and lifting properties

In this section we recall the lifting properties and the small object argument which provides functorial factorizations.

Definition 2.1.20. Let $\mathcal{C}$ be a category and let $i\colon A \to B$ and $p\colon X \to Y$ be two morphisms in $\mathcal{C}$. We say that i has the *left lifting property* (LLP) with respect to p, and equivalently that p has the *right lifting property* (RLP) with respect to i, if for any pair of morphisms $(u\colon A \to X, v\colon B \to Y)$ rendering the solid square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "v"] \arrow[ur, dashed, "f"] & Y \end{tikzcd}$$

commutative, there exists a morphism $f\colon B \to X$, such that $u = fi$ and $v = pf$.

Definition 2.1.21. Let $\mathcal{C}$ be a category. A morphism $f\colon X \to Y$ of $\mathcal{C}$ is a *retract* of a morphism $g\colon A \to B$ if there is a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] & B \arrow[r] & Y \end{tikzcd}$$

where the horizontal composites are identities.

Lemma 2.1.22 (The Retract Argument, [Hov07] Lemma 1.1.9)**.** *Let $f\colon X \to Y$, $i\colon X \to Z$ and $p\colon Z \to Y$ be morphisms in a category $\mathcal{C}$ such that $f = pi$. If p has the RLP with respect to f, then f is a retract of i.*

Definition 2.1.23. For a small set I of morphisms in $\mathcal{C}$ we write $\mathrm{rlp}(I)$ (resp. $\mathrm{llp}(I)$) for the class of morphisms having the RLP (resp. the LLP) with respect to all morphism in I.

We write $\mathrm{cell}(I)$ for the class of transfinite compositions of pushouts of morphisms in I and we write $\mathrm{cof}(I)$ for the class of retracts of $\mathrm{cell}(I)$.

Definition 2.1.24. A class I of morphisms in a cocomplete category C is called *saturated* if it is closed under retracts, transfinite compositions and pushouts.

Proposition 2.1.25. *Let $\mathcal{C}$ be a complete and cocomplete category and let I be a class of morphisms in $\mathcal{C}$.*

i) The class $\mathrm{llp}(I)$ *is saturated.*

ii) The class $\mathrm{cof}(I)$ *equals the class* $\mathrm{llp}(\mathrm{rlp}(I))$.

Lemma 2.1.26. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors ($F \dashv G$). Let $i\colon A \to B$ be a morphism in $\mathcal{C}$ and $p\colon X \to Y$ a morphism in $\mathcal{D}$. Then i has the LLP with respect to Gp if and only if Fi has the LLP with respect to p.*

Definition 2.1.27. Let $\mathcal{C}$ be a cocomplete category. A small set I of morphisms *permits the small object argument* if the domains of the morphisms of I are small relative to *cell*(I).

Proposition 2.1.28 (The Small Object Argument, [Hov07] Theorem 2.1.14.)**.** *Let $\mathcal{C}$ be a cocomplete category and let I be a small set of morphisms that permits the small object argument. Then there is a functorial factorization T on $\mathcal{C}$, $T(f) = (\beta(f), \alpha(f))$,*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\alpha(f)"'] & & Y \\ & Z_f \arrow[ur, "\beta(f)"'] & \end{tikzcd}$$

such that $\alpha(f)$ is in $\mathrm{cell}(I)$ *and $\beta(f)$ is in* $\mathrm{rlp}(I)$.

2.1.4 Quillen model structures

In this section we give an overview of the theory of closed model categories and provide a list of results that we can refer to in the later chapter. We will omit most of the proofs as the subject is widely presented in the standard literature. A nice introduction to the subject is [DS95]. For further reference the reader may consult the fundamental papers [Qui67] and [Qui69] and more recent textbooks [Hov07] and [Hir02].

Definition 2.1.29 (D. Quillen, [Qui69])**.** A *(closed) model category* is a category $\mathcal{M}$ endowed with three classes of morphisms W, F and C satisfying the following axioms.

CM1. (Limit axiom) The category $\mathcal{M}$ is complete and cocomplete.

CM2. (Two out of three axiom) $(\mathcal{M}, W)$ is a category with weak equivalences, i.e. if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms in C such that two of the three morphisms f, g and gf are in W, then so is the third.

CM3. (Retract axiom) The classes W, C and F are closed under retracts.

CM4. (Lifting axiom) For morphisms $i \in C$ and $p \in F$, i has the LLP with respect to p whenever i or p is in W.

CM5. (Factorization axiom) Any morphism $f \in \mathcal{M}$ admits

a) a factorization $f = pi$ with $i \in C$ and $p \in F \cap W$;

b) a factorization $f = qj$ with $j \in C \cap W$ and $q \in F$.

We also say that the category $\mathcal{M}$ is endowed with a *model structure* given by the three classes W, C and F. The elements of the class W are called *weak equivalences*, of the class C *cofibrations* and of the class F *fibrations*. A cofibration (resp. a fibration) that is also a weak equivalence is called a *trivial cofibration* (resp. a *trivial fibration*).

Remark 2.1.30. In [Qui67], Quillen made a distinction between a notion of a model category and of a *closed* model category. We will only use closed model structures and omit the adjective closed.

Remark 2.1.31. In some references (e.g. [Hir02] and [Hov07]) the factorization in CM5 is required to be functorial. We do not make this assumption, but we want to emphasize that in the examples of our interest the factorizations are constructed by the Small Object Argument (Prop. 2.1.28) and hence they are functorial.

Lemma 2.1.32 ([Hov07] 1.1.10)**.** *In a model category, any two of the three classes W, C, F determine the third. More precisely,*

i) a morphism is a cofibration (resp. trivial cofibration) if and only if it has the LLP with respect to all trivial fibrations (resp. fibrations);

ii) a morphism is a fibration (resp. trivial fibration) if and only if it has the RLP with respect to all trivial cofibrations (resp. cofibrations);

iii) a morphism is a weak equivalence if and only if it factors as a trivial cofibration followed by a trivial fibration.

Note that Prop. 2.1.25 implies that the class of cofibrations and the class of trivial cofibrations are closed under pushouts. Dually, fibrations and trivial fibrations are closed under pullbacks.

Remark 2.1.33. The axioms of a model structure are *self-dual*, i.e. for a model category $\mathcal{M}$ the opposite category $\mathcal{M}^{op}$ is also a model category with the classes C and F interchanged. Hence every statement has a dual statement, e.g. the statements (i) and (ii) of Lemma 2.1.32 are dual to each other. For this reason, we will write only one of the two dual statements.

In section 2.3, we will outline the construction of the homotopy category associated to a model category. That outline will show that the "important information" for the construction is carried by the objects which are cofibrant and fibrant.

Definition 2.1.34. An object X of a model category is *cofibrant* if the unique morphism $0 \to X$ (from the initial object 0) is a cofibration. Dually, an object X is *fibrant* if the unique morphism $X \to 1$ (to the terminal object 1) is a fibration. An object X is *bifibrant* if it is fibrant and cofibrant.

We denote by $\mathcal{M}_c$ (resp. $\mathcal{M}_f$, $\mathcal{M}_{cf}$) the full subcategory of $\mathcal{M}$ on cofibrant (resp. fibrant, bifibrant) objects.

2.2 Examples of homotopy theories

Example 2.2.1. Any category $\mathcal{M}$ admits the *trivial model structure* where W is the class of isomorphisms, while C and F are classes of all morphisms.

Remark 2.2.2 (Objects with basepoints, [Hov07], 1.1.8)**.** Let $\mathcal{M}$ be a model category and let us denote by $\mathcal{M}_*$ the category under the terminal object 1. The objects of $\mathcal{M}_*$ are objects of $\mathcal{M}$ equipped with a basepoint $1 \to \mathcal{M}$. The forgetful functor $G\colon \mathcal{M}_* \to \mathcal{M}$ admits a left adjoint $F\colon \mathcal{M} \to \mathcal{M}_*$ that assigns a disjoint basepoint to each object. The category $\mathcal{M}_*$ admits a model structure in which a morphism f is a weak equivalence (resp. a (co)fibration) if and only if Gf is a weak equivalence (resp. a (co)fibration) in $\mathcal{M}$.

2.2.1 Small categories

Definition 2.2.3. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *isofibration* if for any object c in $\mathcal{C}$ and each isomorphism $g\colon Fc \to d$ in $\mathcal{D}$, there is an isomorphism $f\colon c \to c'$ in $\mathcal{C}$ such that $F(f) = g$.

The following model structure has been known to exist since the 1980s. A proof of the existence is written in an unpublished note by Charles Rezk.

Theorem 2.2.4 (Folk model structure)**.** *The category Cat admits a model structure such that the class W consists of equivalences of categories, the class C consists of functors that are injective on objects and the class F consist of isofibrations.*

2.2.2 Chain complexes

Theorem 2.2.5 ([Hov07], Theorem 2.3.11)**.** *Let R be a commutative ring. The category of nonnegatively graded chain complexes of R-modules admits a model structure with the class W consisting of quasi-isomorphisms, the class F consisting of epimorphisms in each nonzero degree and the class C consisting of monomorphisms with projective cokernel in each degree. This model structure is called the* projective model structure.

Remark 2.2.6. Dually, there is the *injective model structure*, with the same class of weak equivalence, but with the class C consisting of the monomorphisms in each nonzero degree and the class F consisting of the epimorphisms with injective kernel in each degree.

Remark 2.2.7. Note that a projective resolution is a typical example of a cofibrant replacement and an injective resolution is a fibrant replacement.

Remark 2.2.8. The category of (unbounded) chain complexes of R-modules admits a model structure with the class W being the chain homotopy equivalences, the class F being the epimorphisms that are split as morphisms of underlying R-modules and the class C being the monomorphisms that are split as morphisms of underlying R-modules.

2.2.3 Topological spaces

Definition 2.2.9. A map between topological spaces $f\colon X \to Y$ is a *weak homotopy equivalence* if the induced maps

$$f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

are isomorphisms for every $n \geq 0$ and every point $x_0 \in X$.

Definition 2.2.10. Let D^n and S^n denote the n-dimensional disk and the n-dimensional sphere, respectively. A map between topological spaces is a *Serre fibration* if it has the RLP with respect to all inclusions of the form $j_n\colon D^n \times \{0\} \to D^n \times [0,1]$.

Let I be the set of inclusions $i_n\colon S^{n-1} \to D^n$. The class $\mathrm{cell}(I)$ consists of retracts of relative cell complexes. In particular, every relative CW-complex is in $\mathrm{cell}(I)$.

Theorem 2.2.11 ([Qui67], II.3.1)**.** *The category of topological spaces admits a model structure, called the* Quillen model structure*, with the class W being the weak homotopy equivalences, the class F being the Serre fibrations and the class C being the retracts of relative cell complexes.*

Remark 2.2.12. Every object is fibrant in this model structure. Cofibrant objects are retracts of CW complexes and CW approximation is a typical example of a cofibrant replacement.

Remark 2.2.13. This model structure is an example of a cofibrantly generated model structure (see Definition 2.3.24). The set $I = \{i_n\colon n \geq 0\}$ is the set of generating cofibrations, while the set $J = \{j_n\colon n \geq 0\}$ is the set of generating trivial cofibrations.

Remark 2.2.14. There are two more model structures that are standard examples in the literature. The *Hurewicz (or Strøm) model structure* has homotopy equivalences as weak equivalences and Hurewicz fibrations as fibrations. Here, a map between topological spaces is a *Hurewicz fibration* if it has the RLP with respect to all inclusions $A \times \{0\} \to A \times [0,1]$.

The *mixed model structure* has weak homotopy equivalences as the weak equivalences and Hurewicz fibrations as fibrations.

Remark 2.2.15. As we have indicated in 2.2.2, each of these model structures can be transferred to the category of pointed topological spaces.

Remark 2.2.16. The category of topological spaces is Cartesian, but it is not closed Cartesian. Hence it is practical to consider a category with less objects but having properties that are nice from the technical point of view. We say that a full replete subcategory $\mathcal{C}$ of topological spaces is *convenient* if it is complete and cocomplete, Cartesian closed and contains all CW-complexes. One quite common choice of such a category is the category of compactly generated Hausdorff spaces.

Theorem 2.4.25. in [Hov07] states that there is a model structure on the category of compactly generated Hausdorff spaces, where a map is a weak equivalence (resp. cofibration, fibration) if and only if it is a weak equivalence (resp. cofibration, fibration) as a morphism in the Quillen model structure on Top. Moreover, the two model structures are Quillen equivalent. Hence, from the homotopic point of view it is not a serious drawback to work in the more convenient category.

2.2.4 Simplicial sets

For each integer $n \geq 0$, let us denote by $[n]$ the linear order $\{0 < 1 < \ldots < n\}$ with $n+1$ elements. Let Δ be the small category whose objects are linear orders $[n], n \in \mathbb{N}_0$ and order-preserving morphisms. There is an equivalent (large) category Δ_{big} whose objects are all nonempty finite linear objects and morphisms are order-preserving maps. Obviously, Δ is the skeleton of Δ_{big}.

Definition 2.2.17. A *simplicial object* in a category $\mathcal{C}$ is a functor $X\colon \Delta^{op} \to \mathcal{C}$. A *simplicial set* is a simplicial object in the category of sets.

In Δ, there are unique injective maps $d_i\colon [n-1] \to [n]$, $i = 0, 1, \ldots, n$ such that i is not in the image of d_i and unique surjective maps $s_j\colon [n+1] \to [n]$, $j = 0, 1, \ldots, n$ such that j there are two elements mapped to j. The maps d_i are called *elementary face maps* and maps s_j are called *elementary degeneracy maps*.

Every map in Δ can be factored into elementary face and degeneracy maps and the factorization is unique up to the following *(co)simplicial relations*

$$d_i d_j = d_{j-1} d_i \quad \text{if } i < j, \quad s_i s_j = s_{j+1} s_i \quad \text{if } i < j,$$
$$d_i s_j = \begin{cases} s_j d_i, & \text{if } i < j, \\ 1, & \text{if } i = j \text{ or } i = j+1, \\ s_j d_{i-1}, & \text{if } i > j+1. \end{cases}$$

We denote by $s\mathcal{C}$ the category of simplicial objects in a category $\mathcal{C}$ (with natural transformations as morphisms) and, in particular, the category of simplicial sets by sSet.

Definition 2.2.18. For each integer $n \geq 0$, let $\Delta[n]$ be the representable presheaf $\mathrm{Hom}(-, [n])\colon \Delta^{op} \to \mathrm{Set}$.

For a simplicial set X, the Yoneda lemma implies that the set $X_n = X([n])$ is naturally isomorphic to the set $\mathrm{Hom}_{\mathrm{sSet}}(\Delta[n], X)$.

Example 2.2.19. Kan extensions along the Yoneda embedding $\Delta \to \mathrm{sSet}$ provide many important examples of adjunctions between various categories and the category of simplicial sets. Any functor $F\colon \Delta \to \mathcal{C}$ to a cocomplete category $\mathcal{C}$ gives rise to an adjunction $|-|_F \dashv N_F$ such that

$$N_F(C)_n = \mathrm{Hom}_{\mathcal{C}}(F[n], C), \quad \text{and} \quad |X|_F = \operatorname*{colim}_{\Delta[n]\to X} F[n].$$

a) Let Δ^n be the geometric n-simplex, i.e. a convex hull of $n+1$ affine independent points in $\mathbb{R}^{n+1}$. The functor $F\colon [n] \mapsto \Delta^n$ induces a left adjoint $|-|\colon \mathrm{sSet} \to \mathrm{Top}$ called the *geometric realization functor* and a right adjoint $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSet}$ called the *singular complex functor*.

b) We can consider $[n]$ as a category with $n+1$ objects $0, 1, \ldots, n$ having a unique morphism from i to j if and only if $i \leq j$. This gives an embedding $\Delta \to \mathrm{Cat}$. The induced right adjoint functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$ is called the *nerve functor*. The left adjoint $\tau\colon \mathrm{sSet} \to \mathrm{Cat}$ is called the *fundamental category functor*.

c) Another important example for higher category theory comes from simplicial categories. Let us denote $W[n]$ the simplicial category whose objects are $0, 1, \ldots, n$ and the simplicial hom-sets are given as nerves

$$\mathrm{Hom}_{W[n]}(i, j) = N(P_{i,j})$$

where $P_{i,j}$ is the poset of subsets of $[i, j] = \{i, i+1, \ldots, j\}$ that contain i and j, ordered by inclusions. Functor $W\colon \Delta \to \mathrm{sCat}$ gives an adjunction and we call the right adjoint functor $hcN\colon \mathrm{sCat} \to \mathrm{sSet}$ the *homotopy coherent nerve*.

We now recall the ingredients of the model structure on simplicial sets which gives a model for the homotopy theory of topological spaces.

Definition 2.2.20. Let the *boundary* $\partial\Delta[n] \subseteq \Delta[n]$ be the union of all the faces of $\Delta[n]$, i.e. the union of images of the induced maps $d_i^*\colon \Delta[n-1] \to \Delta[n]$ for all $i = 0, 1, \ldots, n$. Hence $(\partial\Delta[n])_m$ consists of all the maps $[m] \to [n]$ that are not surjective.

Analogously, let the *k-th horn* $\Lambda^k[n] \subseteq \Delta[n]$ be the union of all faces except the one induced by d_k.

Proposition 2.2.21. *The class of monomorphisms in sSet is the smallest saturated class containing all boundary inclusions $\partial\Delta[n] \to \Delta[n]$.*

Definition 2.2.22. The smallest saturated class containing horn inclusions $\Lambda^k[n] \to \Delta[n]$ is called the class of *anodyne extensions*.

Definition 2.2.23. A map $p\colon X \to Y$ in the category of simplicial sets is called a *Kan fibration* if it has the RLP with respect to all horn inclusions $\Lambda^k[n] \to \Delta[n]$. A simplicial set X is a *Kan complex* if the unique map from X to the terminal object is a Kan fibration.

In other words, X is a Kan complex if every inner horn inclusion $\Lambda^k[n] \to \Delta[n]$, for $0 \leq k \leq n$, induces a surjection

$$\mathrm{Hom}(\Delta[n], X) \to \mathrm{Hom}(\Lambda^k[n], X).$$

Theorem 2.2.24 ([Qui67], II.3.3)**.** *There is a model structure on the category of simplicial sets such that fibrations are Kan fibrations, cofibrations are monomorphisms and a map f is a weak equivalence if the induced map of geometric realizations $|f|$ is a weak equivalence of topological spaces.*

The model structure of Theorem 2.2.24 is called the *Kan-Quillen model structure.*

Remark 2.2.25. The Kan-Quillen model structure has the following properties:

(i) it is cofibrantly generated with the set of generating cofibrations being the boundary inclusions $\partial\Delta[n] \to \Delta[n]$ and the set of generating trivial cofibrations being the horn inclusions $\Lambda^k[n] \to \Delta[n]$, hence the trivial cofibrations are exactly the anodyne extensions;

(ii) it is combinatorial since the presheaf categories are presentable;

(iii) it is a symmetric monoidal model category with the tensor product being the Cartesian product and (this also amounts to say that) it is a simplicial model category;

(iv) cylinder objects are obtained by tensoring with $\Delta[1]$;

(v) the adjunction given by the singular complex and the geometric realization functors is a Quillen equivalence.

Proposition 2.2.26. *A simplicial set X is isomorphic to a nerve of category if and only if every inner horn inclusion $\Lambda^k[n] \to \Delta[n]$, $0 < k < n$ induces a bijection*

$$\mathrm{Hom}(\Delta[n], X) \to \mathrm{Hom}(\Lambda^k[n], X).$$

Definition 2.2.27. A simplicial set X is an *inner Kan complex* (or an ∞-*category*) if every inner horn inclusion $\Lambda^k[n] \to \Delta[n], 0 < k < n$ induces a surjection

$$\mathrm{Hom}(\Delta[n], X) \to \mathrm{Hom}(\Lambda^k[n], X).$$

The category of simplicial sets also admits another model structure due to A. Joyal which models the homotopy theory of ∞-categories.

Theorem 2.2.28 (Joyal model structure, [Joy08])**.** *There is a model structure on the category of simplicial sets such that the cofibrations are monomorphisms and fibrant objects are inner Kan complexes.*

Theorem 2.2.29 (Thomason model structure, [Tho80])**.** *There is a model structure on Cat such that a functor f is a fibration (resp. weak equivalence) if the nerve $N(f)$ is a fibration (resp. weak equivalence) in the Kan-Quillen model structure on sSet. In that model structure every cofibrant object is a poset, which implies that every simplicial set is weakly equivalent to a nerve of a poset.*

Theorem 2.2.30 (Bergner model structure, [Ber04])**.** *There is a model structure on the category sCat of simplicially enriched categories in which a simplicially enriched functor $F\colon \mathcal{C} \to \mathcal{D}$ is*

- *a weak equivalence if for all objects x and y in $\mathcal{C}$, the map $\underline{\mathrm{hom}}(x, y) \to \underline{\mathrm{hom}}(F(x), F(y))$ is a weak equivalence in the Kan-Quillen model strucutre and the induced functor $\pi_0(F)\colon Ho(\mathcal{C}) \to Ho(\mathcal{D})$ is essentially surjective;*
- *a fibration if for all objects x and y in $\mathcal{C}$, the map $\underline{\mathrm{hom}}(x, y) \to \underline{\mathrm{hom}}(F(x), F(y))$ is a Kan fibration and the induced functor $\pi_0(F)\colon Ho(\mathcal{C}) \to Ho(\mathcal{D})$ is an isofibration.*

2.2.5 Operads

We introduce (coloured) operads as a generalization of the notion of categories where the morphisms have a sequence of domains.

Definition 2.2.31. A *(coloured) operad* P enriched in a symmetric monoidal category $\mathcal{M}$ consists of a set of colours $\mathrm{col}(P)$, an object $P(c_1, \dots, c_n; c)$ in $\mathcal{M}$ for each sequence $c_1, \dots, c_n, c \in \mathrm{col}(P)$ and the following further structure

- composition maps
$$P(c_{11}, \dots, c_{1k_1}; c_1) \otimes \dots \otimes P(c_{n1}, \dots, c_{nk_n}; c_n) \otimes P(c_1, \dots, c_n; c) \to P(c_{11}, \dots, c_{nk_n}; c);$$
- identities, given by a morphism $I \to P(c; c)$ for every $c \in \mathrm{col}(P)$;
- the action of the symmetric groups, given for each $\sigma \in \Sigma_n$ by a morphism $\sigma^*\colon P(c_1, \dots, c_n, c) \to P(c_{\sigma(1)}, \dots, c_{\sigma(n)}; c)$

such that the composition is associative, the identities are neutral for the composition (from left and right) and the action of symmetric groups is compatible with the compositions.

Definition 2.2.32. Let P and Q be coloured operads enriched in $\mathcal{M}$. A *morphism of operads* $F\colon P \to Q$ consists of a map $F\colon \operatorname{col}(P) \to \operatorname{col}(Q)$ and morphisms

$$P(c_1, \ldots, c_n; c) \to Q(Fc_1, \ldots, Fc_n; Fc), \quad c_1, \ldots, c_n, c \in \operatorname{col}(P)$$

respecting composition, identities and the action of the symmetric groups.

Remark 2.2.33. If we do not include the action of the symmetric group in Definition 2.2.31 we obtain a notion of a *nonsymmetric operad*. The forgetful functor from the category of symmetric operads to the category of nonsymmetric operads admits a left adjoint Σ called the *symmetrization functor*.

Remark 2.2.34. A *collection* C enriched in a symmetric monoidal category $\mathcal{M}$ consists of a set of colours $\operatorname{col}(C)$ and objects $C(c_1, \ldots, c_n; c)$ in $\mathcal{M}$ for all $c_1, c_2, \ldots, c_n, c \in \operatorname{col}(C)$. There is a forgetful functor from the category of operads to the category of collections and it admits a left adjoint which we denote by Free. For a collection C, the operad $\operatorname{Free}(C)$ is called *the free operad generated by* C. We will not give the explicit construction of $\operatorname{Free}(C)$ explicitly, but we want to emphasize that operations of $\operatorname{Free}(C)$ are described in terms of planar trees whose vertices are labelled by elements of C.

We will also use the construction of an operad given by *generators and relations*. The standard references include [GK94], [MSS07], [BM03].

Remark 2.2.35. Every small category enriched in $\mathcal{M}$ is an example of an operad enriched in $\mathcal{M}$ having the set of objects as a set of colours and only unary operations. Also, every functor is a morphism of operads. Hence the theory of (enriched) categories is subsumed in the theory of operads. Let Oper be the category of small operads enriched in sets. The embedding $j_!\colon \mathrm{Cat} \to \mathrm{Oper}$ has a right adjoint functor $j^*\colon \mathrm{Oper} \to \mathrm{Cat}$ sending an operad P to a category j^*P by forgetting non-unary operations.

Example 2.2.36. Every symmetric monoidal category $\mathcal{M}$ induces an operad $O_{\mathcal{M}}$. The set[2] of colours of $O_{\mathcal{M}}$ is the set of objects of $\mathcal{M}$ and the operation sets are given by

$$O_{\mathcal{M}}(X_1, \ldots, X_n; X) = \operatorname{Hom}_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n, X).$$

If $\mathcal{M}$ is a closed monoidal category then it also induces an operad enriched in itself.

Definition 2.2.37. Let P be a coloured operad enriched in $\mathcal{M}$. A *P-algebra* A consists of a family of objects $(A_c)_{c \in \operatorname{col}(P)}$ in $\mathcal{M}$ together with the structure maps

$$P(c_1, \ldots, c_n; c) \otimes A_{c_1} \otimes \ldots \otimes A_{c_n} \to A_c$$

satisfying the obvious action axiom.

A *morphism of P-algebras* $f\colon A \to B$ consists of a family of morphisms $f_c\colon A_c \to B_c$ compatible with all the structure maps.

[2] If necessary we pass to a bigger universe.

Remark 2.2.38. Note that a P-algebra is equivalently given by a morphism of operads $A\colon P \to O_{\mathcal{M}}$.

Example 2.2.39. Let n be a positive integer. We will describe the operad E_n of little n-cubes. A little n-cube is an embedding $\alpha\colon I^n \to I^n$, where I is the segment $[0,1]$. A k-configuration of little n-cubes is a k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of little n-cubes such that the images of α_i have disjoint interiors. Let $E_n(k)$ be the subspace of the space $\mathrm{Hom}((I^n)^k, I^n)$ (with the compact-open topology) consisting of k-configurations of little n-cubes.

The sequence of spaces $(E_n(k))_k$ has the structure of an operad with the composition $E_n(m_1) \times \ldots \times E_n(m_k) \times E_n(k) \to E_(m_1 + \ldots + m_k)$ given by

$$(\alpha_1, \ldots, \alpha_k) \circ ((\beta_{1,1}, \ldots, \beta_{1,m_1}), \ldots, (\beta_{k,1}, \ldots, \beta_{k,m_k})) =$$
$$(\alpha_1 \circ \beta_{1,1}, \ldots, \alpha_1 \circ \beta_{1,m_1}, \ldots, \alpha_k \circ \beta_{k,1}, \ldots, \alpha_k \circ \beta_{k,m_k})$$

The identity in $E_n(1)$ is given by the identity map $I^n \to I^n$ and the action of the symmetric groups is given by permutations of the components of configurations.

Let (Y, y_0) be a pointed space. We consider the n-fold loop space $\Omega^n Y$ as the space of maps of pairs $f\colon (I^n, \partial I^n) \to (Y, y_0)$. The space $\Omega^n Y$ is an algebra for the operad of little n-cubes E_n.

If α is a little n-cube, then it induces a homeomorphism $I^n \to \mathrm{Im}\ \alpha$. Let $\overline{\alpha}\colon \mathrm{Im}\ \alpha \to I^n$ be the inverse of that homeomorphism. Let us describe the action $E_n(k) \times (\Omega^n Y)^k \to \Omega^n Y$. Let $(\alpha_1, \ldots, \alpha_k) \in E_n(k)$ be a k-configuration and consider maps $f_i\colon (I^n, \partial I^n) \to (Y, y_0)$ for $i = 1, \ldots, k$. We describe the result $f\colon (I^n, \partial I^n) \to (Y, y_0)$ of the action,

$$f = (\alpha_1, \ldots, \alpha_k) \cdot (f_1, \ldots, f_k)$$

by defining it on the image of α_i as the composition $f_i \circ \overline{\alpha_i}$ and extend it to the rest of the cube I^n by sending all points to y_0. Note that this defines a continuous map since the boundaries of the images of α_i are sent to y_0.

Theorem 2.2.40 (May's recognition principle for iterated loop spaces, [May72])**.** *Let n be a positive integer. A topological space X is weakly equivalent to a n-fold loop space $\Omega^n Y$ of another space Y if and only if X is a grouplike E_n-algebra.*

Let us now describe the monoidal structure on the category of coloured operads given by Boardman and Vogt in [BV73].

Definition 2.2.41. Let P and Q be coloured operads. The *Boardman-Vogt tensor product* $P \otimes_{BV} Q$ of P and Q is the coloured operad with $\mathrm{col}(P \otimes_{BV} Q) = \mathrm{col}(P) \times \mathrm{col}(Q)$ generated by the operations

$$\begin{aligned} p \otimes d, &\quad \text{for } p \in P(c_1, \ldots, c_n; c), d \in \mathrm{col}(Q) \qquad \text{and} \\ c \otimes q, &\quad \text{for } c \in \mathrm{col}(P), q \in Q(d_1, \ldots, d_n; d) \end{aligned}$$

with the following relations

(i) for a colour $d \in \mathrm{col}(Q)$, $- \otimes d$ respects composition and symmetric action in P;

(ii) for a colour $c \in \mathrm{col}(P)$, $c \otimes -$ respects composition and symmetric action in Q;

(iii) (interchange relation)

$$\sigma^*_{n,m}((p \otimes d) \circ ((c_1 \otimes q), \dots, (c_n \otimes q))) = (c \otimes q) \circ ((p \otimes d_1), \dots, (p \otimes d_m))$$

where the permutation $\sigma_{n,m} \colon \{0, 1, \dots, nm-1\} \to \{0, 1, \dots, nm-1\}$ is defined by $\sigma_{n,m}(kn + j) = jm + k$ for all $0 \leq k \leq n-1$ and $0 \leq j \leq m-1$.

For an illustration of the interchange relation see 3.2.13.

Remark 2.2.42. Property (i) implies that, for every colour $d \in \mathrm{col}(Q)$, there is a map of operads $P \cong P \otimes_{BV} \{d\} \to P \otimes_{BV} Q$ given by $p \mapsto p \otimes d$. Similarly, property (ii) implies that, for every colour $c \in \mathrm{col}(P)$, there is a map of operads $Q \cong \{c\} \otimes_{BV} Q \to P \otimes_{BV} Q$ given by $q \mapsto c \otimes q$.

Theorem 2.2.43. *The category Oper with the Boardman-Vogt tensor product is a symmetric closed monoidal category.*

Definition 2.2.44. Let $F, G \colon P \to Q$ be morphisms of coloured operads. A *natural transformation* $\eta \colon F \Rightarrow G$ is a morphism of operads

$$\eta \colon P \otimes_{BV} I \to Q$$

such that the composition with the inclusion $P \otimes_{BV} \{0\} \to P \otimes_{BV} I$ is F and the composition with the inclusion $P \otimes_{BV} \{1\} \to P \otimes_{BV} I$ is G.

Equivalently, a natural transformation $\eta \colon F \Rightarrow G$ is given by a family of unary operations $\eta_c \in Q(Fc; Gc)$ indexed by the colours of operad P such that for any operation $p \in P(c_1, \dots, c_n; c)$ the following "naturality condition" holds $\eta_c \circ F(p) = G(p) \circ (\eta_{c_1}, \dots, \eta_{c_n})$.

Definition 2.2.45. A morphism of operads $F \colon P \to Q$ is

(i) *essentially surjective* if the functor j^*F between the underlying categories is essentially surjective;

(ii) *full* (resp. *faithful*) if the maps $F \colon P(c_1, \dots, c_n; c) \to Q(Fc_1, \dots, Fc_n; Fc)$ are surjective (resp. injective);

(iii) an *operadic equivalence* if it is fully faithful and essentially surjective.

(iv) an *operadic isofibration* if j^*F is a categorical isofibration, cf. 2.2.3.

Theorem 2.2.46. *The category Oper admits a model structure such that the class W consists of operadic equivalences, the class C consists of morphisms of operads that are injective on colours and the class F consists of operadic isofibrations.*

Remark 2.2.47. For an operad P, the tensor product $P \otimes_{BV} I$ is a cylinder object for P and a left homotopy between morphisms F and G is given by a natural transformation $F \Rightarrow G$.

Theorem 2.2.48 ([BM03])**.** *Let $\mathcal{M}$ be the category of compactly generated spaces. There is a cofibrantly generated model structure on the category of operads with one colour enriched in $\mathcal{M}$ in which a map $P \to Q$ is a weak equivalence (resp. fibration) if and only if for each n the map $P(n) \to Q(n)$ is a weak equivalence (resp. fibration) in $\mathcal{M}$.*

Remark 2.2.49. Theorem 3.2. in [BM03] states that the previous theorem holds for any Cartesian closed model category which is cofibrantly generated with a cofibrant terminal object and a symmetric monoidal fibrant replacement functor.

Theorem 2.2.50 ([CM11])**.** *The category sOper of operads enriched in simplicial sets admits a model structure in which a map $F \colon P \to Q$ is*

- *a weak equivalence if for all colours $c_1, \dots, c_n$ and c in P, the map*

  $$\underline{\mathrm{hom}}_P(c_1, \dots, c_n; c) \to \underline{\mathrm{hom}}_Q(F(c_1), \dots, F(c_n); F(c))$$

  is a weak equivalence in the Kan-Quillen model structure and the morphism of operads $\pi_0(F) \colon Ho(P) \to Ho(Q)$ is essentially surjective;

- *a fibration if for all colours $c_1, \dots, c_n$ and c in P, the map*

  $$\underline{\mathrm{hom}}_P(c_1, \dots, c_n; c) \to \underline{\mathrm{hom}}_Q(F(c_1), \dots, F(c_n); F(c))$$

  is a Kan fibration and the induced morphism of operads $\pi_0(F) \colon Ho(P) \to Ho(Q)$ is an operadic isofibration.

Remark 2.2.51. In [BM07], a model structure is established for the category of operads enriched in a monoidal model category with a suitable interval object, with a fixed set of colours. The case of all (small) coloured operads enriched in (a certain class of) monoidal model categories has been recently studied by G. Caviglia.

2.3 Further properties of model categories

2.3.1 Left and right homotopy

Definition 2.3.1. Let $\mathcal{M}$ be a model category and X an object of $\mathcal{M}$.
A *cylinder object* for X is a factorization of the fold morphism

$$\begin{tikzcd} X \sqcup X \arrow[rr, "\nabla"] \arrow[dr, "i_0 \sqcup i_1"'] & & X \\ & \mathrm{Cyl}(X) \arrow[ur, "s"'] & \end{tikzcd}$$

such that $s \in W$. A cylinder object is *good* if $i_0 \sqcup i_1 \in C$ and a good cylinder object is *very good* if s is a trivial fibration.
A *path object* for X is a factorization of the diagonal morphism

$$\begin{tikzcd} X \arrow[rr, "\Delta"] \arrow[dr, "r"'] & & X \times X \\ & \mathrm{Path}(X) \arrow[ur, "(p_0,p_1)"'] & \end{tikzcd}$$

such that $r \in W$. A path object is *good* is $(p_0, p_1) \in F$ and a good path object is *very good* if r is a trivial cofibration.

Remark 2.3.2. Axiom CM5 implies the existence of a *very good* cylinder object, which we denote by[3] $X \times I$. Axiom CM4 implies that for every other good cylinder object $\mathrm{Cyl}(X)$ there is a morphism $\mathrm{Cyl}(X) \to X \times I$, which is a weak equivalence by CM2. From this one can deduce that it does not matter which good cylinder object we use.
Dually, there is a *very good* path object X^I and for every other good path object $\mathrm{Path}(X)$ there is a weak equivalence $X^I \to \mathrm{Path}(X)$.
If the factorizations of CM5 can be made functorial, then we obtain functorial cylinder and path objects.

Definition 2.3.3. Let $\mathcal{M}$ be a model category and $f, g\colon X \to Y$ be two morphisms in $\mathcal{M}$. We say that

(i) f is *left homotopic* to g ($f \sim_l g$) if there is a morphism $H\colon \mathrm{Cyl}(X) \to Y$, for some cylinder object $\mathrm{Cyl}(X)$ of X, such that $Hi_0 = f$ and $Hi_1 = g$;

(ii) f is *right homotopic* to g ($f \sim_r g$) if there is a morphism $K\colon X \to \mathrm{Path}(Y)$, for some path object $\mathrm{Path}(Y)$ of Y, such that $p_0K = f$ and $p_1K = g$;

[3]Note that $X \times I$ is not a product of two objects, but merely a name for a particular cylinder object.

In general, left and right homotopy relations are reflexive and symmetric, but not transitive. We denote $\pi^l(X,Y)$ (resp. $\pi^r(X,Y)$) the set of equivalence classes of $\hom(X,Y)$ with respect to the equivalence relation generated by left (resp. right) homotopy.

The following lemma comprises fundamental results about left and right homotopy from [Qui67] (Ch I, Lemma 4,5,6,7). All statements follow from simple yet clever use of the axioms CM1-CM5 and the definition of cylinder and path objects.

Lemma 2.3.4. *Let $\mathcal{M}$ be a model category and X a cofibrant object in $\mathcal{M}$.*

(i) Left homotopy is an equivalence relation on the set of morphisms from X to Y.

(ii) Let $f, g\colon X \to Y$ be two morphisms in $\mathcal{M}$. If $f \sim_l g$, then $f \sim_r g$.

(iii) The composition in $\mathcal{M}$ induces a map $\pi^r(X,Y) \times \pi^r(Y,Z) \to \pi^r(X,Z)$.

(iv) If $p\colon A \to B$ is a trivial fibration in $\mathcal{M}$, then the induced map

$$p_*\colon \pi^l(X,A) \to \pi^l(X,B)$$

is a bijection.

Lemma 2.3.5. *Let $\mathcal{M}$ be a model category and f and g morphisms in $\mathcal{M}$. Let $\mathcal{N}$ be a category and $F\colon \mathcal{M} \to \mathcal{N}$ a functor which sends weak equivalences in $\mathcal{M}$ to isomorphisms in $\mathcal{N}$. If $f \sim_r g$ or $f \sim_l g$, then $F(f) = F(g)$.*

Lemma 2.3.4 (iii) implies that there is a category $\pi\mathcal{M}_c$ whose objects are objects of $\mathcal{M}_c$ and the set of morphisms from X to Y is $\pi^r(X,Y)$. Dually, there is category $\pi\mathcal{M}_f$ whose objects are objects of $\mathcal{M}_f$ and the set of morphisms from X to Y is $\pi^l(X,Y)$.

Let X be cofibrant and Y fibrant. Lemma 2.3.4 implies that the left and right homotopy relations are equivalence relations on the set of morphisms from X to Y and that they coincide. We write $\sim$ for that equivalence relation. We let $\pi\mathcal{M}_{cf}$ be the category whose objects are bifibrant objects and the set of morphisms between X and Y is given by the set of equivalence classes of $\hom_{\mathcal{M}}(X,Y)$ with respect to $\sim$.

2.3.2 The construction of the homotopy category

It is quite standard to approximate objects by "nice" objects - in topology we use CW approximations and in homological algebras we use projective resolutions. This idea is generalized to the setting of model categories in terms of (co)fibrant replacements.

Definition 2.3.6. A *cofibrant replacement* of an object X is given by a weak equivalence $\lambda_X\colon Q(X) \to X$ from a cofibrant object $Q(X)$.

A *cofibrant replacement* of a morphism $f\colon X \to Y$ is given by a morphism $Q(f)\colon Q(X) \to Q(Y)$ such that the following diagram commutes

$$\begin{tikzcd} Q(X) \arrow[r, "\sim"] \arrow[d, "Q(f)"'] & X \arrow[d, "f"] \\ Q(Y) \arrow[r, "\sim"] & Y \end{tikzcd}$$

Dually, we define a *fibrant replacement* $\rho_X\colon X \to R(X)$ of an object X and a fibrant replacement $R(f)\colon R(X) \to R(Y)$ of a morphism $f\colon X \to Y$.

By Axiom CM5a) we can factor the morphism $0 \to X$ as a cofibration $0 \to Q(X)$ followed by a trivial fibration $\lambda_X\colon Q(X) \to X$. Similarly, we get a trivial fibration $\lambda_Y\colon Q(Y) \to Y$ from a cofibrant object $Q(Y)$. Given a morphism $f\colon X \to Y$ we can find a lift $Q(f)\colon Q(X) \to Q(Y)$ in the following diagram

$$\begin{tikzcd} \emptyset \arrow[rr] \arrow[d] & & Q(Y) \arrow[d, "\lambda_Y"] \\ Q(X) \arrow[r, "\lambda_X"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Since λ_Y is a trivial fibration, Lemma 2.3.4 (iv) implies that the map $Q(f)$ is unique up to left homotopy. Part (ii) of the same lemma implies uniqueness of the lift up to right homotopy, too. This implies that $Q(f)Q(g) \sim Q(fg)$ and $Q(1_X) \sim 1_{Q(X)}$, so there is a well-defined functor $\overline{Q}\colon \mathcal{M} \to \pi\mathcal{M}_c$ sending f to the right homotopy equivalence class $\overline{Q}(f)$ of $Q(f)$. Dually, there is a functor $\overline{R}\colon \mathcal{M} \to \pi\mathcal{M}_f$

Remark 2.3.7. If X is cofibrant we take $Q(X) = X$ and $\lambda_X = 1_X$. Dually, if X is fibrant we take $R(X) = X$ and $\rho_X = 1_X$.

Remark 2.3.8. In most examples the factorizations of CM5 are obtained by The Small Object Argument (2.1.28). In that case we actually have functorial (co)fibrant replacements $Q\colon \mathcal{M} \to \mathcal{M}_c$ and $R\colon \mathcal{M} \to \mathcal{M}_f$.

Let X and Y be cofibrant objects and $f, g\colon X \to Y$ such that $f \sim_r g$. We can choose $\rho_X\colon X \to R(X)$ and $\rho_Y\colon Y \to R(Y)$ to be trivial cofibrations. Then $\rho_Y f \sim_r \rho_Y g$, so $R(f)\rho_X \sim_r R(g)\rho_X$. The dual of Lemma 2.3.4 (iii) implies that $R(f) \sim_r R(g)$. Since $R(X)$ and $R(Y)$ are bifibrant, we have $R(f) \sim R(g)$. This shows that the functor $\overline{R}$ restricts to a functor $\pi\overline{R}\colon \pi\mathcal{M}_c \to \pi\mathcal{M}_{cf}$. We denote $\overline{RQ} = \pi\overline{R} \circ \overline{Q}\colon \mathcal{M} \to \pi\mathcal{M}_{cf}$.

The conclusions that we have drawn from Lemma 2.3.4 enable us to construct one more category. Theorem 2.3.11 expresses that this construction gives the homotopy category.

Definition 2.3.9. Let $Ho(\mathcal{M})$ be the category whose objects are objects of $\mathcal{M}$ and

$$\mathrm{Hom}_{Ho(\mathcal{M})}(X, Y) = \pi(RQX, RQY).$$

Let $\gamma\colon \mathcal{M} \to Ho(\mathcal{M})$ be given by $\gamma(X) = X$ and $\gamma(f) = \overline{RQ}(f)$.

Lemma 2.3.10. *If $f\colon A \to B$ is a weak equivalence in a model category $\mathcal{M}$, then $\gamma(f)$ is an isomorphism in $Ho(\mathcal{M})$.*

Proof. We first assume that A and B are bifibrant. Then $RQ(f) = f$. Note that for a bifibrant object X we have $\pi^r(A, X) = \pi(A, X)$ and $\pi^r(B, X) = \pi(B, X)$. If f is a trivial cofibration, Lemma 2.3.4 implies $f^*\colon \pi(B, X) \to \pi(A, X)$ is a bijection for every bifibrant X. Since f^* is given by postcomposition with $\gamma(f)$, Lemma 2.1.8 implies that $\gamma(f)$ is an isomorphism.

If f is a trivial fibration, dual statements imply that $f_*\colon \pi(X, A) \to \pi(X, B)$ is a bijection for every X. Hence $\gamma(f)$ is an isomorphism. Axioms CM2 and CM5 imply that every weak equivalence can be factored as a trivial cofibration followed by a trivial fibration, so $\gamma(f)$ is an isomorphism for every weak equivalence f between bifibrant objects.

In general (if A and B are not necessarily bifibrant), we have $\gamma(f) = \gamma(RQ(f))$. By CM2 f is a weak equivalence if and only if $RQ(f)$ is a weak equivalence, so the conclusion follows from the first part of the proof. □

The proof of the following fundamental theorem of model category theory now follows from Lemma 2.3.5 and 2.3.10.

Theorem 2.3.11 ([Qui67], I.1.1'). *The category $Ho(\mathcal{M})$ satisfies the universal property of a homotopy category. Hence $\mathcal{M}[W^{-1}]$ exists and it is isomorphic to $Ho(\mathcal{M})$.*

Lemma 2.3.10 shows that there is a fully faithful functor $\overline{\gamma}\colon \pi\mathcal{M}_{cf} \to Ho(\mathcal{M})$ rendering the following diagram commutative

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\overline{RQ}"] \arrow[dr, "\gamma"'] & & \pi\mathcal{M}_{cf} \arrow[dl, "\overline{\gamma}"] \\ & Ho(\mathcal{M}) & \end{tikzcd}$$

The functor $\overline{\gamma}$ is essentially surjective since for every object we have weak equivalences

$$X \leftarrow Q(X) \to RQ(X)$$

which $\overline{\gamma}$ sends to isomorphisms in $Ho(\mathcal{M})$. Hence $\overline{\gamma}$ is an equivalence of categories. Because of this, $\pi\mathcal{M}_{cf}$ is considered an alternative description of the homotopy category (although one must be aware it is only equivalent and not isomorphic to $\mathcal{M}[W^{-1}]$).

2.3.3 Detecting weak equivalences

Lemma 2.3.12 (Factorization Lemma, [Bro73]). *Let $u\colon X \to Y$ be a morphism between fibrant objects. There is a factorization $u = pi$ such that p is a fibration, i is a trivial cofibration, and i has a left inverse which is a trivial fibration.*

There is, as always, a dual statement about factorizations of morphisms between cofibrant objects. We will use the following immediate consequence of the dual Factorization Lemma which often carries Ken Brown's name in the literature.

Corollary 2.3.13 (Ken Brown's Lemma)**.** *Let $\mathcal{M}$ be a model category and let $\mathcal{N}$ be a category with weak equivalences, i.e. with a class of morphisms satisfying the two out of three property. If a functor $F\colon \mathcal{M} \to \mathcal{N}$ maps trivial cofibrations between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$, then F maps all weak equivalences between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$.*

Corollary 2.3.14. *Let $\{f_i\colon A_i \to B_i\}_{i\in I}$ be a set of weak equivalences between cofibrant objects in a model category $\mathcal{M}$. Then $\sqcup_i\colon \sqcup_i A_i \to \sqcup_i B_i$ is a weak equivalence.*

Proof. There is a coproduct functor $\sqcup_i\colon \mathcal{M}^I \to \mathcal{M}$. The product category $\mathcal{M}^I$ carries a model structure with "objectwise" weak equivalences, cofibrations and fibrations. Trivial cofibrations are closed under coproducts, i.e. the functor $\sqcup_i$ sends trivial cofibrations in $\mathcal{M}^I$ to trivial cofibrations in $\mathcal{M}$. So, the statement follows from the Ken Brown's Lemma 2.3.13. $\square$

Definition 2.3.15. A morphism f in a model category $\mathcal{M}$ is a *homotopy equivalence* if there exists a morphisms $g\colon Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$.

Theorem 2.3.16 (Whitehead's Theorem, [Hov07] 1.2.8)**.** *Let A and B be bifibrant objects in a model category $\mathcal{M}$. A morphism $f\colon A \to B$ is a weak equivalence if and only if it is a homotopy equivalence.*

Lemma 2.3.17. *Morphism $f\colon A \to B$ is a weak equivalence in a model category $\mathcal{M}$ if and only if $\gamma(f)$ is an isomorphism in $Ho(\mathcal{M})$.*

Proof. One implication is the content of Lemma 2.3.10. So, let us assume that $\gamma(f)$ is an isomorphism in $Ho(\mathcal{M})$. Since $\gamma(f) = [RQ(f)]$ is the homotopy class of the map $RQ(f)$, we conclude that $RQ(f)$ is a homotopy equivalence. Whitehead's Theorem implies that $RQ(f)$ is a weak equivalence which is equivalent to f being a weak equivalence. $\square$

Lemma 2.3.18. *A morphism $f\colon A \to B$ between cofibrant objects is a weak equivalence if and only if the induced map $f^*\colon \pi(B, X) \to \pi(A, X)$ is a bijection for all fibrant X.*

Proof. Since A is cofibrant and X is fibrant, Lemma 2.3.4 and its dual imply that $\pi^r(A, X) = \pi^l(A, X) = \pi(A, X)$. Also, $RQ(A) = R(A)$ and $RQ(X) = Q(X)$. By Lemma 2.3.4 (iv) the induced maps $\rho_A^*\colon \pi(RA, QX) \to \pi(A, QX)$ and $\lambda_{X*}\colon \pi(A, QX) \to \pi(A, X)$

are bijections. The following commutative diagram

$$\begin{tikzcd}
\pi(RB,QX) \arrow[r, "\gamma(f)^*"] \arrow[d, "\rho_B^*"'] & \pi(RA,QX) \arrow[d, "\rho_A^*"] \\
\pi(B,QX) \arrow[d, "\lambda_{X*}"'] & \pi(A,QX) \arrow[d, "\lambda_{X*}"] \\
\pi(B,X) \arrow[r, "f^*"] & \pi(A,X)
\end{tikzcd}$$

shows that f^* is a bijection if and only if $\gamma(f)^*$ is a bijection. By Lemma 2.3.17 f is a weak equivalence if and only if $\gamma(f)$ is an isomorphism, i.e. if and only if $\gamma(f)^*\colon \pi(RQB,RQX) \to \pi(RQA,RQX)$ is a bijection for all X. $\square$

Lemma 2.3.19. *In a closed model category, a cofibration is trivial if and only if it has the LLP against fibrations between fibrant objects.*

Proof. Note that trivial cofibrations have the LLP with respect to all fibrations by CM4.

For the other implication, let $u\colon A \to B$ be a cofibration having the LLP against fibrations between fibrant objects. Let $\rho_B\colon B \to R(B)$ be a trivial cofibration to a fibrant object $R(B)$. Let us consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\rho_A"] \arrow[d, "u"'] & R(A) \arrow[d, "R(u)"] \\
B \arrow[r, "\rho_B"'] & R(B)
\end{tikzcd}$$

obtained by factoring the map $\rho_B u$ as a trivial cofibration ρ_A and a fibration $R(u)$. Since u is a trivial cofibration and $R(u)$ a fibration, there is a lift $d\colon B \to R(A)$.

Since ρ_A and ρ_B are weak equivalences, $\gamma(\rho_A)$ and $\gamma(\rho_B)$ are isomorphisms in the homotopy category. The identities $du = \rho_A$ and $R(u)d = \rho_B$ imply that $\gamma(d)$ is also an isomorphism. By Lemma 2.3.17, d is a weak equivalence. By CM2, the identity $du = \rho_A$ implies that u is a weak equivalence. $\square$

Proposition 2.3.20. *A model structure is uniquely determined by the class of cofibrations and the class of fibrant objects.*

Proof. It is enough to show that the class of cofibrations and the class of fibrant objects determine the class of weak equivalences. Then the class of trivial cofibrations is determined and hence the class of fibrations, too.

By CM2, a morphism f is a weak equivalence if and only if its cofibrant replacement $Q(f)$ is a weak equivalence. We will use Lemma 2.3.18 to determine weak equivalences between cofibrant objects.

Let $f\colon A \to B$ be a morphism between cofibrant objects. As observed in Remark 2.3.2, there are very good cylinder objects $A \times I$ and $B \times I$, which are obtained by a factorization of the corresponding fold maps into a cofibration followed by a trivial fibrations. Hence such cylinder objects can be determined by cofibrations. It follows that left homotopy relation is also determined by cofibrations.

On the other hand, for a cofibrant A and a fibrant X the left homotopy relation coincides with the homotopy relation and hence the set $\pi(A, X) = \pi^l(A, X)$ is determined by cofibrations and fibrant objects. Analogously, $\pi(B, X)$ is determined by cofibrations and fibrant objects. Hence we can determine if the map $f_*\colon \pi(B, X) \to \pi(A, X)$ is a bijection, so Lemma 2.3.18 finishes the proof. □

Definition 2.3.21. A model category $\mathcal{M}$ is *left proper* if the class of weak equivalences is closed under pushouts along cofibrations. Dually, $\mathcal{M}$ is *right proper* if the class of weak equivalences is closed under pullbacks along fibrations.

Proposition 2.3.22 ([Hir02], 13.1.3.)**.** *If every object of a model category $\mathcal{M}$ is cofibrant (resp. fibrant), then $\mathcal{M}$ is left proper (resp. right proper).*

Example 2.3.23. a) The model structure on topological spaces is both left and right proper. It is right proper since every object is fibrant. That it is left proper is shown in [Hir02] 13.1.10.

b) The Kan-Quillen model structure on simplicial sets is both left and right proper. It is left proper since every object is cofibrant. That it is right proper is shown in [Hir02] 13.1.13.

2.3.4 Cofibrantly generated model categories

Definition 2.3.24. A model category $\mathcal{M}$ is *cofibrantly generated* if there exist (small) sets I and J such that I and J permit the small object argument, $\mathrm{cof}(I)$ is the class of cofibrations, $\mathrm{cof}(J)$ is the class of trivial cofibrations.
We call the elements of I the *generating cofibrations* and the elements of J the *generating trivial cofibrations*. Also, we say that I and J are the *generating sets*.

Theorem 2.3.25 (Kan's Recognition Principle, [Hov07], 2.1.19)**.** *Let $\mathcal{M}$ be a complete and cocomplete category. Assume that*

- *there is a class W of morphisms satisfying two-out-of-three property which is closed under retracts,*
- *there are (small) sets I and J of morphisms permitting the small object argument,*
- $\mathrm{cell}(J) \subseteq \mathrm{cof}(I) \cap W$ *and* $\mathrm{inj}(I) \cap W \subseteq \mathrm{inj}(J)$,

- *one of the following holds*

$$\mathrm{cof}(I) \cap W \subseteq \mathrm{cof}(J) \quad \text{or} \quad \mathrm{inj}(J) \cap W \subseteq \mathrm{inj}(I).$$

Then there is a cofibrantly generated model structure on $\mathcal{M}$ for which W is the class of weak equivalences, and I and J are the sets of generating cofibrations and generating trivial cofibrations, respectively.

Definition 2.3.26. A model category $\mathcal{M}$ is *combinatorial* if it is locally presentable and cofibrantly generated.

Remark 2.3.27. The existence of a combinatorial model structure is proved using an argument due to Jeff Smith. As we do not use this result, we refer the reader to [Beke] for the statement and the proof of Smith's Recognition Principle.

Definition 2.3.28. A model category $\mathcal{M}$ is *tractable* if it is combinatorial and the domains of the maps in the generating sets I and J are cofibrant.

2.3.5 Quillen functors

Proposition 2.3.29. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let*

$$(F \dashv G)\colon\ \mathcal{N} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{M}.$$

be an adjunction. The following statements are equivalent.

(i) F preserves cofibrations and trivial cofibrations.

(ii) G preserves fibrations and trivial fibrations.

(iii) F preserves cofibrations and G preserves fibrations.

(iv) F preserves trivial cofibrations and G preserves trivial fibrations.

Definition 2.3.30. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories. An adjunction $(F \dashv G)$ is a *Quillen pair* if it satisfies the equivalent statements of the previous proposition. We say that F is a *left Quillen functor* and G is a *right Quillen functor*.

Definition 2.3.31. Let $\mathcal{M}$ be a model category, $\mathcal{C}$ a category and $F\colon \mathcal{M} \to \mathcal{C}$ a functor. The *total left derived functor* is a functor $LF\colon \mathrm{Ho}\mathcal{M} \to \mathrm{Ho}\mathcal{N}$ such that $\gamma_{\mathcal{N}} \circ F = LF \circ \gamma_{\mathcal{M}}$.

One of the main applications of Ken Brown's lemma is the following result.

Proposition 2.3.32. *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a left Quillen functor. Then F sends weak equivalences between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$. Analogously, a right Quillen functor sends weak equivalences between fibrant objects in $\mathcal{N}$ to weak equivalences in $\mathcal{M}$.*

Proposition 2.3.32 implies that, for a left Quillen functor F, a total left derived functor is given by

$$LF(X) = FQ(X), \quad LF(f) = [FQ(f)].$$

Similarly, a right Quillen functor G induces a total right derived functor RG and a Quillen pair $F \dashv G$ induces an adjunction $LF \dashv RG$ on the level of homotopy categories.

Proposition 2.3.33. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and $(F \dashv G)\colon \mathcal{N} \rightleftarrows \mathcal{M}$ a Quillen pair. The following statements are equivalent.*

(i) For every cofibrant object A in $\mathcal{M}$, every fibrant object X in $\mathcal{N}$ and every morphism $f\colon A \to GX$, the morphism f is a weak equivalence in $\mathcal{M}$ if and only if the corresponding map $\hat{f}\colon FA \to X$ is a weak equivalence.

(ii) The total derived functors $(LF \dashv RG)\colon Ho\mathcal{N} \rightleftarrows Ho\mathcal{M}$ are equivalences of the homotopy categories.

Definition 2.3.34. A Quillen pair $(F \dashv G)\colon \mathcal{N} \rightleftarrows \mathcal{M}$ is a *Quillen equivalence* if it satisfies the equivalent conditions of the previous proposition.

Definition 2.3.35. Let $\mathcal{M}$ be a model category, $\mathcal{N}$ a complete and cocomplete category and let there be an adjunction

$$(F \dashv G)\colon \mathcal{N} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{M}.$$

A model structure on $\mathcal{N}$ is *transferred* if a morphism f is a weak equivalence (resp. a fibration) if and only if Gf is a weak equivalence (resp. a fibration) in $\mathcal{M}$.

If $\mathcal{M}$ is a cofibrantly generated model category with generating sets I and J there are simple sufficient conditions for the existence of the transferred model structure. First such statement appears in [Qui67], but the explicit study of the transfer is due to Sjoerd Crans, [Cra95].

Theorem 2.3.36 (The Transfer Principle)**.** *Let $\mathcal{M}$ be a cofibrantly generated model category with generating sets I and J . Let $\mathcal{N}$ be a complete and cocomplete category and let there be an adjunction*

$$(F \dashv G)\colon \mathcal{N} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{M}.$$

Assume that

(i) sets $FI = \{Ff \colon f \in I\}$ *and* $FJ = \{Ff \colon f \in J\}$ *permit the small object argument,*

(ii) U *takes relative* FJ*-cell complexes to weak equivalences.*

Then there is a cofibrantly generated model structure on $\mathcal{N}$ *in which* FI *and* FJ *are generating sets and a morphism* f *is a weak equivalence (resp. fibration) if and only if* Uf *is a weak equivalence (resp. fibration) in* $\mathcal{M}$*. Also,* (F, U) *is a Quilen pair with respect to this model structure.*

The following two lemmas allow us to use modify the sufficient conditions. The statement of both lemmas are taken from [BM03].

Lemma 2.3.37. *Let* $(F \dashv G) \colon \mathcal{N} \longrightarrow \mathcal{M}$ *be an adjunction. If* G *preserves filtered colimits, then* F *preserves small objects.*

Lemma 2.3.38. *If* $\mathcal{N}$ *has a fibrant replacement functor and functorial path objects for fibrant objects then any transfinite composition of pushouts of images under* F *of the generating trivial cofibrations in* $\mathcal{M}$ *yields a weak equivalence in* $\mathcal{N}$*.*

Example 2.3.39. As shown by Quillen in [Qui67], the category of simplicial abelian groups and the category of simplicial commutative algebras admit a transferred model structure along the forgetful functor to simplicial sets.

2.3.6 Model structures on categories of diagrams

Theorem 2.3.40. *Let* $\mathcal{M}$ *be a cofibrantly generated model category and* I *a small category. The category of diagrams* $\mathcal{M}^I$ *admits a model structure such that a natural transformation* $f \colon X \to Y$ *is a weak equivalence (resp. a fibration) in* $\mathcal{M}^I$ *if and only if* $f(i) \colon X(i) \to Y(i)$ *is a weak equivalence (resp. fibration) in* $\mathcal{M}$ *for every object* $i \in I$*.*

We call this model structure the *projective* model structure on the category of diagrams.

Theorem 2.3.41. *Let* $\mathcal{M}$ *be a combinatorial model category and* I *a small category. The category of diagrams* $\mathcal{M}^I$ *admits a model structure such that a natural transformation* $f \colon X \to Y$ *is a weak equivalence (resp. a cofibration) in* $\mathcal{M}^I$ *if and only if* $f(i) \colon X(i) \to Y(i)$ *is a weak equivalence (resp. cofibration) in* $\mathcal{M}$ *for every object* $i \in I$*.*

We call this model structure the *injective* model structure on the category of diagrams.

Chris Reedy (in [Ree74]) showed that the category $\mathcal{M}^{\Delta^{op}}$ of simplicial objects in a model category $\mathcal{M}$ can be endowed with a one more model structure. There is a larger class of small categories with no non-identity isomorphisms, called *Reedy categories*, that one can take instead of Δ, for which the same argument works. In [BM11], Berger and Moerdijk introduced a generalization of a Reedy category allowing non-identity isomorphisms.

Definition 2.3.42. A *generalized Reedy category* is given by a small category $\mathbb{R}$ equipped with a degree function $d\colon \mathrm{Ob}(\mathbb{R}) \to \mathcal{N}_0$ and two wide subcategories $\mathbb{R}_-$ and $\mathbb{R}_+$ such that

(i) non-invertible morphisms in $\mathbb{R}_-$ (resp. $\mathbb{R}_+$) lower (resp. raise) the degree;

(ii) isomorphisms in $\mathbb{R}$ preserve the degree;

(iii) $\mathbb{R}_+ \cap \mathbb{R}_- = \mathrm{Iso}(\mathbb{R})$;

(iv) every morphism f of $\mathbb{R}$ factors as $f = gh$ with $g \in \mathbb{R}_+$ and $h \in \mathbb{R}_-$ in a unique way up to isomorphism;

(v) if $\theta f = f$ for an isomorphism θ and $f \in \mathbb{R}_+$, then θ is an identity.

A generalized Reedy category $\mathbb{R}$ is *dualizable* if $f\theta = f$ for an isomorphism θ and $f \in \mathbb{R}_+$ implies that θ is an identity.

Definition 2.3.43. For an object r of a generalized Reedy category, the category $\mathbb{R}_+(r)$ has as objects the non-invertible morphisms in $\mathbb{R}_+$ with codomain r. For each functor $F\colon \mathbb{R} \to \mathcal{M}$ and each object r or $\mathbb{R}$, the *r-th latching object* $L_r(X)$ of X is defined to be

$$L_r(X) = \operatorname*{colim}_{s \to r} X_s$$

where the colimit is taken over the category $\mathbb{R}_+(r)$.

Dually, the category $\mathbb{R}_-(r)$ has as objects non-invertible morphisms in $\mathbb{R}_-$ with domain r. The *r-th matching object* $M_r(X)$ is defined to be

$$M_r(X) = \lim_{r \to s} X_s$$

where the limit is taken over the category $\mathbb{R}_-(r)$.

In the following definition, for a given object r of $\mathbb{R}$, we consider the group $\mathrm{Aut}(r)$ as a category with one object and endow $\mathcal{M}^{\mathrm{Aut}(r)}$ with the projective model structure.

Definition 2.3.44. Let $\mathcal{M}$ be a cofibrantly generated category. A morphism $f\colon X \to Y$ in $\mathcal{M}^{\mathbb{R}}$ is a

(i) a *Reedy weak equivalence* if for each r, the induced map $f_r\colon X_r \to Y_r$ is a weak equivalence in $\mathcal{M}^{\mathrm{Aut}(r)}$.

(ii) a *Reedy cofibration* if for each r, the relative latching map $X_r \cup_{L_r(X)} L_r(Y) \to Y_r$ is a cofibration in $\mathcal{M}^{\mathrm{Aut}(r)}$.

(iii) a *Reedy fibration* if for each r, the relative matching map $X_r \to M_r(X) \times_{M_r(Y)} Y_r$ is a fibration in $\mathcal{M}^{\mathrm{Aut}(r)}$.

Theorem 2.3.45 ([BM11], 1.6)**.** *Let $\mathcal{M}$ be a cofibrantly generated model category and $\mathbb{R}$ a generalized Reedy category. With the classes of Reedy weak equivalences, Reedy cofibrations and Reedy fibrations, the category of diagrams $\mathcal{M}^{\mathbb{R}}$ is a model category.*

2.3.7 Simplicial model categories and function complexes

Recall that a category $\mathcal{M}$ *enriched in sSet* (in the sense of [Kel82]) consists of a simplicial set $\hom(X,Y)$ for any two objects X and Y in $\mathcal{M}$ and compositions map

$$\hom(Y,Z) \times \hom(X,Y) \to \hom(X,Z)$$

for any three objects X, Y and Z, such that the composition is associative and unital and

$$\mathrm{Hom}_{\mathcal{M}}(X,Y) = \hom(X,Y)_0 = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^0, \hom(X,Y)).$$

In that case, there is a bifunctor $\hom\colon \mathcal{M}^{op} \times \mathcal{M} \to \mathrm{sSet}$.

Definition 2.3.46. Let $\mathcal{M}$ be a category enriched in sSet. We say that $\mathcal{M}$ is *weakly tensored* if for every object X the functor $\hom(X,-)\colon \mathcal{M} \to \mathrm{sSet}$ has a left adjoint functor $-\otimes X\colon \mathrm{sSet} \to \mathcal{M}$.
We say that $\mathcal{M}$ is *weakly cotensored* if for every object Y the functor $\hom(-,Y)\colon \mathcal{M}^{op} \to \mathrm{sSet}$ has a left adjoint functor $Y^{(-)}\colon \mathrm{sSet} \to \mathcal{M}$.

Remark 2.3.47. If $\mathcal{M}$ is weakly tensored, the Yoneda lemma implies that

$$\hom(X,Y)_n = \mathrm{Hom}_{\mathcal{M}}(\Delta^n, \hom(X,Y)) = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \otimes X, Y).$$

Remark 2.3.48. If $\mathcal{M}$ is a category enriched in sSet which is weakly tensored and weakly cotensored, then for every simplicial set K, there are adjoint functors $K \otimes -\colon \mathcal{M} \to \mathcal{M}$ and $(-)^K\colon \mathcal{M} \to \mathcal{M}$, i.e. there are natural bijections

$$\mathrm{Hom}_{\mathcal{M}}(K \otimes X, Y) \cong \mathrm{Hom}_{\mathrm{sSet}}(K, \hom(X,Y)) \cong \mathrm{Hom}_{\mathcal{M}^{op}}(Y^K, X) = \mathrm{Hom}_{\mathcal{M}}(X, Y^K)$$

Remark 2.3.49. If $\mathcal{M}$ is a weakly tensored category, then for any two simplicial sets K and L and for any object X in $\mathcal{M}$ there is a map

$$a\colon (K \times L) \otimes X \to K \otimes (L \otimes X).$$

These maps satisfy the associativity coherence law coming from the associativity of the composition for enriched hom-spaces.

The condition that all the maps $a\colon (K \times L) \otimes X \to K \otimes (L \otimes X)$ are isomorphisms is equivalent to the condition that there are natural isomorphisms of simplicial sets

$$\hom(K \otimes X, Y) \cong \hom(K, \hom(X,Y))$$

where on the right hand side we use that the category sSet is enriched in itself. In that case, we say that $\mathcal{M}$ is (strongly) tensored. Analogously, we define the notion of (strongly) cotensored category.

Note that some authors define a simplicial category to be the category which is both strongly tensored and strongly cotensored. On the other hand, we will say that $\mathcal{M}$ is a *simplicial category* if it is enriched in sSet, but we do not even require that it is weakly tensored or weakly cotensored.

Definition 2.3.50. A *(weakly) simplicial model category* is a category $\mathcal{M}$ which is enriched in sSet, weakly tensored and weakly cotensored, which has a model structure and satisfies the following two axioms

(i) If $i\colon A \to B$ is a cofibration in $\mathcal{M}$ and $j\colon L \to K$ is a cofibration of simplicial sets, then the pushout product map

$$A \otimes K \sqcup_{A\otimes L} B \otimes L \to B \otimes K$$

is a cofibration in $\mathcal{M}$ that is a trivial cofibration if either i or j is.

(ii) If $A \to B$ is a cofibration in $\mathcal{M}$ and K and L are simplicial sets, then the map

$$(K \times L) \otimes B \cup_{(K\times L)\otimes A} K \otimes (L \otimes A) \to K \otimes (L \otimes B)$$

is a trivial cofibration.

Remark 2.3.51. Property (i) in the previous definition is called the *pushout-product axiom*. For a tensored category the pushout-product axiom is equivalent to the following axiom which was denoted in [Qui67] axiom SM7:

- If $i\colon A \to B$ is a cofibration in $\mathcal{M}$ and $p\colon X \to Y$ is a fibration in $\mathcal{M}$, then the map of simplicial sets $\hom(B,X) \to \hom(A,X) \times_{\hom(A,Y)} \hom(B,Y)$ is a fibration that is a trivial fibration if either i or p is a weak equivalence.

Remark 2.3.52. Many references that we have come across define simplicial model categories as categories which are strongly tensored and cotensored and satisfy the pushout-product property, (i). Note that the property (ii) is then automatically satisfied, since the considered maps a are isomorphisms.

However, we need to consider the weaker notions since our main example, the category of dendroidal sets with the stable model structure, will be a weakly simplicial model category which is not strongly tensored.

We have adopted the definitions of weakly tensored and weakly cotensored simplicially enriched categories and of a weakly simplicial model category from [HHM13], Section 3.5.

Remark 2.3.53. The axiom SM7 implies that $\hom(B,X)$ is a Kan complex for every cofibrant B and fibrant X.

Corollary 2.3.54. *If X is a cofibrant object in $\mathcal{M}$ and $j\colon L \to K$ is a cofibration of simplicial sets, then the map $1_B \otimes j\colon B \otimes L \to B \otimes K$ is a cofibration that is a trivial cofibration if j is a weak equivalence.*

Definition 2.3.55. We say that morphisms f and $g\colon X \to Y$ are *simplicially homotopic* ($f \sim_s g$) if they are in the same connected component as vertices of $\hom(X, Y)$.

Remark 2.3.56. We can define $f \sim g$ if there is a map $\Delta[1] \to \hom(X, Y)$ taking ends of $\Delta[1]$ to f and g. The relation $\sim$ is not an equivalence relation unless $\hom(X, Y)$ is a Kan complex, so $\sim_s$ is the smallest equivalence relation generated by the relation $\sim$.

Remark 2.3.57. The axioms of a weakly simplicial model category ensure that if X is a cofibrant object, then $X \otimes \Delta[1]$ is a cylinder object for X. This provides an important connection between simplicial homotopy and left homotopy which makes it possible to detect weak equivalences in terms of simplicial hom spaces.

If $f, g\colon X \to Y$ are simplicially homotopic maps, then the maps $Q(f), Q(g)\colon Q(X) \to Q(Y)$ between cofibrant replacements are also simplicially homotopic maps. Since $Q(X)$ is cofibrant, $Q(X) \otimes \Delta[1]$ is a cylinder object and hence $Q(f)$ and $Q(g)$ are left homotopy equivalent. The proof of the following lemma, now follows from the results of section 2.3.2.

Lemma 2.3.58 ([Hir02], 9.5.15 and 9.5.16.)**.** *Let $f, g\colon X \to Y$ be two morphisms in a simplicial model category. Then $f \sim_s g$ if and only if $\gamma(f) = \gamma(g)$ in $Ho(\mathcal{M})$. Consequently, a simplicial homotopy equivalence in a simplicial model category is a weak equivalence.*

Theorem 2.3.59. *Let $f\colon X \to Y$ be a morphism in a weakly simplicial model category. Then f is a weak equivalence if and only if for every cofibrant replacement $Q(f)\colon Q(X) \to Q(Y)$ and every fibrant object Z the map of simplicial sets*

$$Q(f)^*\colon \hom(Q(Y), Z) \to \hom(Q(X), Z)$$

is a weak equivalence.

Proof. If f is a trivial cofibration between cofibrant objects, Axiom SM7 implies that f^* is a trivial fibration. Hence, Ken Brown's Lemma implies that if f is a weak equivalence between cofibrant objects, f^* is a weak equivalence. In general, if f is a weak equivalence then $Q(f)$ is a weak equivalence between cofibrant objects, so $Q(f)^*$ is a weak equivalence.

Let $Q(f)$ be a cofibrant replacement such that $Q(f)^*$ is a weak equivalence for every fibrant object Z. For every fibrant object we have isomorphisms

$$Q(f)^*\colon \pi_0 \hom(R(Y), Z) \to \pi_0 \hom(R(X), Z).$$

Considering first $Z = Q(X)$ and then $Z = Q(Y)$, we get that $Q(f)$ is a simplicial homotopy equivalence. Lemma 2.3.58 implies that $Q(f)$ is a weak equivalence and hence by CM2 f is a weak equivalence. $\square$

Remark 2.3.60. Theorem 2.3.59 shows that the Kan complexes $\hom(X, Y)$ for X cofibrant and Y fibrant hold the homotopy information of a model structure of $\mathcal{M}$. Hence, if

Q and R are respectively cofibrant and fibrant replacement functors, for any two objects X and Y we define

$$\underline{\mathrm{hom}}(X, Y) = \mathrm{hom}(QX, RY).$$

We call the Kan complexes $\underline{\mathrm{hom}}(X, Y)$ the *derived mapping spaces* or *homotopy function complexes.*

Note that the complexes $\underline{\mathrm{hom}}(X, Y)$ carry much more information about the homotopies than it is visible from the homotopy category. Indeed, the homotopy category "can see" only the set of connected components of $\underline{\mathrm{hom}}(X, Y)$ for objects X and Y:

$$\pi_0 \underline{\mathrm{hom}}(X, Y) \cong \mathrm{Hom}_{Ho(\mathcal{M})}(X, Y).$$

Example 2.3.61. Let $\mathcal{M}$ be a simplicial model category. If we restrict to bifibrant objects we obtain a category enriched in Kan complexes. Taking the homotopy coherent nerve yields an ∞-category. The homotopy theory of ∞-categories has been developed by A. Joyal and further theory extending the well-known theorems from category theory to ∞-categorical setting has been carried out by J. Lurie. For further reference, we will use [Lur09].

Remark 2.3.62. For model categories that are not simplicial, homotopy function complexes also exist, but the details are more technically involved. In subsequent chapters we will construct and study only simplicial model categories, so there is no reason for us to go into details about the general case and we only give the basic idea.

For a model category $\mathcal{M}$, we consider the category $s\mathcal{M} = [\Delta^{op}, \mathcal{M}]$ of simplicial objects in $\mathcal{M}$ and the category $c\mathcal{M} = [\Delta, \mathcal{M}]$ of cosimplicial objects in $\mathcal{M}$. The categories $s\mathcal{M}$ and $c\mathcal{M}$ have a Reedy model structure described in subsection 2.3.6. For an object X in $\mathcal{M}$ we denote by cX the constant cosimplicial object, i.e. such that $(cX)_n = X$ and all structure maps are 1_X. Analogously, let sX be the constant simplicial object.

Definition 2.3.63. A *cosimplicial resolution* of an object X in a model category $\mathcal{M}$ is a cofibrant replacement $\tilde{X} \to cX$ in the Reedy model structure on $c\mathcal{M}$. A *simplicial resolution* of an object X in a model category $\mathcal{M}$ is a fibrant replacement $sX \to \hat{X}$ in the Reedy model structure on $s\mathcal{M}$.

For two objects X and Y the function complex is given by the simplicial set

$$\underline{\mathrm{hom}}(X, Y) = \mathrm{diag}\,\mathrm{hom}(cX, sY)$$

such that $\mathrm{diag}\,\mathcal{M}(\tilde{X}, \hat{Y})_n = \mathcal{M}(\tilde{X}_n, \hat{Y}_n)$ with the structure maps induced by the structure maps of $\tilde{X}$ and $\hat{Y}$.

There are other approaches to construct simplicially enriched category starting from a model category $\mathcal{M}$ with weak equivalences W. We call such a simplicially enriched category a *simplicial localization* if the homotopy function complexes have the same homotopy type as the one constructed by (co)simplicial resolutions. The most standard construction is called the *hammock localization* and it is due to W. G. Dwyer and D. Kan, see [DK80c], [DK80a] and [DK80b].

2.3.8 Left Bousfield localization

Localization of a category is a procedure which turns morphisms into isomorphisms. Localization of a model category is a procedure which turns morphisms into weak equivalences and hence gives a localization of the homotopy category.

Definition 2.3.64. Let $\mathcal{M}$ be a model category and S a (small) set of morphisms in $\mathcal{M}$. A *left localization* of $\mathcal{M}$ with respect to S is given by a model category $L_S\mathcal{M}$ and a left Quillen functor $j\colon \mathcal{M} \to L_S\mathcal{M}$ such that

(i) The total left derived functor $Lj\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(L_S\mathcal{M})$ sends the classes represented by morphisms in S to isomorphisms in $\mathrm{Ho}(L_S\mathcal{M})$.

(ii) If $\mathcal{N}$ is a model category and $F\colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor such that $LF\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ sends the classes represented by morphisms in S to isomorphisms in $\mathrm{Ho}(\mathcal{N})$, then there is a unique left Quillen functor $\tilde{F}\colon L_S\mathcal{M} \to \mathcal{N}$ such that $\tilde{F}j = F$.

Definition 2.3.65. Let $\mathcal{M}$ be a model category and S a set of morphisms in $\mathcal{M}$. An object Z of $\mathcal{M}$ is *S-local* if Z is fibrant and for every morphism $f\colon A \to B$ in S the induced map

$$f^*\colon \underline{\mathrm{hom}}(B, Z) \to \underline{\mathrm{hom}}(A, Z)$$

is a weak equivalence of simplicial sets.

A map $g\colon X \to Y$ in $\mathcal{M}$ is an *S-local equivalence* if for every S-local object Z the induced map

$$g^*\colon \underline{\mathrm{hom}}(Y, Z) \to \underline{\mathrm{hom}}(X, Z)$$

is a weak equivalence of simplicial sets.

Remark 2.3.66. For a (weakly) simplicial model category, if $g\colon X \to Y$ is a cofibration, then $g^*\colon\ \mathrm{hom}(Y, Z) \to \mathrm{hom}(X, Z)$ is a fibration for every fibrant object Z by Axiom SM7. Hence g is an S-local equivalence if and only if g^* is a trivial fibration of simplicial sets.

Theorem 2.3.59 implies the following S-local analog of Whitehead's theorem in the case of simplicial model categories, the general case is given as 3.1.5 and 3.2.13 in [Hir02].

Theorem 2.3.67. *Let $\mathcal{M}$ be a model category and S a class of morphisms in $\mathcal{M}$.*

(i) Every weak equivalence in $\mathcal{M}$ is an S-local equivalence.

(ii) If X and Y are S-local objects and $f\colon X \to Y$ is an S-local equivalence, then f is a weak equivalence.

Definition 2.3.68. A *left Bousfield localization* of a model category $\mathcal{M}$ with respect to a class of morphisms S of $\mathcal{M}$ is a model structure $L_S\mathcal{M}$ on the category $\mathcal{M}$ such that

(i) The class of weak equivalences of $L_S\mathcal{M}$ equals the class of S-local equivalences of $\mathcal{M}$.

(ii) The class of cofibrations of $L_S\mathcal{M}$ equals the class of cofibrations of $\mathcal{M}$.

(iii) The class of fibrations of $L_S\mathcal{M}$ is the class of maps with the right lifting property with respect to those maps that are both cofibrations and S-local equivalences.

Remark 2.3.69. Theorem 2.3.59 implies that every weak equivalence in $\mathcal{M}$ is an S-local equivalence with respect to any set S of morphisms in $\mathcal{M}$. This shows that a left Bousfield localization $L_S\mathcal{M}$ of a model category $\mathcal{M}$ is another model structure on the same underlying category having the same class of cofibrations and a bigger class of weak equivalence. Consequently, the class of fibrations of $L_S\mathcal{M}$ is a subclass of the class of fibrations of $\mathcal{M}$ and every fibrant object in $L_S\mathcal{M}$ is fibrant in $\mathcal{M}$.

Theorem 2.3.70. *If $L_S\mathcal{M}$ is a left Bousfield localization of $\mathcal{M}$ with respect to S, then the identity functor $1_{\mathcal{M}}\colon \mathcal{M} \to L_S\mathcal{M}$ is a left localization of $\mathcal{M}$ with respect to S.*

Lemma 2.3.71. *Let $L_S\mathcal{M}$ be a left Bousfield localization of a left proper model category $\mathcal{M}$ with respect to S. A fibrant object Z of $\mathcal{M}$ is fibrant in $L_S\mathcal{M}$ if and only if it is S-local.*

We will use the following restatement of 3.3.16 and 3.4.1 from [Hir02].

Lemma 2.3.72. *Let M be a left proper model category and C a set of morphisms such that the left Bousfield localization L_CM exists. If X and Y are fibrant objects in L_CM then a morphism $f : X \to Y$ is a fibration in L_CM if and only if it is a fibration in M.*

Remark 2.3.73. A left Bousfield localization $L_S\mathcal{M}$ does not exist always, but there are two large classes of examples. Left proper cellular model structures are studied in [Hir02] and for such model categories a left Bousfield localization exists with respect to any class of morphisms. For our interest, more important examples are left proper combinatorial model structures. The proof of existence of a left Bousfield localization is due to J. Smith. The proof is written down in [Lur09] A.3.7.3 and [Bar07] 2.11 and 4.7.

Theorem 2.3.74 (Smith)**.** *Let $\mathcal{M}$ be a left proper simplicial combinatorial model category and S a (small) set of morphisms in $\mathcal{M}$. Then the left Bousfield localization $L_S\mathcal{M}$ exists and it is a left proper simplicial combinatorial model category.*

Theorem 2.3.75 (Dugger)**.** *Let $\mathcal{M}$ be a combinatorial simplicial model category. There is a small category $\mathcal{C}$ and a set of morphisms S in the category $sPSh(\mathcal{C})$ of simplicial presheaves on $\mathcal{C}$ such that the Bousfield localization of the global projective model structure on $sPSh(\mathcal{C})$ with respect to S is Quillen equivalent to $\mathcal{M}$.*

Example 2.3.76. The Kan-Quillen model structure on simplicial sets is a Bousfield localization of the Joyal model structure with respect to $S = \{\Lambda^0[n] \to \Delta[n] \colon n \in \mathbb{N}\}$.

Chapter 3

Dendroidal sets

This chapter contains preliminaries on dendroidal sets. As we mentioned in the introduction, dendroidal sets are presheaves on the category Ω of finite rooted trees. In the first section, we will review the definition of trees and the category Ω, as well as the notions of face and degeneracy maps between trees. In the second section, we will present results from the literature about various aspects of the category of dendroidal sets relevant to this thesis. These aspects are: the relations to simplicial sets and operads, the monoidal structure and (the key ingredients of) the known model structures.

3.1 The formalism of trees

3.1.1 Definition of a tree

Definition 3.1.1. A *tree* is a triple $(T, \leq, L)$ consisting of a finite non-empty set T, a partial order $\leq$ on T and a subset L of minimal elements of T such that

- there is a unique maximal element $r \in T$, called the *root* of T;
- for every $e \in T$, the order $\leq$ induces a total order on the set $\{f \in T : e \leq f\}$.

We usually denote such a triple $(T, \leq, L)$ simply by T. The elements of the set L are called *leaves*. Elements of T are called *edges*. *Inner edges* are edges other than the root and the leaves. We define the *height* of an edge e as the cardinality of the (totally ordered) set $\{f \in T : e \leq f\}$.

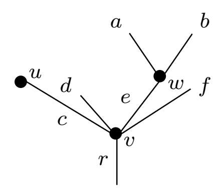
For an edge e which is not a leaf, the set v of all of its immediate predecessors is called a *vertex*. We say that e is the *output* of v. Elements of a vertex are also called *inputs* of v. A *sibling* of an edge e is any other edge f such that e and f are both inputs of the same vertex. We say that an edge e is *attached* to a vertex v if e is the output or an input of v.

The unique vertex whose output is the root is called the *bottom vertex*. We say that a vertex is a *top vertex* if all of its inputs are leaves. A top vertex may be empty and then it is called a *stump*. Note that the union of the set of leaves and the set of stumps is in bijection with the set of minimal elements of T. A tree with no stumps is called an *open* tree.

A tree with exactly one vertex is called a *corolla* and denoted C_n where n is the number of leaves. A *linear tree* is a tree whose all vertices have exactly one input. A linear tree with exactly n vertices is denoted L_n. A tree with no vertices is called the *unit tree* and it is denoted by L_0.

To draw a tree on a paper we must put a total order on the inputs of every vertex. This gives additional structure to the tree called a *planar structure*.

Example 3.1.2. Here is a picture of a (planar) tree with a root r, the set of leaves $L = \{a, b, d, f\}$, inner edges c and e, a stump u, another top vertex $w = \{a, b\}$ and a bottom vertex $v = \{c, d, e, f\}$.



Definition 3.1.3. Let S be a tree with the set of leaves $L(S) = \{l_1, ..., l_m\}$. Let $T_1, ..., T_m$ be trees with pairwise disjoint underlying sets such that for every index $i \in \{1, ..., m\}$ the root l_i of T_i is the only common element of S and T_i. We define a new tree $S \circ (T_1, ..., T_m)$ such that

- the underlying set is the union $S \cup T_1 \cup ... \cup T_m$,
- the partial order extends the partial orders of $S, T_1, ..., T_m$ in the sense that $t \leq s$ for all $s \in S$ such that $l_i \leq s$ and all $t \in T_i, i = 1, ..., m$
- the set of leaves is $L(T_1) \cup ... \cup L(T_m)$.

We say that we have obtained $S \circ (T_1, ..., T_m)$ by *grafting* the trees $T_1, ..., T_m$ on top of S.

3.1.2 Operads associated with trees and the category Ω

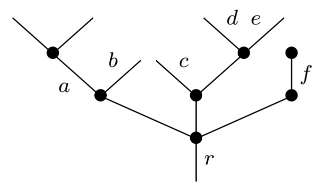
Definition 3.1.4. Let $(T, \leq, L)$ be a tree, $n \geq 0$ an integer and $t_1, ..., t_n, t$ elements of T such that $t_i \leq t$ for $i = 1, ..., n$. A pair $(\{t_1, ..., t_n\}, t)$ is an *operation* of T if

- for every leaf $l \in L, l \leq t$ there exists a unique $i \in \{1, 2, ..., n\}$ such that $l \leq t_i$;
- for every stump v with an output edge f there exists at most one $i \in \{1, 2, ..., n\}$ such that $f \leq t_i$.

We also write $(t_1, ..., t_n; t)$ for such an operation.

Example 3.1.5. Let v be a vertex of a tree T with an output edge e. Then (v, e) is an operation of T.

Example 3.1.6. The following tree has an operation $(a, b, c, d, e, f; r)$.



Note that the same tree also has an operation $(a, b, c, d, e; r)$ and many others.

Lemma 3.1.7. *Let T be a tree.*

(i) For every $t \in T$, $(t; t)$ is an operation of T.

(ii) If $(t_1, ..., t_n; t)$ and $(t_{i,1}, ..., t_{i,k_i}; t_i)$ for $i \in \{1, ..., n\}$ are operations of T, then

$$(t_{1,1}, ..., t_{1,k_1}, t_{2,1}, ..., t_{n,k_n}; t)$$

is also an operation of T.

(iii) If $(t_1, ..., t_n; t)$ is an operation of T then $(t_{\sigma(1)}, t_{\sigma(2)}, ..., t_{\sigma(n)}; t)$ is also an operation of T for any permutation $\sigma \in \Sigma_n$.

Proof. All statements follow directly from the definition and their verification is left to the reader. $\square$

Definition 3.1.8. To every tree T we associate a coloured operad $\Omega(T)$ with a set of colours being T and

$$\Omega(T)(t_1, ..., t_n; t) = \begin{cases} *, & \text{if } (t_1, ..., t_n; t) \text{ is an operation of } T; \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $*$ denotes a fixed singleton. The structure maps are uniquely determined and Lemma 3.1.7 shows they are well-defined.

Remark 3.1.9. Note that $\Omega(T)$ is the free operad generated by the vertices of T.

Lemma 3.1.10. *Let S and T be trees. A map of sets $f : S \to T$ extends to a morphism of operads $f : \Omega(S) \to \Omega(T)$ if and only if $(f(s_1), ..., f(s_n); f(s))$ is an operation in T for every operation $(s_1, ..., s_n; s)$ in S.*

Proof. A morphism of operads $f : \Omega(S) \to \Omega(T)$ consists of component maps

$$\Omega(S)(s_1, ..., s_n; s) \to \Omega(T)(f(s_1), f(s_2), ..., f(s_n); f(s))$$

compatible with the structure maps. The component maps are either the unique maps $\emptyset \to *$ or identities on $\emptyset$ or identities on $*$. Compatibility follows directly since all structure maps are uniquely determined by their domains and codomains. $\square$

Definition 3.1.11. The category Ω of trees is a category whose objects are trees and the morphism sets are given by

$$\mathrm{Hom}_{\Omega}(S,T) = \mathrm{Hom}_{\mathrm{Oper}}(\Omega(S), \Omega(T)).$$

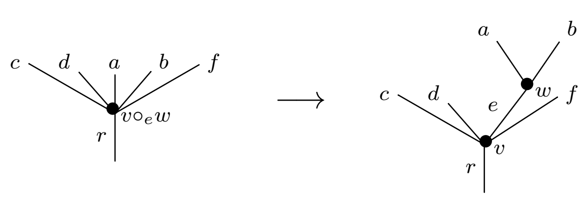
Hence Ω is a full subcategory of the category of (coloured) operads.
The category Ω has a full subcategory Ω° of open trees .

Each planar tree is considered to be a non-symmetric coloured operad and each (non-planar) tree to be a (symmetric) coloured operad. The category Ω_p is the full subcategory of the category of non-symmetric operads and Ω is the full subcategory of symmetric operads. Note that the symmetrization functor from the category of non-symmetric operads to symmetric operads restricts to a functor $\Sigma : \Omega_p \to \Omega$ which on objects forgets the planar structure.

3.1.3 Elementary face and degeneracy maps

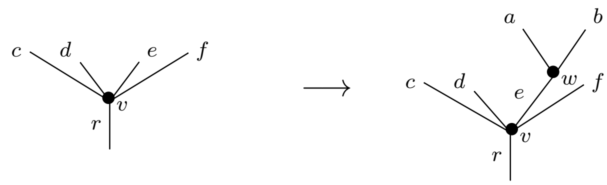
Definition 3.1.12. There are three types of elementary face maps : *inner*, *top* and *bottom*.

Let e be an inner edge of a tree T. We define $\partial_e T$ to be a tree whose underlying set is $T \setminus \{e\}$, the partial order is induced from the one on T and the set of leaves is the same as of T. There is an *inner elementary face map* $\partial_e \colon \partial_e T \to T$ which is an inclusion of partially ordered sets. Note that if e is an input of a vertex v and the output of a vertex w, the tree $\partial_e T$ has a vertex $v \circ_e w := (w \cup v) \setminus \{e\}$ instead of v and w. In terms of graphs, we obtain $\partial_e T$ by contracting the edge e:

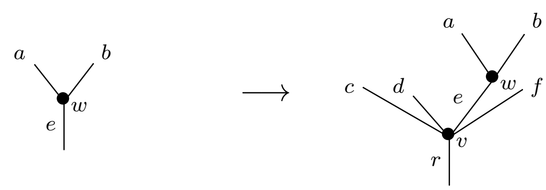


Let w be a top vertex of a tree T. We define $\partial_w T$ to be a tree whose underlying set is $T \setminus w$, the partial order is induced from the one on T and the set of leaves is obtained by deleting the inputs and adding the output of w to the set of leaves of T. There is a *top elementary face map* $\partial_w \colon \partial_w T \to T$ which is an inclusion of partially ordered sets.

Note that if T is a corolla with the root r there is a unique top elementary face map and $\partial_w T = L_0^r$. In terms of graphs, we chop off the vertex w and its inputs:



Let v be a bottom vertex of a tree T and e an input of v such that all other inputs of v are leaves. We define a tree $\partial_{v,e}T$ with the underlying set $\{f \in T : f \leq e\}$, the induced partial order from T and the set of leaves obtained by deleting the siblings of e from the set of leaves of T. There is a *bottom elementary face map* $\partial_{v,e} \colon \partial_{v,e}T \to T$ which is an inclusion of partially ordered sets. Note that if T is a corolla C_n, then $\partial_{v,e}T = L_0^e$ and we have n bottom elementary face maps. If T has at least two vertices, then e is an inner edge and there is only one bottom elementary face map so we may write ∂_v for $\partial_{v,e}$. In terms of graphs, we chop off v with the root and all inputs of v other than e:



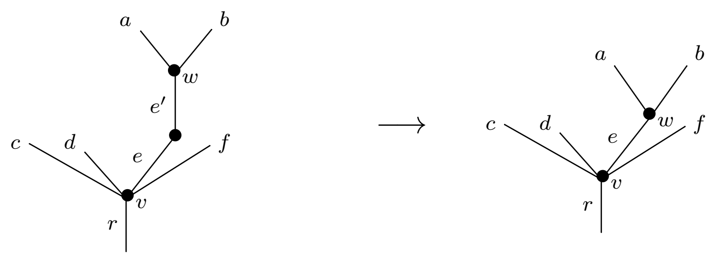
Remark 3.1.13. Note that the above definition also applies to the corolla. A corolla with n inputs has a unique top face and n bottom faces.

Remark 3.1.14. If $\partial_f T \to T$ is an elementary face map, then every operation of $\partial_f T$ is also an operation of T, hence by Lemma 3.1.10 elementary face maps are morphisms of operads. In fact, this implies that the elementary face maps are monomorphisms in Ω. One can check that every monomorphism in Ω can be decomposed as a composition of elementary face maps. This decomposition is not unique since there are certain relations between elementary face maps, called dendroidal relations.

The dendroidal relations express in which sense two elementary face maps commute. The dendroidal relation $\partial_f \partial_g T = \partial_g \partial_f T$ holds for any pair of elementary face maps ∂_f and ∂_g except in the following case.

Let v be a top or a bottom vertex of a tree T and e an inner edge attached to v. Let w be the other vertex attached to e. The face $\partial_e T$ (obtained by contracting the edge e) will have a vertex that corresponds to the composition of the vertices v and w. We denote this vertex of $\partial_e T$ by u. We can chop off the vertex w in $\partial_v T$ if and only if we can chop off u in $\partial_e T$. In this case the dendroidal relation is $\partial_w \partial_v T = \partial_u \partial_e T$.

Definition 3.1.15. Let e be an inner edge of a tree T. We define $\sigma_e T$ to be a tree whose underlying set is $T \cup \{e'\}$, where e' is not an element of T. The partial order is induced from the one on T with the addition that $e' \leq e$ and for any edge $f \leq e$, $f \neq e$ we have $f \leq e'$. The set of leaves of $\sigma_e T$ is the same as of T. There is an *elementary degeneracy map* $\sigma_e \colon \sigma_e T \to T$ which is the unique surjection of partially ordered sets sending e and e' to e. Note that the tree $\sigma_e T$ has a vertex with an input e and otuput e'. If we consider σ_e as a map of operads, then the operation represented by this vertex is sent to the identity on e. In terms of graphs, we obtain $\sigma_e T$ by adding a copy e' of the edge e:



Remark 3.1.16. For any pair of elementary degeneracy maps there is an obvious dendroidal relation - the two maps commute in the sense that $\sigma_e \sigma_f T = \sigma_f \sigma_e T$. Also, the relations between elementary face maps and elementary degeneracy maps are the obvious ones.

Lemma 3.1.17 ([MW07]). *Every morphism in Ω can be factored in a unique way as a composition of elementary face maps followed by an isomorphism and followed by a composition of elementary degeneracy maps.*

Remark 3.1.18. Note that the domain of an elementary face map has exactly one vertex less than the codomain. Also, the domain of an elementary degeneracy map has exactly one vertex more than the codomain. In fact, the category Ω is a generalized Reedy category with the degree function given by the number of vertices, cf. Section 2.3.6.

We discuss the combinatorics of elementary face maps in more detail in Chapter 4.

3.2 Dendroidal sets

3.2.1 Relating dendroidal sets to simplicial sets and operads

Definition 3.2.1. A *dendroidal set* is a presheaf on the category Ω and we denote the category of dendroidal sets

$$\mathrm{dSet} := [\Omega^{\mathrm{op}}, \mathrm{Set}].$$

Remark 3.2.2. We denote by $\Omega[T]$ the dendroidal set represented by a tree T and by η the representable $\Omega[L_0]$. If X is a dendroidal set, we denote $X_T := X(T)$. The Yoneda lemma implies $X_T \cong \mathrm{Hom}_{\mathrm{dSet}}(\Omega[T], X)$.

Definition 3.2.3. A dendrex is called *degenerate* if it is in an image of σ^* where σ is an elementary degeneracy map. A dendrex which is not degenerate is called a *non-degenerate* dendrex.

Lemma 3.2.4 ([MT10], Lemma 3.4.1)**.** *Let X be a dendroidal set and $x \in X_T$ a dendrex of X, for some tree T. There is a unique degeneracy maps $\sigma\colon T \to S$ (a composition of elementary degeneracy maps) and a unique nondegenerate dendrex $x^\# \in X_S$ such that $x = \sigma^*(x^\#)$.*

Remark 3.2.5. There is a fully faithful functor $i\colon \Delta \to \Omega$ sending the linear order $[n]$ to the linear tree L_n. It induces an adjunction

$$i_! : \text{ sSet} \rightleftarrows \text{dSet} : i^* .$$

The functor $i_!$ is fully faithful. This and other good properties of this adjunction make dendroidal sets a generalization of simplicial sets.

Remark 3.2.6. The inclusion of the subcategory Ω° of open trees into Ω, also induces an embedding of the category of *open dendroidal sets* (presheaves on Ω°) into the category of dendroidal sets. Where there is no danger of confusion we will consider open dendroidal sets as dendroidal sets.

Remark 3.2.7. We have already discussed the relation between the four categories Oper, Cat, dSet and sSet in the Introduction. Let us emphasize once more the existence of the nerve functor in the dendroidal setting.

By the general arguments of left Kan extensions along the Yoneda embedding the inclusion $\Omega \to$ Oper induces an adjunction

$$\tau_d : \text{ dSet} \rightleftarrows \text{Oper} : N_d .$$

We call N_d the *dendroidal nerve functor* and for every coloured operad P we have $N_d(P)_T = \mathrm{Hom}_{\mathrm{Oper}}(\Omega(T), P)$. Note that N_d is also a fully faithful functor.

3.2.2 A tensor product of dendroidal sets

Recall that on the category of (coloured) operads there is a tensor product $\otimes_{BV}$, called the Boardman-Vogt tensor product, making it a closed symmetric monoidal category (cf. Definition 2.2.41).

Definition 3.2.8. Let S and T be two trees. We define the binary tensor product of the representables $\Omega[T]$ and $\Omega[S]$ by

$$\Omega[S] \otimes \Omega[T] = N_d(\Omega(S) \otimes_{BV} \Omega(T)).$$

We can extend this definition to a functor $\otimes\colon \mathrm{dSet} \times \mathrm{dSet} \to \mathrm{dSet}$ which is preserving colimits in each variable.

Remark 3.2.9. Let P and Q be operads. There is a canonical map $N_dP \otimes N_dQ \to N_d(P \otimes_{BV} Q)$. Indeed, since

$$N_dP \otimes N_dQ = \operatorname*{colim}_{\Omega[S]\to N_dP} \operatorname*{colim}_{\Omega[T]\to N_dQ} \Omega[S] \otimes \Omega[T] = \operatorname*{colim}_{\Omega(S)\to P, \Omega(T)\to Q} N_d(\Omega(S) \otimes_{BV} \Omega(T))$$

the canonical map is obtained by the universal property of the colimit.

Remark 3.2.10. This binary tensor product is not strictly associative. We will not show this now, but after a few more remarks we will give a more explicit description of the binary tensor product (in terms of shuffles) and from that description it is easy to see that $(\Omega[L_1] \otimes \Omega[L_1]) \otimes \Omega[C_2]$ is not isomorphic to $\Omega[L_1] \otimes (\Omega[L_1] \otimes \Omega[C_2])$.

So, the binary tensor product is not part of a structure of a monoidal category, but it is a part of an *unbiased monoidal category with symmetric associativity constraints* as defined in [HHM13], Section 6.3. We do not give the whole definition here because we will only use binary and ternary tensor products.

In general, for dendroidal sets X_1, X_2 and X_3 there is no map between $(X_1 \otimes X_2) \otimes X_3$ and $X_1 \otimes (X_2 \otimes X_3)$, but we can relate these two terms to ternary tensor products.

In general, we define n-fold tensor products for representables by

$$\Omega[T_1] \otimes \ldots \otimes \Omega[T_n] = N_d(\Omega(T_1) \otimes_{BV} \ldots \otimes_{BV} \Omega(T_n))$$

and we extend this to a functor on arbitrary dendroidal sets preserving colimits in each variable. Note that we used here that the Boardman-Vogt tensor product $\otimes_{BV}$ is associative. By the previous remark, there are maps

$$(\Omega[T_1] \otimes \Omega[T_2]) \otimes \Omega[T_3] \to \Omega[T_1] \otimes \Omega[T_2] \otimes \Omega[T_3]$$

and

$$\Omega[T_1] \otimes (\Omega[T_2] \otimes \Omega[T_3]) \to \Omega[T_1] \otimes \Omega[T_2] \otimes \Omega[T_3]$$

for any trees T_1, T_2 and T_3. Hence, there are maps $(X_1 \otimes X_2) \otimes X_3 \to X_1 \otimes X_2 \otimes X_3$ and $(X_1 \otimes X_2) \otimes X_3 \to X_1 \otimes X_2 \otimes X_3$ for any dendroidal sets X_1, X_2 and X_3. These are examples of associativity constraints.

For our purpose we will only need the following result from [HHM13]: the associativity constraint $\Omega[T_1] \otimes (\Omega[T_2] \otimes \Omega[T_3]) \to \Omega[T_1] \otimes \Omega[T_2] \otimes \Omega[T_3]$ is an isomorphism if T_1 is a linear tree.

Remark 3.2.11. Using the binary tensor products we can show that the category of dendroidal sets is enriched in simplicial sets and weakly tensored and cotensored. If K is a simplicial set and X and Y are dendroidal sets we define

$$\underline{\mathrm{hom}}(X,Y)_n = \mathrm{Hom}_{\mathrm{dSet}}(X \otimes i_!\Delta[n], Y), \quad X \otimes K = X \otimes i_! K,$$

$$(Y^K)_T = \mathrm{Hom}_{\mathrm{dSet}}(\Omega[T] \otimes i_! K, Y).$$

To obtain a composition we use the associativity constrain $X \otimes (\Delta[n] \times \Delta[n]) \to X \otimes \Delta[n] \otimes \Delta[n]$ and the fact that $X \otimes \Delta[n] \otimes \Delta[n] \cong (X \otimes \Delta[n]) \otimes \Delta[n]$. If $f \in \underline{\mathrm{hom}}(X,Y)_n$ and $g \in \underline{\mathrm{hom}}(Y,Z)_n$ we define $gf \in \underline{\mathrm{hom}}(X,Z)_n$ as the map

$$\begin{tikzcd}
X \otimes \Delta[n] \arrow[r, "X\otimes\delta"] & X \otimes (\Delta[n] \times \Delta[n]) \arrow[d] & \\
& X \otimes \Delta[n] \otimes \Delta[n] \cong (X \otimes \Delta[n]) \otimes \Delta[n] \arrow[d, "f\otimes\Delta[n]"] & \\
& Y \otimes \Delta[n] \arrow[r, "g"] & Z
\end{tikzcd}$$

where $\delta\colon \Delta[n] \to \Delta[n] \times \Delta[n]$ is the diagonal. The fact that the composition is associative can be reduced to the case when X, Y and Z are representables and in that case it can be checked by a straightforward calculation using the explicit description of the diagonal and the associativity constraint.

Definition 3.2.12. Let S and T be trees, let r_S and r_T be the roots of S and T respectively, and $L(S) = \{l_1, ..., l_m\}$ be the set of leaves of S. We let $S \otimes r_T$ be a tree isomorphic to S with the underlying set $S \times \{r_T\}$. Similarly, we let $l_i \otimes T$ be a tree isomorphic to T for $i = 1, 2, ..., m$. We define

$$R_1 = (S \otimes r_T) \circ (l_1 \otimes T, l_2 \otimes T, ..., l_m \otimes T)$$

to be a tree obtained by grafting copies of T on top of S.

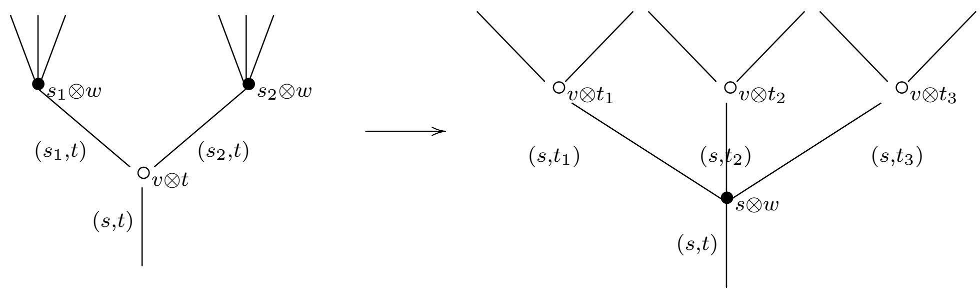
Definition 3.2.13. We inductively define trees that we call *shuffles of S and T*. Each shuffle consists of a tree and a decoration of each stump by one of two colours: black or white. The tree R_1 with all its stumps coloured black is a shuffle. Let $v = \{s_1, ..., s_m\}$ be a vertex of S with the output s, $w = \{t_1, ..., t_n\}$ be a vertex of T with the output t and a shuffle R such that $v \otimes t = \{(s_1, t), ..., (s_m, t)\}$ and $s_i \otimes w = \{(s_i, t_1), ..., (s_i, t_n)\}$ are vertices of R.

If $m \neq 0$ and $n \neq 0$, then we form a new shuffle consisting of a tree R' such that

- the underlying set of R' is $(\{(s, t_1), ..., (s, t_n)\} \cup R) \setminus \{(s_1, t), ..., (s_m, t)\}$;
- the partial order of R' is uniquely determined by $(s,t) \leq (s', t')$ in R' if and only if $s \leq s'$ in S and $t \leq t'$ in T;

- the set of leaves of R' is the same as the set leaves of R, i.e. $L(R') = L(R)$;

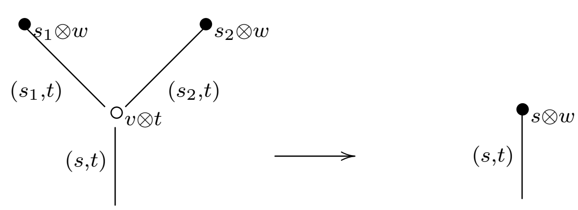
and the colours associated to the stumps of R' are the same as for R.



If $n = 0$ and $m \neq 0$, then we form a new shuffle in the following way. The tree of that shuffle is R' such that

- the underlying set of R' is $R \setminus \{(s_1, t), ..., (s_m, t)\}$;
- the partial order of R' is uniquely determined by $(s, t) \leq (s', t')$ in R' if and only if $s \leq s'$ in S and $t \leq t'$ in T;
- the set of leaves of R' is the same as the set leaves of R, i.e. $L(R') = L(R)$.

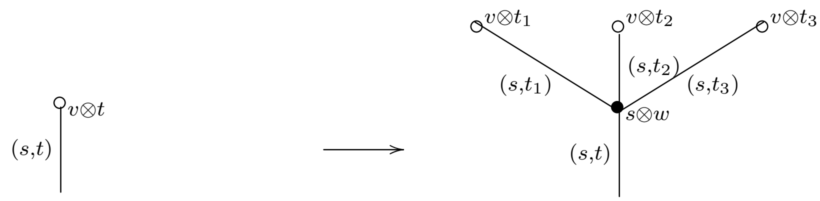
For all the stumps of R which are are also the stumps of R' the associated colour does not change. The new stump of R' with the output edge (s, t) is associated with the black colour.



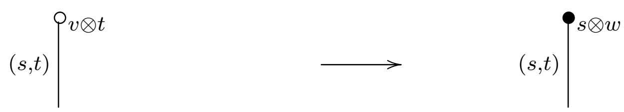
If $m = 0$, we will define a new shuffle only if the colour associated to the stump (s, t) is white. If this is the case and $n \neq 0$, then we form a new shuffle in the following way. The tree of that shuffle is R' such that

- the underlying set of R' is $\{(s, t_1), ..., (s, t_n)\} \cup R$;
- the partial order of R' is uniquely determined by $(s, t) \leq (s', t')$ in R' if and only if $s \leq s'$ in S and $t \leq t'$ in T;
- the set of leaves of R' is the same as the set leaves of R, i.e. $L(R') = L(R)$;

For all the stumps of R which are are also the stumps of R' the associated colour does not change. The new stumps of R' with the output edges (s, t_1), ..., (s_m, t) will be associated with the white colour.



If $m = 0$ and $n = 0$, then we form a new shuffle consisting of the same tree R with only the colour of the stump with output edge (s, t) changed from white to black.



We say that R' is obtained from R by a *percolation step*. In general, we will call the vertices of the form $v \otimes t$ *white* (i.e. those vertices having the same T-component for all inputs and the output, or a white stump) and the vertices of the form $s \otimes w$ *black* (i.e. those vertices having the same S-component for all inputs and the output, or a black stump).

Remark 3.2.14. Every shuffle R of S and T has the following properties:

(i) the set of edges of R is a subset of $S \times T$,

(ii) the root of R is (r_S, r_T),

(iii) the set of leaves $L(R)$ of R is equal to the set $L(S) \times L(T)$.

Note that the number of vertices of R_1 is finite, so there are finitely many shuffles of S and T and we obtain all shuffles from R_1 by letting the black vertices percolate towards the root in all possible ways.

Proposition 3.2.15 ([MW09], Lemma 9.5)**.** *Every shuffle R of S and T comes with a canonical monomorphism $m : \Omega[R] \to \Omega[S] \otimes \Omega[T]$. If R_i, $i = 1, ..., N$ are all shuffles of S and T then the dendroidal set $\Omega[S] \otimes \Omega[T]$ is isomorphic to the union of all $\Omega[R_i]$, i.e.*

$$\Omega[T] \otimes \Omega[S] \cong \bigcup_{i=1}^{N} \Omega[R_i].$$

In this context we also call an edge of a tree T a colour of T. Let P be a face of a shuffle R of S and T. We say that a colour t of a tree T *appears* in P if there is at least one edge (s, t) of P for some colour s of S.

Remark 3.2.16. If a shuffle R' is obtained from R by a percolation step we define $R \preceq R'$ and say that R is an *immediate predecessor* of R'. This defines a natural partial order on the set of all shuffles of S and T with R_1 being the unique minimal element. We call this a *left percolation poset* of S and T. Note that there is a unique maximal element in this partial set, namely the shuffle R_N obtained by grafting copies of S on top of T.

Remark 3.2.17. Symmetrically, there is a reversed poset called the *right percolation poset* of S and T, in which the smallest element is obtained by grafting copies of S on top of T.

Definition 3.2.18. We call a percolation step *regular* if none of the vertices that percolate are stumps. On the set of all shuffles we define a relation $\sim$ as the smallest equivalence relation such that $R_i \sim R_j$ if R_j is obtained from R_i by a regular percolation step.

Let R and R' be shuffles of S and T such that $R \preceq R'$ in the left percolation poset. If $R \sim R'$, then the minimal elements of the poset R are equal to minimal elements of the poset R'. Furthermore, if R has a vertex of type $v \otimes t$ for a vertex v of S and $t \in T$, then R' has a vertex of type $v \otimes t'$ for some $t' \in T$.

Let M be the Cartesian product of the set of minimal elements of S and the set of minimal elements of T. Note that R_1 is a unique shuffle having M as the set of minimal elements and no white vertex above a black vertex. Hence, if R has M as the set of minimal elements we have $R \sim R_1$ (we can apply inverse percolation steps to percolate black vertices to the bottom). This shows the following lemma.

Lemma 3.2.19. *For a shuffle R of S and T we have $R \sim R_1$ if and only if the set of minimal elements of the poset R is exactly the Cartesian product of the set of minimal elements of S and the set of minimal elements of T.*

Proposition 3.2.20. *Let S and T be trees and v a vertex of S. Every shuffle R of S and T either contains a vertex of type $v \otimes t, t \in T$ or there exists a shuffle R' which has a vertex of type $v \otimes t, t \in T$ and $R \subseteq R'$ (i.e. there is a mono $R \to R'$ in Ω given by inclusion of edges).*

Proof. Shuffle R_1 has a vertex of type $v \otimes r_T$. For $R \sim R_1$ the statement then follows from Lemma 3.2.19. If $R \nsim R_1$ we have obtained R by using percolation steps with stumps. We can perform only inverse percolation steps with stumps and we will obtain a shuffle R' which has the same set of minimal elements as R_1, hence $R' \sim R_1$. The statement now follows from the first part of the proof. □

3.2.3 Normal monomorphisms and normalizations

Definition 3.2.21. Any elementary face map $\partial_f : \partial_f T \to T$ induces a map of representable dendroidal sets $\partial_f : \Omega[\partial_f T] \to \Omega[T]$. The union of all images of maps $\partial_f : \Omega[\partial_f T] \to \Omega[T]$ is denoted by $\partial\Omega[T]$. The inclusion $\partial\Omega[T] \to \Omega[T]$ is called a *boundary inclusion.*

Definition 3.2.22. A monomorphism $f\colon A \to B$ of dendroidal sets is called *normal* if the action of the automorphism group $\mathrm{Aut}(T)$ on $B_T \setminus f(A_T)$ is free, for every tree T. A dendroidal set A is *normal* if $\emptyset \to A$ is a normal monomorphism. We also call normal monomorphisms *cofibrations*.

Remark 3.2.23. Let us give a characterization of normal objects in terms of skeletal filtrations. Consider the full subcategory $\Omega^{\leq n}$ of Ω on trees with at most n vertices. The inclusion $i_n\colon \Omega^{\leq n} \to \Omega$ induces an adjunction

$$(i_n)_! : \mathrm{dSet}^{\leq n} \rightleftarrows \mathrm{dSet} : i_n^* .$$

between the corresponding categories of presheaves. We denote by $\mathrm{Sk}_n\colon \mathrm{dSet} \to \mathrm{dSet}$ the composition $\mathrm{Sk}_n = (i_n)_! i_n^*$. For each dendroidal set X, the counit of the above adjunction $\mathrm{Sk}_n(X) \to X$ is a monomorphism ([MT10], Lemma 3.3.4). So, we have a presentation $X = \bigcup_{n\geq 0} \mathrm{Sk}_n(X)$ which is called the *skeletal filtration*.

Consider the commutative squares

$$\begin{tikzcd}
\coprod \partial\Omega[T] \arrow[r] \arrow[d] & \mathrm{Sk}_{n-1}(X) \arrow[d] \\
\coprod \Omega[T] \arrow[r] & \mathrm{Sk}_n(X)
\end{tikzcd}$$

where the coproducts ranges over all isomorphism classes of dendrices $\Omega[T] \to \mathrm{Sk}_n(X)$ for trees T with exactly n vertices. We say that X *has a normal skeletal filtration* if these squares are pushout diagrams for all $n \geq 1$.

Proposition 3.2.24 ([MT10], Proposition 3.4.4)**.** *A dendroidal set X is normal if and only if it has a normal skeletal filtration.*

Proposition 3.2.25. *The class of all normal monomorphisms is the smallest class of monomorphisms closed under pushouts and transfinite compositions that contains all boundary inclusions $\partial\Omega[T] \to \Omega[T]$.*

Proposition 3.2.26 ([CM11], Corollary 1.7 and 1.8)**.** *If $f\colon A \to B$ is any morphism of dendroidal sets and B is normal, then A is also normal. If f is a monomorphism and B is normal, then f is a normal monomorphism.*

The Small Object Argument (Proposition 2.1.28) gives the following result.

Corollary 3.2.27. *Every morphism $f\colon X \to Y$ of dendroidal sets can be factored as $f = hg$, $g\colon X \to Z, h\colon Z \to Y$ where g is a normal monomorphism and h has the right lifting property with respect to all normal monomorphisms.*

Definition 3.2.28. A *normalization* of a dendroidal set X is a morphism $X' \to X$ from a normal object X' having the right lifting property with respect to all normal monomorphisms.

By Corollary 3.2.27 for every morphism $f\colon X \to Y$ we can first construct a normalization $X' \to X$ and then factor the map $X' \to Y$ to obtain the following diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Note that both vertical maps are normalizations and $f'\colon X' \to Y'$ is a cofibration. We say that f' is a *normalization* of f.

Definition 3.2.29. A normal monomorphism is *linear* (resp. *open*) if it is in the saturated class generated by boundary inclusion $\partial\Omega[T] \to \Omega[T]$ for linear (resp. open) trees T.

Proposition 3.2.30 ([CM13a], Erratum)**.** *Let $f\colon A \to B$ and $K \to L$ be normal monomorphisms of dendroidal sets. If one of them is linear or both are open, then the map*

$$g\colon A \otimes K \sqcup_{A\otimes L} B \otimes L \to B \otimes K \tag{3.1}$$

is again a normal monomorphism.

3.2.4 Dendroidal Kan fibrations

Definition 3.2.31. For an elementary face map $\partial_f : \partial_f T \to T$ we denote by $\Lambda^f[T]$ the union of images of all elementary face maps $\partial_g : \Omega[\partial_g T] \to \Omega[T], g \neq f$. The inclusion $\Lambda^f[T] \to \Omega[T]$ is called a *horn inclusion*. A horn inclusion is called *inner* (respectively, *top* or *bottom*) if ∂_f is an inner (resp. top or bottom) elementary face map. For a bottom horn we will also use the term *root horn*.

Note that most trees do not have a bottom horn. A bottom horn can only exist if the tree is a corolla or the whole tree is concentrated over a single input of the root vertex.

Definition 3.2.32 ([MW07])**.** A dendroidal set X is called *inner Kan* if it admits fillers for all inner horns, i.e. for any inner edge e of a tree T the horn inclusion $\Lambda^e[T] \to \Omega[T]$ induces a bijection

$$\mathrm{Hom}(\Omega[T], X) \to \mathrm{Hom}(\Lambda^e[T], X).$$

Definition 3.2.33 ([Heu11a])**.** A *dendroidal Kan complex* is a dendroidal set that admits fillers for all inner and top horns.

Definition 3.2.34 ([BN14])**.** A dendroidal set X is called *fully Kan* if it has fillers for all horn inclusions. A dendroidal set X is called *strictly fully Kan* if additionally all fillers for trees T with more than one vertex are unique.

Remark 3.2.35. a) A fully Kan dendroidal set is also a dendroidal Kan complex and an inner Kan dendroidal set.

b) The reader might wonder why we do not impose uniqueness for corolla fillers in the strictly fully Kan condition. The reason is that this forces the underlying simplicial set to be discrete as we will see in Proposition 6.1.4.

Proposition 3.2.36. *(i) Any map $X \to Y$ can be factored as an anodyne map $X \to Z$ followed by a fully Kan fibration $Z \to Y$. In particular, for any dendroidal set X there exists an anodyne map $X \to X_K$ such that X_K is a fully Kan dendroidal set.*

(ii) For a morphism $X \to Y$ and any choice of $X \to X_K$ as in (i) there exists an anodyne map $Y \to Y_K$ such that Y_K is a fully Kan dendroidal set and there is a commutative square as follows

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X_K \arrow[r] & Y_K \end{tikzcd}$$

(iii) If X is countable, then X_K is countable as well.

(iv) Let X be a normal dendroidal set. If $U \subseteq X_K$ is countable, then there exists a countable $A \subseteq X$ with $U \subseteq A_K$.

Proof. (i) The statement follows from the small object argument applied to the set of horn inclusions. Since we wish to modify the proof in the rest of the proposition we will briefly describe the small object argument in this case.

Let us consider the set I of all isomorphism classes of squares

$$\begin{tikzcd} \Lambda^f[T] \arrow[r] \arrow[d] & X \arrow[d] \\ \Omega[T] \arrow[r, "\alpha_i"'] & Y \end{tikzcd}$$

such that there is no lift $\Omega[T] \to X$ which would fit in the diagram.

We construct X' as the pushout

$$\begin{tikzcd} \coprod_{i\in I} \Lambda^f[T] \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod_{i\in I} \Omega[T] \arrow[r] & X' \end{tikzcd}$$

where the coproducts vary over all isomorphism classes in I. Note that $X \to X'$ is a dendroidal anodyne extension. By the universal property of the pushout, there is a unique map $X' \to Y$.

The map $X' \to Y$ is not necessarily a fully Kan fibration, so we continue inductively. We denote $Z_{(1)} = X'$ and we construct $Z_{(n)} = Z'_{(n-1)}$ and the maps $Z_{(n)} \to Y$ for all positive integers n. In this way we obtain a sequence of anodyne extensions

$$X \to Z_{(1)} \to Z_{(2)} \to \dots$$

Let $Z = \bigcup_n Z_{(n)}$. Then the composition $X \to Z$ is a dendroidal anodyne extension and the map $Z \to Y$ is a fully Kan fibration (because dendroidal sets $\Omega[T]$ are small, so the lifting problem is reduced to a lifting problem at some stage $Z_{(n)}$ where it has a solution by construction).

The second statement is a special case for $Y = \emptyset$.

(ii) Let $X \to X_K$ be an anodyne extension and X_K a fully Kan dendroidal set. We form a pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X_K \arrow[r] & Y' \end{tikzcd}$$

and then consider an anodyne extension $Y' \to Y'_K$ to a fully Kan dendroidal set Y'_K. Since anodyne extensions are closed under pushouts, $Y \to Y'$ is anodyne and also $Y' \to Y'_K$ is anodyne. So, we set $Y_K = Y'_K$ and let $X_K \to Y_K$ be the composition of $X_K \to Y'$ and $Y' \to Y_K$.

(iii) The statement follows because in the small object argument we glue in only countably many cells in each step and there are countably many steps.

(iv) The statement follows by showing the corresponding statement for each intermediate step $X \to X_{(n)}$ in the small object argument yielding $X \to X_K$. Since $X_{(n)} = X'_{(n-1)}$, it is enough to show the statement for $X \to X' = X_{(1)}$.

Let $U \subseteq X_K$ be a countable subpresheaf and $X_K = \bigcup_n X_{(n)}$. We first prove an auxiliary statement that for a countable $V \subseteq X_{(n)}$ there is a countable $B \subseteq X$ such that $V \subset B_{(n)}$. We proceed by induction on n.

For $n = 1$, we let $V \subseteq X'$ be a countable subpresheaf and consider how X' is constructed in the first part of the proof. We consider the subset $J \subseteq I$ of those elements that factor through V and define B to be the union of $V \cap X$ and the images of maps α_i for $i \in J$. From the construction it follows that $V \subseteq B'$.

For $n > 1$ and $V \subseteq X_{(n)}$, we first construct a countable $C \subseteq X_{(n-1)}$ as in the starting case ($n = 1$) such that $V \subseteq C'$. By the inductive hypothesis there is a countable $B \subseteq X$ such that $C \subseteq B_{(n-1)}$. Then $V \subseteq C' \subseteq B'_{(n-1)} = B_{(n)}$.

This completes the proof of the auxiliary statement. We now prove the general statement. We denote $U_{(n)} = U \cap X_{(n)}$. Since $U_{(n)}$ is countable by the auxiliary statement there is some countable $B^n \subseteq X$ such that $U_{(n)} \subseteq B^n_{(n)}$. We let $B = \bigcup_n B^n \subseteq X$. Then B is countable and

$$U = \bigcup_n U_{(n)} \subseteq \bigcup_n B^n_{(n)} \subseteq \bigcup_n \left(\bigcup_m B^m\right)_{(n)} = \bigcup_n B_{(n)} = B_K.$$

□

3.2.5 Homotopy theories of dendroidal sets

The class of inner Kan dendroidal sets has been introduced and studied in [MW09, CM13a] and the class of dendroidal Kan complexes in [Heu11b]. The main results are the constructions of the following model structures.

Theorem 3.2.37 (Cisinski-Moerdijk)**.** *There is a left proper, combinatorial model structure on the category of dendroidal sets with cofibrations given by normal monomorphisms and fibrant objects given by inner Kan dendroidal sets. This model category is Quillen equivalent to the model category of (coloured) operads enriched in simplicial sets.*

Theorem 3.2.38 (Heuts)**.** *There is a simplicial left proper, combinatorial model structure on the category of dendroidal sets with cofibrations given by normal monomorphisms and fibrant objects given by dendroidal Kan complexes. This model structure is Quillen equivalent to the standard model category of E_∞-algebras in simplicial sets (i.e. the model structure transferred from the Kan-Quillen model structure on simplicial set).*

The slogan is that inner Kan dendroidal sets are a combinatorial model for topological operads and dendroidal Kan complexes are a model for E_∞-spaces. The Cisinski-Moerdijk model structure is also called the operadic and the Heuts model structure is also called the covariant structure.

In the following chapters we will construct one more model structure on dendroidal sets. In Chapter 4 we develop the combinatorics of the dendroidal anodyne extensions that we need for the construction. In Chapter 5 we carry out the construction and in Chapter 6 we prove, using and extending the results of Heuts, that the corresponding homotopy theory is equivalent to the homotopy theory of connective spectra.

Chapter 4

Combinatorics of dendroidal anodyne extensions

This chapter is devoted to the proof of the pushout-product property for dendroidal sets. This property (or at least one of its special cases) is essential for the construction of the model structures on dendroidal sets. The need to prove that an inclusion of a subobject of a representable dendroidal set is an anodyne extension reoccurs many times in the proof of this property. To deal with all of these occurrences efficiently we develop a formalism which we call the method of canonical extensions. This method relies on a delicate study of the combinatorics of elementary face maps. We emphasize this method because it can be applied to show other results that are not related to the pushout-product property.

We will prove the pushout-product property in two cases: one concerning the top horns and one concerning the root horns. The case concerning top horns is easier and it is already considered in [Heu11a]. We spell out this case in detail to illustrate our method and prepare the reader for the more complicated case concerning root horns. The case concerning the root horns is the main result of this chapter.

Both cases come in two variants that stem from the Erratum to [CM11]. The same proof applies to both variants. The variant involving linear trees is the one that we use in the construction of the stable model structure. The other variant involves open trees and it will not be used in subsequent chapters, although it is an interesting result in its own right. Indeed, the variants of the covariant and stable model structure on open dendroidal sets have not been studied before, but the pushout-product property for open trees shows that the monoidal structure should be compatible with these model structures on open dendroidal sets (which is not the case for dendroidal sets).

4.1 Elementary face maps

4.1.1 Planar structures and a total order of face maps

Recall from Chapter 3, that a *planar structure* on a tree T is a family of total orders $(v, \preceq_v)$, one for each vertex v of T.

Lemma 4.1.1. *Let $(T, \leq, L)$ be a tree and e, f two distinct elements of T other than the root r_T. There exist unique siblings $e', f' \in T$ such that $e \leq e'$ and $f \leq f'$.*

Proof. If $e \leq f$ then $e' = f' = f$. Similarly, if $f \leq e$ then $e' = f' = e$.

Otherwise, let us assume that e and f are not comparable. The finite totally ordered set $\{h \in T : e \leq h\} \cap \{h \in T : f \leq h\}$ is non-empty since it contains the root of T. Hence there exists a unique maximal element g such that $e \leq g$ and $f \leq g$. Then e' is the maximal element such that $e \leq e' < g$ and f' is the maximal element such that $f \leq f' < g$. By maximality e' and f' are immediate successors of g and hence siblings. □

For every planar structure on $(T, \leq, L)$ given by a family of total orders $\preceq_v$, we can define a relation $\preceq$ on the set T by

$$e \preceq f \Leftrightarrow e' \preceq_v f'$$

for e' and f' associated to e and f by the previous lemma. One easily checks that $\preceq$ is a total order on T which extends the partial order $\leq$.

Example 4.1.2. In terms of graphs this total order is given by traversing the tree T from left to right and from bottom to top. We have $\{r \preceq c \preceq d \preceq e \preceq a \preceq b \preceq f\}$ for the planar tree in Example 3.1.2.

Let T be a tree with the root r. To every elementary face map of T we can assign an operation of T - we assign $(\{e\}; e)$ to an inner elementary face map ∂_e, $(w; e)$ to a top elementary face map ∂_w where e is the output of w, and $(v; r)$ to a bottom face map $\partial_{v,e}$.

Definition 4.1.3. For a tree S we say that it is a *face* of a tree T if there is a sequence of elementary face maps $\partial_{f_1}, ..., \partial_{f_r}$ such that $S = \partial_{f_1} ... \partial_{f_r} T$. We let $\mathrm{Sub}(T)$ be the family of all faces of T.

Remark 4.1.4. Let S be a face of a tree T. If f is an inner edge or a top vertex of T which is not an inner edge or a top vertex of S, then there is at most one face R of T such that $S = \partial_f R$.

Remark 4.1.5. By the previous considerations, any operation of a face S is an operation of T as well.

Remark 4.1.6. Every total order $\preceq$ extending the partial order $\leq$ of a tree T induces a total order $\leq$ on the set of operations of T such that $(A, t) \leq (B, s)$ if

- $t \preceq s$ or
- $t = s$ and A is empty or
- $t = s$, $A = \{t_1, ..., t_m\}$, $B = \{s_1, ..., s_n\}$ and there is a positive integer k such that $t_i = s_i$ for $1 \leq i \leq k-1$ and $t_k \preceq s_k$, $t_k \neq s_k$.

For any face S of T, this gives a total order on the set

$$\mathcal{F}_S = \{\partial_f | \partial_f : \partial_f S \to S\}$$

because to any elementary face map ∂_f we associated an operation in S which is also an operation in T. Also, we get a total order on the set

$$\mathcal{E}_S = \{\partial_f | \partial_f : \partial_f R \to R, R \in \mathrm{Sub}(T), S = \partial_f R\}$$

because to any elementary face map ∂_f we associated an operation in R which is also an operation in T. These considerations have the following important consequence.

Remark 4.1.7. For faces S and R of a tree T and a commutative square

$$\begin{tikzcd} S \arrow[r, "\partial_g"] \arrow[d, "\partial_f"'] & \partial_f R \arrow[d, "\partial_f"] \\ \partial_g R \arrow[r, "\partial_g"'] & R \end{tikzcd}$$

of maps in F with $S = \partial_f \partial_g R = \partial_g \partial_f R$, we have $\partial_f \leq \partial_g$ in $\mathcal{E}_S$ if and only if $\partial_f \leq \partial_g$ in $\mathcal{F}_R$. Moreover if S and R are faces of T such that $\partial_f, \partial_g \in \mathcal{F}_S$ and $\partial_f, \partial_g \in \mathcal{F}_R$ then $\partial_f \leq \partial_g$ in $\mathcal{F}_S$ if and only if $\partial_f \leq \partial_g$ in $\mathcal{F}_R$.

4.1.2 Combinatorial aspects of elementary face maps

Definition 4.1.8. Let S be a face of T. For an inner edge e of T we say that $\partial_e S$ *exists* if e is also an inner edge of S. Analogously, we define when $\partial_w S$ and $\partial_{v,f} S$ exist for a top vertex w and the bottom vertex v with all inputs except possibly f being a leaf.

Definition 4.1.9. Let v be a top vertex and e the unique inner edge of T attached to v and w. We say that a pair $\{\partial_v, \partial_e\}$ is a *mixed pair of elementary face maps.* For a pair of the form $\{\partial_v, \partial_w\}$ or of the form $\{\partial_e, \partial_{w \circ_e v}\}$ we say that it is an *adjacent pair of elementary face maps.*

We analogously define a mixed pair $\{\partial_v, \partial_e\}$ and adjacent pairs $\{\partial_v, \partial_w\}$, $\{\partial_e, \partial_{v \circ_e w}\}$ for the bottom vertex v and the unique inner edge e attached to v and w.

Remark 4.1.10. In these cases the faces $\partial_e\partial_v T$ and $\partial_v\partial_e T$ do not exist. In case v is a top vertex there exist faces $\partial_{w\circ_e v}\partial_e T$ and $\partial_w\partial_v T$ and they are equal. In case v is a bottom vertex there exist faces $\partial_{v\circ_e w}\partial_e T$ and $\partial_w\partial_v T$ and they are equal.

Remark 4.1.11. If T is a tree and $\partial_f\colon \partial_f T \to T$ and $\partial_g\colon \partial_g T \to T$ are two elementary face maps not forming a mixed pair (f, g may be top or bottom vertices or inner edges), then there exist faces $\partial_f\partial_g T$ and $\partial_g\partial_f T$. Actually, $\partial_f\partial_g T = \partial_g\partial_f T$ and we have a commuting diagram

$$\begin{tikzcd} \partial_f\partial_g T = \partial_g\partial_f T \arrow[r, "\partial_g"] \arrow[d, "\partial_f"'] & \partial_f T \arrow[d, "\partial_f"] \\ \partial_g T \arrow[r, "\partial_g"'] & T. \end{tikzcd} \tag{4.1}$$

Let $\partial_f\colon \partial_f T \to T$ and $\partial_g\colon \partial_g\partial_f T \to \partial_f T$ be two elementary face maps not forming an adjacent pair. Then there exist $\partial_g T$, $\partial_f\partial_g T$ and the diagram (4.1) commutes.

Lemma 4.1.12. *Let T be a tree. Consider faces P, P_1, P_2 of T and elementary face maps $\partial_f\colon P \to P_1$ and $\partial_g\colon P \to P_2$. The set of all faces S of T such that there exist elementary face maps $\partial_{f_1}, ..., \partial_{f_r}, \partial_{g_1}, ...\partial_{g_r}$ satisfying*

$$P_1 = \partial_{g_1}\partial_{g_2}\dots\partial_{g_r}S, \quad P_2 = \partial_{f_1}\partial_{f_2}\dots\partial_{f_r}S$$

is non-empty and has a unique minimal element P_3 with respect to the induced partial order from $\mathrm{Sub}(T)$.

Proof. Let $\mathcal{S}$ be the set of all faces S of T such that there exist elementary face maps $\partial_{f_1}, ..., \partial_{f_r}, \partial_{g_1}, ...\partial_{g_r}$ satisfying

$$P_1 = \partial_{g_1}\partial_{g_2}\dots\partial_{g_r}S, \quad P_2 = \partial_{f_1}\partial_{f_2}\dots\partial_{f_r}S.$$

Since P_1 and P_2 are faces of T, T itself is an element of $\mathcal{S}$. Since $\mathcal{S}$ is finite, it has minimal elements.

Assume S_1 and S_2 are elements of $\mathcal{S}$. Their intersection (as operads) is a disjoint union of faces of T. Every connected component of the intersection is a face of S_1 and a face of S_2.

Also, the intersection contains P_1 and P_2. Since P is connected there is a unique tree S in the intersection of S_1 and S_2 which contains P. Both P_1 and P_2 are connected, so they are also contained in S. Note that S is a face of both S_1 and S_2. Also, P_1 and P_2 are faces of S, so S is an element of $\mathcal{S}$. This shows that the set $\mathcal{S}$ is direct, hence it has a minimal element. $\square$

Definition 4.1.13. Under the assumptions of the previous lemma, we say that a pair (∂_f, ∂_g) is *good* if $r = 1$ and $f_1 = f, g_1 = g$.

Remark 4.1.14. Every edge of the face P_3 is an edge of P_1 or an edge of P_2. Furthermore, (∂_f, ∂_g) is good except if f and g are both top vertices or both bottom vertices attached to the same unique inner edge.

4.2 The method of canonical extensions

In this section we give a technique for showing that certain monomorphisms of dendroidal sets are (operadic, covariant or stable) anodyne extensions.

Let R be a tree. Under certain conditions on a dendroidal subset A_0 of the representable dendroidal set $\Omega[R]$ we will show that the inclusion $A_0 \to \Omega[R]$ is an anodyne extension. The approach that we will present has the advantage of being applicable to obtain many new and old results and that these conditions on A_0 are easily verified in the concrete cases that we consider.

The idea is to form a filtration

$$A_0 \subset A_1 \subset ... \subset A_{N-1} \subset A_N = \Omega[R] \tag{4.2}$$

in which every inclusion $A_n \to A_{n+1}$ is a pushout of a coproduct of a family of horn inclusions of faces of R, i.e. fits into a pushout diagram

$$\begin{tikzcd} \coprod \Lambda^f[P] \arrow[r] \arrow[d] & A_n \arrow[d] \\ \coprod \Omega[P] \arrow[r] & A_{n+1} \end{tikzcd}$$

where the coproduct ranges over pairs $(\partial_f P, P)$ of faces of R that will be carefully formed and ordered in the way we now describe in detail.

Definition 4.2.1. Let R be a tree and A_0 be a dendroidal subset of $\Omega[R]$. We say that a face P of the tree R is *missing* if the monomorphism $\Omega[P] \to \Omega[R]$ does not factor through A_0. In that case we write $P \nsubseteq A_0$.

Definition 4.2.2. Let F be a subset of the set

$$\{\partial_f \colon \partial_f P \to P | P \in \mathrm{Sub}(R); \quad P, \partial_f P \nsubseteq A_0\}. \tag{4.3}$$

For every missing face P of R we define the set of F*–extensions* of P

$$\mathcal{E}_F(P) = \{\partial_f \colon \partial_f P' \to P' | \partial_f \in F, \partial_f P' = P\}$$

and the set of F*–faces* of P

$$\mathcal{F}_F(P) = \{\partial_f \colon \partial_f P \to P | \partial_f \in F\}.$$

Definition 4.2.3. We say that a subset F of the set (4.3) is an *extension set* with respect to A_0 if the following Axioms (F1)-(F6) are satisfied.

The Pullback Axiom (F1). For any two distinct elementary face maps $\partial_f, \partial_g \in \mathcal{F}_F(P)$, faces $\partial_f\partial_g P$ and $\partial_g\partial_f P$ exist and $\partial_f \in \mathcal{F}_F(\partial_g P), \partial_g \in \mathcal{F}_F(\partial_f P)$.

The Composition Axiom (F2). For any two distinct elementary face maps $\partial_f \in \mathcal{F}_F(P), \partial_g \in \mathcal{F}_F(\partial_f P)$ faces $\partial_g P$, $\partial_f\partial_g P$ exist and $\partial_g \in \mathcal{F}_F(P), \partial_f \in \mathcal{F}_F(\partial_g P)$.

The Amalgamation Axiom (F3). For any $\partial_g \colon P \to P_2$ (not necessarily in F) and any $\partial_f \colon P \to P_1, \partial_f \in F$, the elementary face maps $\partial_{f_1}, ..., \partial_{f_r}$ of Lemma 4.1.12 can be chosen as elements of F.

The Good Pair Axiom (F4). For any missing face P and any two distinct elementary face maps $\partial_g, \partial_f \in \mathcal{E}_F(P)$, the pair (∂_f, ∂_g) is a good pair (i.e. in terms of Lemmma 4.1.12 $r = 1$ and $f_1 = f, g_1 = g$).

The Bad Pair Axiom (F5). For any $\partial_g \colon P \to P_2$ which is not an element of F there is at most one element $\partial_f \in \mathcal{E}_F(P)$ such that (∂_f, ∂_g) is not a good pair. If (∂_f, ∂_g) is not a good pair, then in terms of Lemma 4.1.12, $\partial_{f_1} \in \mathcal{E}_F(P_2)$, but $\partial_{f_1} \notin \mathcal{E}_F(P)$.

The Existence Axiom (F6). For any missing face P, at least one of the sets $\mathcal{F}_F(P)$ and $\mathcal{E}_F(P)$ is non-empty.

For the rest of this section let us fix a tree R, a planar structure on R, a dendroidal subset A_0 and an extension set F with respect to A_0. By the considerations in subsection 4.1.1, the planar structure on R induces a total order on every set $\mathcal{E}_F(P)$ and $\mathcal{F}_F(P)$ for every face P of R. By Remark 4.1.7 these total orders are compatible in the sense that for two elementary face maps ∂_f and ∂_g if $\partial_f \leq \partial_g$ in any of these sets $\mathcal{E}_F(P)$ or $\mathcal{F}_F(P)$ then the same relation holds in all sets $\mathcal{E}_F(P)$ and $\mathcal{F}_F(P)$ that contain both ∂_f and ∂_g.

Definition 4.2.4. Let P be a face of R such that $\mathcal{F}_F(P)$ is non-empty. We say that an elementary face map $\partial_f \colon \partial_f P \to P$ is *a canonical extension* if $\partial_f = \min \mathcal{F}_F(P)$ and $\partial_f = \min \mathcal{E}_{\partial_f P}$. Since an elementary face map is determined by its domain and codomain we also say that the pair $(\partial_f P, P)$ is a canonical extension.

Remark 4.2.5. If $(\partial_f P, P)$ is a canonical extension then $\partial_f P$ and P are missing faces.

Remark 4.2.6. It might happen that any one of the conditions $\partial_f = \min \mathcal{F}_F(P)$ and $\partial_f = \min \mathcal{E}_{\partial_f P}$ holds, while the other does not hold.

Recall that, for any $\partial_f \in \mathcal{F}_F(P)$ and any $\partial_g \colon P \to P_1$ belonging to $\mathcal{F}_F(P_1)$, Axiom (F2) implies that there exists $\partial_f \in \mathcal{F}_F(P_1)$.

Lemma 4.2.7. *(Characterization of canonical extensions) Let P be a missing face and let $\partial_f = \min \mathcal{F}_F(P)$. The pair $(\partial_f P, P)$ is a canonical extension if and only if for every $\partial_g \colon P \to P_1$ belonging to F, the relation $\partial_f \leq \partial_g$ holds in $\mathcal{F}_F(P_1)$.*

Proof. Let us assume $(\partial_f P, P)$ is a canonical extension and let $\partial_g \colon P \to P_1, \partial_g \in F$. The Composition Axiom implies that there exist $\partial_f \in \mathcal{F}_{P_1}$ and $\partial_g \in \mathcal{F}_{\partial_f P_1}$, so $\partial_g \in \mathcal{E}_{\partial_f P}$. Since $\partial_f = \min \mathcal{E}_{\partial_f P}$ we have $\partial_f \leq \partial_g$ in $\mathcal{E}_{\partial_f P}$. By Remark 4.1.7 $\partial_f \leq \partial_g$ in $\mathcal{F}_{P_1}$.

Conversely, let us assume that $\partial_f \leq \partial_g$ in $\mathcal{F}_{P_1}$ for every $\partial_g \colon P \to P_1$, $\partial_g \in F$. Consider $\partial_h \colon \partial_f P = \partial_h P_2 \to P_2$ such that $\partial_h \in \mathcal{E}_{\partial_f P}$. If $f \neq h$, The Good Pair Axiom there exists a missing tree P_1 and there exist $\partial_f \colon P_2 \to P_1$ and $\partial_h \colon P \to P_1$. By the Amalgamation Axiom $\partial_f, \partial_h \in \mathcal{F}_{P_1}$, so by assumption $\partial_f \leq \partial_h$ in $\mathcal{F}_{P_1}$. By Remark 4.1.7 we have $\partial_f = \min \mathcal{E}_{\partial_f P}$, so $(\partial_f P, P)$ is a canonical extension. □

Lemma 4.2.8. *Canonical extensions of different trees are different. More precisely, if $(\partial_{f_1} P_1, P_1)$ and $(\partial_{f_2} P_2, P_2)$ are two canonical extensions, then $P_1 \neq P_2$ if and only if $\partial_{f_1} P_1 \neq \partial_{f_2} P_2$.*

Proof. If $P_1 = P_2$, then by definition of a canonical extensions $\partial_{f_1} = \min \mathcal{F}_F(P_1) = \min \mathcal{F}_F(P_2) = \partial_{f_2}$.

Conversely, assume $\partial_{f_1} P_1 = \partial_{f_2} P_2$ and $P_1 \neq P_2$. Then $f_1 \neq f_2$ by Remark 4.1.4. On the other hand $\partial_{f_1} = \min \mathcal{E}_F(\partial_{f_1} P_1) = \min \mathcal{E}_F(\partial_{f_2} P_2) = \partial_{f_2}$, so we have obtained a contradiction. □

Lemma 4.2.9. *Canonical extensions do not compose. More precisely, if $(\partial_f P, P)$ is a canonical extension and $\partial_g = \min \mathcal{F}_{\partial_f P}$, then $(\partial_g \partial_f P, \partial_f P)$ is not a canonical extension.*

Proof. By the Composition Axiom there exists $\partial_g \in \mathcal{F}_F(P)$ and $\partial_f \in \mathcal{F}_{\partial_g P}$. If $(\partial_g \partial_f P, \partial_f P)$ is a canonical extension, then Lemma 4.2.7 implies that $\partial_g \leq \partial_f$ in $\mathcal{F}_F(P)$. Since $g \neq f$, we have $\partial_g < \partial_f$. This is a contradiction with the fact that $\partial_f = \min \mathcal{F}_F(P)$. □

Lemma 4.2.10. *A face which is not the codomain of a canonical extension is the domain of a canonical extension. More precisely, if P is a missing face of R such that $\mathcal{F}_F(P)$ is empty or the pair $(\partial_f P, P)$ is not a canonical extension for $\partial_f = \min \mathcal{F}_F(P)$, then there exists a face P_1 of R and an elementary face map $\partial_g \colon P \to P_1$ such that (P, P_1) is a canonical extension.*

Proof. First, assume that $\mathcal{F}_F(P)$ is empty. The Existence Axiom implies that $\mathcal{E}_F(P) \neq \emptyset$. Let $\partial_g = \min \mathcal{E}_F(P)$, $\partial_g : P \to P_1$. For any $\partial_h \in \mathcal{F}_F(P_1), h \neq g$, the Pullback Axiom implies $\partial_h \in \mathcal{F}_F(P)$, which contradicts $\mathcal{F}_F(P) = \emptyset$. Hence $\partial_g = \min \mathcal{F}_F(P_1)$ and (P, P_1) is a canonical extension.

Otherwise, assume that $\mathcal{F}_F(P)$ is non-empty and for $\partial_f = \min \mathcal{F}_F(P)$ the pair $(\partial_f P, P)$ is not a canonical extension. By Lemma 4.2.7, there exists a face P_1 of R and an elementary face map $\partial_g \colon P \to P_1$ such that $\partial_g < \partial_f$ in $\mathcal{F}_F(P_1)$. Without loss of generality we may assume $\partial_g = \min \mathcal{E}_F(P)$.

For $\partial_h \in \mathcal{F}_F(P_1), h \neq g$, the Pullback Axiom implies that $\partial_h \in \mathcal{F}_F(P)$. By minimality of ∂_f, we have $\partial_f \leq \partial_h$, so $\partial_g < \partial_h$ in $\mathcal{F}_F(P_1)$ by Remark 4.1.7. Hence $\partial_g = \min \mathcal{F}_F(P_1)$ and (P, P_1) is a canonical extension. $\square$

Lemma 4.2.11. *Let $(\partial_f P, P)$ be a canonical extension and let $\partial_g : \partial_g P \to P$, $g \neq f$ (∂_g not necessarily in F) then one of the following holds:*

- *$\partial_g P$ is not a missing face of R;*
- *the set $\mathcal{F}_{\partial_g P}$ is non-empty and for $\partial_h = \min \mathcal{F}_{\partial_g P}$, the pair $(\partial_h \partial_g P, \partial_g P)$ is a canonical extension;*
- $\operatorname{card} \mathcal{E}_{\partial_g P} < \operatorname{card} \mathcal{E}_{\partial_f P}$.

Proof. Let us assume that $\partial_g P$ is a missing face of R.

If $\partial_g \colon \partial_g P \to P$ is an element of F, then by the Pullback Axiom there exists $\partial_g \partial_f P$ and $\partial_f \in \mathcal{F}_{\partial_g P}$. Let $\partial_h = \min \mathcal{F}_{\partial_g P}$, so in particular $\partial_h \leq \partial_f$. For every $\partial_k \colon \partial_g \partial_f P = \partial_k P_2 \to P_2$ such that $\partial_k \in \mathcal{E}_{\partial_g \partial_f P}$, the Good Pair Axiom implies there exists a face P_3 of R and elementary face maps $\partial_k \colon \partial_g P \to P_3$ and $\partial_h \colon P_2 \to P_3$, which are by the Amalgamation Axiom elements of F.

By Lemma 4.2.7, $\partial_f \leq \partial_k$ in $\mathcal{F}_{P_3}$. Hence, by Remark 4.1.7 we have $\partial_h \leq \partial_f \leq \partial_k$, so $\partial_h = \min \mathcal{E}_{\partial_g \partial_f P}$.

Otherwise, assume $\partial_g \colon \partial_g P \to P$ is not an element of F. By the Amalgamation Axiom for any element $\partial_k \colon \partial_g P = \partial_k P_1 \to P_1, \partial_k \in F$, there is a face P_2 of R and an elementary face map $\partial_{k_1} \colon P = \partial_{k_1} P_2 \to P_2, \partial_{k_1} \in F$. Let us choose one such map ∂_{k_1} for every $\partial_k \in \mathcal{E}_{\partial_g P}$ and denote this assignment $\psi : \mathcal{E}_{\partial_g P} \to \mathcal{E}_P$. The Bad Pair Axiom implies that there is at most one element $\partial_k \in \mathcal{E}_{\partial_g P}$ such that $\psi(\partial_k) \neq \partial_k$, so ψ is injective and so $\operatorname{card}(\mathcal{E}_{\partial_g P}) \leq \operatorname{card} \mathcal{E}_P$.

Furthermore, if ∂_{k_1} is an element of $\mathcal{E}_P$, then the Composition Axiom implies that ∂_{k_1} is an element of $\mathcal{E}_{\partial_f P}$, so $\operatorname{card}(\mathcal{E}_F(P)) \leq \operatorname{card}(\mathcal{E}_{\partial_f P})$.

Since $\partial_f \colon \partial_f P \to P$ and $\partial_g \colon \partial_g P \to P$ are elementary face maps of the same tree P, there is an edge e of P which appears in $\partial_g P$ but does not appear in $\partial_f P$. For every elementary face map $\partial_f \colon \partial P' \to P'$, edge e is an edge of P', but it is not an edge of $\partial_f P'$. Hence ∂_f is not an element of $\mathcal{E}_{\partial_g P}$ and we have $\operatorname{card}(\mathcal{E}_{\partial_g P}) < \operatorname{card} \mathcal{E}_{\partial_f P}$. $\square$

Proposition 4.2.12. *Let R be a tree and A_0 a dendroidal subset of $\Omega[R]$ such that there exists an extension set F. The inclusion $A_0 \to \Omega[R]$ is a composition of pushouts of horns $\Lambda^f P \to P$ with $\partial_f \in F$.*

Hence, the inclusion $A_0 \to \Omega[R]$ is a stable anodyne extension, which is moreover a covariant anodyne extension if all elements of F are either inner or top elementary face maps and an operadic anodyne extension if all elements of F are inner elementary face maps.

Proof. By Lemma 4.2.10 every missing face of R with respect to A_0 is either first or the second component of a canonical extension (P, P'). If (P_1, P_1') and (P_2, P_2') are two canonical extensions, Lemma 4.2.9 implies that $P_1' \neq P_2$ and $P_2' \neq P_1$. Since Lemma 4.2.8 states that $P_1 = P_2$ if and only if $P_1' = P_2'$, all such pairs are mutually disjoint.

Let $\mathcal{P}_{n,c}$ be the family of all canonical extensions $(\partial_f P, P)$ such that $\partial_f P$ has n vertices and $\mathrm{card}(\mathcal{E}_{\partial_f P}) = c$.

Let m be the minimal number of vertices over all missing faces. Let d be the minimal cardinality of the set $\mathcal{E}_F(P)$ over all missing faces P with number of vertices being m. We define $A_{m,d}$ to be the union of the dendroidal set A_0 with the representables of all missing faces P and their canonical extensions such that P has m vertices and $\mathrm{card}(\mathcal{E}_F(P)) = d$.

For notational convenience, we define $A_{n,c} = A_{m,d}$ if $1 \leq n < m$ or if $n = m$ and $1 \leq c < d$. We inductively define dendroidal sets $A_{n,c}$ as the union of

- all dendroidal sets $A_{n',c'}$ such that $n' < n$,
- all dendroidal sets $A_{n',c'}$ such that $n' = n$ and $c' < c$,
- all representables $\Omega[P]$ and $\Omega[\partial_f P]$ such that $(\partial_f P, P) \in \mathcal{P}_{n,c}$.

For a fixed $n \geq 1$, if c is the maximum of $\mathrm{card}(\mathcal{E}_F(P))$ over all faces P with n vertices, we define $A_{n+1,0} = A_{n,c}$.

Lemma 4.2.11 implies that there is an inclusion

$$\coprod_{(\partial_f P, P) \in \mathcal{P}_{n,c}} \Lambda^f P \to A_{n,c-1}.$$

Since all canonical extensions are mutually disjoint, for any $(\partial_f P, P) \in \mathcal{P}_{n,c}$ the representable $\Omega[P]$ does not factor through $A_{n,c-1}$ so we have a pushout diagram

$$\begin{tikzcd}
\coprod\limits_{(\partial_f P, P) \in \mathcal{P}_{n,c}} \Lambda^f[P] \arrow[r] \arrow[d] & A_{n,c-1} \arrow[d] \\
\coprod\limits_{(\partial_f P, P) \in \mathcal{P}_{n,c}} \Omega[P] \arrow[r] & A_{n,c}.
\end{tikzcd}$$

□

Remark 4.2.13. We mention one example of an extension set. The *Segal core* $Sc[R]$ of a tree R is the union of images of all monomorphisms $\Omega[C_n] \to \Omega[R]$ which are compositions of only top and bottom elementary face maps (no inner elementary face maps). If R is a tree and $A_0 = Sc[R]$ then the set $F = \{\partial_e \colon \partial_e P \to P | P \in \mathrm{Sub}(R), e \text{ an inner edge of } P\}$ is an extension set. Proposition 4.2.12 gives one more proof of Proposition 2.4. in [CM13a] that the inclusion $Sc[R] \to \Omega[R]$ is an operadic anodyne map.

4.3 The pushout-product property

In this section we would like to prove that for trees T and S, and f being an inner edge, (resp. a top vertex, a root vertex) of the tree S the morphism

$$\Lambda^f[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \Omega[T]$$

is an operadic (resp. covariant, stable) anodyne extension. We restrict to the cases in which one of the trees S and T is linear or both T and S are open trees because only in these cases we can be certain that the above map is a monomorphism.

Theorem 4.3.1. *Let S and T be trees and v be a top vertex of the tree S. If S or T is linear or both S and T are open trees, then the morphism*

$$\Lambda^v[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \Omega[T]$$

is a covariant anodyne extension.

Proof. If $T = \eta$ the statement is equivalent to saying that the horn inclusion $\Lambda^v[S] \to \Omega[S]$ is a covariant anodyne extension, which is true by definition. Hence we assume that T has at least one vertex.

We denote by k the output edge and by $l_1, ..., l_m$ the input edges of the vertex v in S. If v is a stump then $m = 0$.

We fix a total order $R_1 \preceq R_2 \preceq ... \preceq R_N$ extending the left percolation partial order on the set of shuffles as described in 3.2.16, so R_1 is obtained by grafting copies of T on top of S. Let $B_0 = \Lambda^v[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T]$ and we define $B_i = B_{i-1} \cup \Omega[R_i]$.

Because of our assumptions on T and S maps $B_{i-1} \to B_i$ are monomorphisms and the proof is very similar in all of these cases. We will show that the maps $B_{i-1} \to B_i$ are covariant anodyne extensions for all $i = 1, 2, ..., N$. Proposition 3.2.20 implies that for every $i \in \{1, 2, ..., N\}$, $B_i = B_{i-1}$ or the shuffle R_i contains a vertex of type $v \otimes t, t \in T$.

Fix a shuffle R_i such that $\Omega[R_i]$ does not factor through B_{i-1} and let

$$X := \{x \in T : v \otimes x \text{ is a vertex of } R_i\}.$$

If we denote $A_0 := B_{i-1} \cap \Omega[R_i]$ then we have a pushout diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & B_{i-1} \arrow[d] \\ \Omega[R_i] \arrow[r] & B_i. \end{tikzcd}$$

From this it follows that it is enough to show that $A_0 \to \Omega[R_i]$ is a covariant anodyne extension.

Characterization of missing faces. With this notation a face P of R_i is missing (i.e. $\Omega[P] \to \Omega[R_i]$ does not factor through A_0) if and only if

- every colour of T appears in P;
- every colour of $\partial_v S$ appears in P;
- for each predecessor R_j of R_i there is at least one edge of P which is not an edge of R_j.

Claim Every missing face P has an edge $k \otimes x$ for some $x \in X$.

Proof: Since there is at least one white vertex $v \otimes x, x \in X$ there is at least one edge $k \otimes x$ in the shuffle R_i.

First, let us assume that for every $x \in X$ the edge $k \otimes x$ in the shuffle R_i connects two white vertices. In this case colour k appears only on these edges. Colour k must appear in P, so P must contain at least one edge $k \otimes x$, $x \in X$.

Otherwise, we consider an edge $k \otimes x, x \in X$ of maximal height in the shuffle R_i which is an input of a black vertex $k \otimes w$ and the output of a white vertex $v \otimes x$. Because $k \otimes x$ has maximal height, every sibling $k \otimes y$ of that edge $k \otimes x$ is the output of a white vertex $v \otimes y$ (and an input of the black vertex $k \otimes w$).

Hence the shuffle R_i has a predecessor R_j which does not contain inputs of $k \otimes w$ (we can just apply an inverse percolation to R_i at this black vertex to get R_j), so by the characterization of the missing faces at least one of these inputs $k \otimes y, y \in X$ must appear in the missing face P. ■

Hence, for a missing face P the following set is non-empty

$$X_P := \{x \in X : k \otimes x \text{ appears in } P\}.$$

The fact that $A_0 \to \Omega[R_i]$ is a covariant anodyne extension follows from Proposition 4.2.12 by taking $R = R_i$ and the extension set F to consist of elementary face maps $\partial_f P \to P$ such that $\partial_f P$ and P are missing faces of R and such that f is

- an inner edge of P denoted by $l_j \otimes x$, $x \in X_P$, $j = 1, ..., m$ or
- a top vertex of P denoted by $v \otimes x, x \in X_P$.

We fix any planar structure on R and it induces a total order on the sets $\mathcal{F}_F(P)$ and $\mathcal{E}_F(P)$ as in Remark 4.1.7. It remains to show that the set F satisfies Axioms (F1)-(F6).

The Pullback Axiom First, note that the set F does not contain a mixed pair of elementary face maps. Hence if there is a missing tree P and $\partial_f, \partial_g \in \mathcal{F}_F(P)$, the faces $\partial_f\partial_g P$ and $\partial_g\partial_f P$ exist, and they are in fact the same. Denote $P' = \partial_f\partial_g P = \partial_g\partial_f P$. We need to check that P' is a missing face of R, i.e. that P' satisfies the three conditions in the characterization of missing faces. The tree P is missing so it satisfies those three conditions and it contains an edge $k \otimes x$ for $x \in X$. The only edges deleted from P to obtain $\partial_f P$ and from $\partial_f P$ to obtain P' are of the form $l_j \otimes x$. Hence P' will also contain $k \otimes x$ and satisfy the three conditions in the characterization of missing faces. It follows directly from the definition of F that $\partial_f \in \mathcal{F}_{\partial_g P}$ and $\partial_g \in \mathcal{F}_{\partial_f P}$, so the Pullback Axiom is satisfied.

The Composition Axiom Note that F does not contain an adjacent pair of elementary face maps. Hence for any pair of elementary face maps $\partial_f \in \mathcal{F}_F(P), \partial_g \in \mathcal{F}_{\partial_f P}$ the faces $\partial_g P$, $\partial_f \partial_g P$ exist. The face P is missing and contains $k \otimes x$. Since the only edges deleted from P to obtain $\partial_g P$ are of the form $l_j \otimes x$ for some $x \in X$, the face $\partial_g P$ is also missing. Again, from the definition of F it follows that $\partial_g \in \mathcal{F}_F(P)$ and $\partial_f \in \mathcal{F}_{\partial_g P}$, so the Composition Axiom is satisfied.

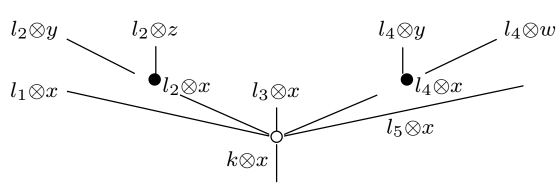
The Amalgamation Axiom Let $\partial_g \colon P \to P_2$ be an elementary face map (not necessarily in F) and $\partial_f \colon P \to P_1$, $\partial_f \in \mathcal{E}_F(P)$. If ∂_f and ∂_g are not elementary face maps corresponding to top vertices with the same output, then (∂_f, ∂_g) is a good pair; i.e. in terms of Lemma 4.1.12 we have $r = 1$ and $\partial_{f_1} = \partial_f \colon P_2 \to P_3$. The faces P_2 and P_3 are missing since they contain all the edges of the missing face P, so $\partial_f \colon P_1 \to P_3$ is an element of F by definition of the set F. Hence, in this case the Amalgamation Axiom is satisfied.

If $\partial_g \colon P \to P_2$ is an elementary face map corresponding to a top vertex g of P_2 and $\partial_f \colon P \to P_1$ corresponding to a top vertex with the same output, then that output edge is $k \otimes x$ for some $x \in X$ and $f = v \otimes x$. One can easily see how P_3 is constructed in this case and that in terms of Lemma 4.1.12 we have

$$\{f_1, .., f_r\} = \{l_1 \otimes x, ..., l_m \otimes x\} \setminus g.$$

Hence $\partial_{f_1}, ..., \partial_{f_r}$ are elements of F and this proves the remaining of the Amalgamation Axiom.

Here is an example of (the relevant part of) P_3 in the case $f = v \otimes x$ and $g = \{l_1 \otimes x, l_2 \otimes y, l_2 \otimes z, l_3 \otimes x, l_4 \otimes y, l_4 \otimes w, l_5 \otimes x\}$:



The Good Pair Axiom From the definition of F, it is clear that there are no two elementary face maps $\partial_f, \partial_g \in \mathcal{E}_F(P)$ such that f and g are top vertices with the same output. Hence (by Remark 4.1.14 and the definition of F) the Good Pair Axiom is satisfied.

The Bad Pair Axiom Let $\partial_g \colon P \to P_2$ be an elementary face map. There is an elementary face map $\partial_f \colon P \to P_1$ in F such that (∂_f, ∂_g) is not a good pair only if f is a vertex of the form $v \otimes x$ and g is a top vertex of P_2 with the output $k \otimes x$. Hence there is at most one such $\partial_f \in F$. For such a pair (∂_g, ∂_f) we have, in terms of Lemma 4.1.12, that $\{f_1, .., f_r\} = \{l_1 \otimes x, ..., l_m \otimes x\} \setminus g$. Obviously $\partial_{f_1} \notin \mathcal{E}_P$ because it is an inner elementary face map and P has $k \otimes x$ as a leaf (there is no tree P' for which f_1 would be an inner edge and $\partial_{f_1} P' = P$).

The Existence Axiom Let P be a missing face and $\mathcal{F}_F(P) = \emptyset$. By Claim 2., P has an edge $k \otimes x$ for some $x \in X$. If $k \otimes x$ is a leaf of P then there exists a face P' of R_i such that $P = \partial_{v \otimes x} P'$, i.e. $\partial_{v \otimes x} \in \mathcal{E}_F(P)$.

If $k \otimes x$ is not a leaf of P, then it is impossible that all $l_j \otimes x, j = 1, ...m$ appear in P. Indeed, if all of them are leaves then we would have $\partial_{v \otimes x} \in \mathcal{F}_F(P)$ and if $l_j \otimes x$ is inner for some j then we would have $\partial_{l_j \otimes x} \in \mathcal{F}_F(P)$. This implies that there is at least one $j \in \{1, ..., m\}$ such that the edge $l_j \otimes x$ is an inner edge of the shuffle R_i. The structure of shuffles implies actually that for every j the edge $l_j \otimes x$ is an inner edge of R_i.

Since $k \otimes x$ is not a leaf and not all $l_j \otimes x$ appear in P, there is an edge $l_j \otimes y$ for some $j \in \{1, ..., m\}$ and some y in T such that $y \leq x$. Since $l_j \otimes x$ is an inner edge of R_i and it lies between edges $k \otimes x$ and $l_j \otimes y$, which are both edges of P, there is a face P' of R_i such that $P = \partial_{l_j \otimes x} P'$, i.e. $\partial_{l_j \otimes x} \in \mathcal{E}_F(P)$. □

Theorem 4.3.2. *Let T and S be trees, let v be the bottom vertex of S with inputs $l_1, l_2, ..., l_m$ such that $l_2, ..., l_m$ are leaves. If S or T is linear or both S and T are open trees, then the morphism*

$$\Lambda^{v,l_1}[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \Omega[T]$$

is a stable anodyne extension.

Remark 4.3.3. The following proof goes equally if S is a corolla or a tree with more than one vertex. If we consider the case when S is linear, then $m = 1$.

Proof. If $T = \eta$ the statement is equivalent to saying that the horn inclusion $\Lambda^{v,l_1}[S] \to \Omega[S]$ is a stable anodyne extension, which is true by definition. Hence we assume that T has at least one vertex.

We denote by r_S (respectively r_T) the root of S (respectively T).

We fix a total order $R_1 \preceq R_2 \preceq ... \preceq R_N$ extending the right percolation partial order on the set of shuffles as described in 3.2.17, with reversed indexing so R_1 is obtained by grafting copies of S on top of T.

Let $B_0 = \Lambda^{v,l_1}[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T]$ and we define $B_i = B_{i-1} \cup \Omega[R_i]$. The assumptions on T and S imply that all maps $B_{i-1} \to B_i$ are monomorphisms.

If T has no leaves, then R_1 is $r_S \otimes T$ and $B_1 = B_0$. In that case we will show that the inclusions $B_{i-1} \to B_i$ are stable anodyne extensions for all $i = 2, ..., N$. In the case T has at least one leaf, we will show that the inclusions $B_{i-1} \to B_i$ are stable anodyne extensions for all $i = 1, 2, ..., N$.

Since the colour r_S appears in every shuffle R_i, each R_i (except R_1 for T with no leaves) contains at least one vertex $v \otimes x$, $x \in T$ (these are vertices above the highest edges of the form $r_S \otimes x$). Let us fix a shuffle R_i and denote

$$X := \{x \in T : v \otimes x \text{ appears in } R_i\}.$$

If we denote $A_0 := B_{i-1} \cap \Omega[R_i]$ then we have a pushout diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & B_{i-1} \arrow[d] \\ \Omega[R_i] \arrow[r] & B_i. \end{tikzcd}$$

From this it follows that it is enough to show that $A_0 \to \Omega[R_i]$ is a stable anodyne extension.

Characterization of missing faces. With this notation a face P of R_i is missing (i.e. $\Omega[P] \to \Omega[R_i]$ does not factor through A_0) if and only if

- every colour of T appears in P;
- every colour of $\partial_{v,l_1} S$ appears in P;
- for each predecessor R_j of R_i there is at least one edge of P which is not an edge of R_j.

We distinguish two cases. In the first case $r_T \notin X$ and the strategy of the proof is very similar to the strategy of the previous theorem. In the other case $r_T \in X$, so the white vertex $v \otimes r_T$ is the bottom vertex of R_i and hence $X = \{r_T\}$. In that case we have to modify our strategy but the proposition 4.2.12 will again be the key ingredient of the proof.

Case 1. Let $r_T \notin X$.

Claim. Every missing face P has an edge $l_j \otimes x$ for some $x \in X$ and $j \in \{1, ..., m\}$.

Proof: First, let us assume that for every $x \in X$ the edge $l_1 \otimes x$ is a leaf (which might happen if S is a corolla) or connects two white vertices in the shuffle R_i. In this case the colour l_1 appears only on these edges. Colour l_1 must appear in P, so P must contain such an edge $l_1 \otimes x$.

Otherwise, consider an occurrence $l_1 \otimes x$ of colour l_1 in the shuffle R_i of minimal height. Then $x \in X$ and $l_1 \otimes x$ is an output of a black vertex. Since $l_2, ..., l_m$ are leaves of T, all edges $l_j \otimes x$ are outputs of black vertices and inputs of a white vertex $v \otimes x$. Hence the shuffle R_i has a predecessor R_j which does not contain $l_j \otimes x$, $j = 1, ..., m$ (we can just apply an inverse percolation to R_i at this white vertex $v \otimes x$). By the characterization of the missing faces at least one of these edges must appear in P. ■

By the Claim, for every missing face P there is a non-empty set

$$X_P := \{x \in X : l_j \otimes x \text{ appears in } P \text{ for some } j = 1, ..., m\}.$$

The fact that $A_0 \to \Omega[R_i]$ is a stable anodyne extension will follow from proposition 4.2.12 by taking $R = R_i$ and F to consist of elementary face maps $\partial_f P \to P$, such that $\partial_f P$ and P are missing faces of R_i (i.e. $\Omega[\partial_f P] \to \Omega[R]$ and $\Omega[P] \to \Omega[R]$ do not factor through A_0) and f is an inner edge of P denoted by $r_S \otimes x, x \in X_P, j = 1, ..., m$.

Note that for a missing face P with an inner edge $r_S \otimes x, x \in X_P$ the face $\partial_{r_S \otimes x} P$ is also missing because the colour x appears with the edge $l_j \otimes x$ by definition of X_P.

We fix any planar structure on R_i. By Remark 4.1.7 this induces compatible total orders on $\mathcal{F}_F(P)$ and $\mathcal{E}_F(P)$ for every face P of R_i. We will show that the set F satisfies Axioms (F1)-(F6).

These axioms are easily verified because F contains only inner elementary face maps. Indeed, F does not contain a mixed pair of elementary face maps, so the Pullback Axiom is satisfied. Also, F does not contain an adjacent pair of elementary face maps, so the Composition Axiom is satisfied. Similarly Axioms (F3), (F4) and (F5) follow because there are no outer elementary face maps in F. Finally, the Existence Axiom is satisfied since for a missing face P every edge $r_S \otimes x, x \in X_P$ is inner in R_i and lies between $l_j \otimes x$ and $r_S \otimes r_T$, so it will be either an inner edge of P and then $\mathcal{F}_F(P) \neq \emptyset$ or it will not appear in P and there exists P' such that $\partial_{r_S \otimes x} P' = P$, so $\mathcal{E}_F(P) \neq \emptyset$.

Case 2. Let $X = \{r_T\}$.

Let R_i' be the maximal face of R_i with the edge $l_1 \otimes r_T$ being the root. Then R_i is obtained by grafting $R_i', l_2 \otimes T, ..., l_m \otimes T$ on (the leaves $l_1 \otimes r_T, ..., l_m \otimes r_T$ of) the corolla with the vertex $v \otimes r_T$, i.e.

$$R_i' = v \otimes r_T \circ (R_i', l_2 \otimes T, ..., l_m \otimes T).$$

For any face R' of R_i' with the root $l_1 \otimes r_T$ we can form a face R of R_i by grafting $R', l_2 \otimes T, ..., l_m \otimes T$ on the corolla with the vertex $v \otimes r_T$, i.e.

$$R = v \otimes r_T \circ (R', l_2 \otimes T, ..., l_m \otimes T).$$

Let us consider the family of all such faces R for all faces R' of R_i' with the root $l_1 \otimes r_T$. The idea is to proceed in two steps. In the first step we add all faces $\partial_{l_1 \otimes r_T} R$ to the filtration and in the second step we add all faces R of the considered family.

Step 1. Let us denote by B_{i-1}' the union of B_{i-1} with the representables of all $\partial_{l_1 \otimes r_T} R$, for all faces R' of R_i' with the root $l_1 \otimes r_T$. We will show first that the inclusion $B_{i-1} \to B_{i-1}'$ is a covariant anodyne extension. Note that if the edge $l_1 \otimes r_T$ is connecting two white vertices in R_i, then $\partial_{l_1 \otimes r_T} R$ is already in B_0 (i.e. $\Omega[\partial_{l_1 \otimes r_T}] \to \Omega[R_i]$ factors through B_0) for every face R of the considered family, so $B_{i-1}' = B_{i-1}$ and there is nothing to prove.

Let us, thus, assume that the edge $l_1 \otimes r_T$ is the output of a black vertex in R_i.

Let K be the number of vertices of R_i' and let us define inductively a filtration

$$B_{i-1} = C_0 \subseteq C_1 \subseteq .... \subseteq C_k \subseteq ... \subseteq C_K = B_{i-1}'$$

where C_k is the union of B_{i-1} with the representables of all $\partial_{l_1 \otimes T} R$, for all possible faces R' with at most k vertices.

It is enough to show that the inclusion $C_{k-1} \cap \Omega[\partial_{l_1 \otimes r_T} R] \to \Omega[\partial_{l_1 \otimes r_T} R]$ is a covariant anodyne extension, for every face R of the considered family such that $\Omega[\partial_{l_1 \otimes r_T} R] \to B_i$ does not factor through C_{k-1}, since we have a pushout diagram

$$\begin{tikzcd}
\coprod C_{k-1} \cap \Omega[\partial_{l_1 \otimes r_T} R] \arrow[r] \arrow[d] & C_{k-1} \arrow[d] \\
\coprod \Omega[\partial_{l_1 \otimes r_T} R] \arrow[r] & C_k
\end{tikzcd}$$

where the coproduct is taken over all faces R in the considered family.

Let us fix a face R' of R_i' with the root $l_1 \otimes r_T$.

We define Y to be the subset of T such that x is in Y if there is a leaf of R' denoted by $x \otimes s$, $s \in S$. Since Y is a subset of T, it has the induced partial order.

We define X to be the set of minimal elements of Y with respect to that induced partial order. Note that selecting just these minimal elements ensures that (X, r_T) is an operation of T which will be important in verification of Axioms (F1)-(F6).

Let T' be an arbitrary face of T. We call a top vertex w of T' an X*–vertex*

- if w is non-empty and all inputs of w are elements of X (i.e. w is a non-empty subset of X) or
- if w is empty (a stump) and for its output y there is no $x \in X$ such that $y \leq x$.

We denote $A_0 := C_{k-1} \cap \Omega[\partial_{l_1 \otimes r_T} R]$ and we apply the Proposition 4.2.12 to the tree $\partial_{l_1 \otimes r_T} R$ and the extension set F consisting of elementary face maps $\partial_f P \to P$ such that P is a face of $\partial_{l_1 \otimes r_T} R$, faces $\partial_f P$ and P are missing (i.e. $\partial_f P, P \nsubseteq A_0$) and such that f is

- an inner edge of P denoted by $l_j \otimes x, x \in X, j \in \{2, ..., m\}$ or
- a top vertex of P denoted by $l_j \otimes w$ such that w is an X–vertex and $j \in \{2, ..., m\}$.

Note that the missing trees are exactly those faces of $\partial_{l_1 \otimes r_T} R$ that contain all the edges of R' except $l_1 \otimes r_T$, in which all colours of T and all colours of S appear and there is at least one edge which does not appear in any of the preceding shuffles.

We fix a planar structure on $\partial_{l_1 \otimes r_T} R$. By Remark 4.1.7 this induces compatible total orders on $\mathcal{F}_F(P)$ and $\mathcal{E}_F(P)$.

It remains to show that the set F satisfies Axioms (F1)-(F6).

The Pullback Axiom First, note that the set F does not contain a mixed pair of elementary face maps. Hence if there is a missing tree P and $\partial_f, \partial_g \in \mathcal{F}_F(P)$, the faces $\partial_f \partial_g P$ and $\partial_g \partial_f P$ exist, and they are in fact the same. Denote $P' = \partial_f \partial_g P = \partial_g \partial_f P$.

We need to check that P' is a missing tree with respect to A_0. The edges deleted from P to obtain $\partial_f P$ and from $\partial_f P$ to obtain P' are of the form $l_j \otimes x$, $x \in X$, $j \in \{2, ..., m\}$.

Since P contains all the edges of R' except $l_1 \otimes r_T$, the same is true for P'. Hence if a colour of T appears in P it will also appear in P'. If a colour of S appears in P, it will obviously appear in P' as well. Since P is a missing tree it follows that P' is a missing tree.

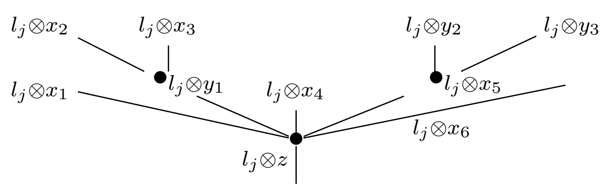
It follows directly from the definition of F that $\partial_f \in \mathcal{F}_{\partial_g P}$ and $\partial_g \in \mathcal{F}_{\partial_f P}$, so the Pullback Axiom is satisfied.

The Composition Axiom Note that F does not contain an adjacent pair of elementary face maps. Hence for any pair of elementary face maps $\partial_f \in \mathcal{F}_F(P), \partial_g \in \mathcal{F}_{\partial_f P}$ the faces $\partial_g P$, $\partial_f \partial_g P$ exist. Since the only edges deleted from P to obtain $\partial_g P$ are of the form $l_j \otimes x$ for some $x \in X$, $j \in \{2, ..., m\}$ the face $\partial_g P$ is also missing. Again, from the definition of F it follows that $\partial_g \in \mathcal{F}_F(P)$ and $\partial_f \in \mathcal{F}_{\partial_g P}$, so the Composition Axiom is satisfied.

The Amalgamation Axiom Let $\partial_g \colon P \to P_2$ be an elementary face map and $\partial_f \colon P \to P_1$, $\partial_f \in \mathcal{E}_F(P)$. If ∂_f and ∂_g are not elementary face maps corresponding to top vertices with the same output, then (∂_f, ∂_g) is a good pair, i.e. in terms of Lemma 4.1.12 we have $r = 1$ and $\partial_{f_1} = \partial_f \colon P_2 \to P_3$. The faces P_2 and P_3 are missing since they contain all the edges of the missing face P, so $\partial_f \colon P_1 \to P_3$ is an element of F by definition of the set F. Hence, in this case The Amalgamation Axiom is satisfied.

If $\partial_g \colon P \to P_2$ is an elementary face map corresponding to a top vertex g of P_2 and $\partial_f \colon P \to P_1$ corresponding to a top vertex with the same output, then $f = l_j \otimes w$ for some X-vertex w and $j \in \{2, ..., m\}$. One can easily see how P_3 is constructed in this case and that in terms of Lemma 4.1.12 $f_1, ..., f_r$ are either inner edges of the form $l_j \otimes x, x \in w$ with $l_j \otimes x \notin g$ or of the form $l_j \otimes w', w' \subseteq w$. Hence $\partial_{f_1}, ..., \partial_{f_r}$ are elements of F and this proves the remaining of the Amalgamation Axiom.

Here is an example of (the relevant part of) P_3 in which $g = \{l_j \otimes x_1, l_j \otimes y_1, l_j \otimes x_4, l_j \otimes y_2, l_j \otimes y_3, l_j \otimes x_6\}$, $w = \{x_1, x_2, x_3, x_4, x_5, x_6\}$:



The Good Pair Axiom From the definition of F, it is clear that there are no two elementary face maps $\partial_f, \partial_g \in \mathcal{E}_F(P)$ such that f and g are top vertices with the same output. Hence (by Remark 4.1.14 and the definition of F) the Good Pair Axiom is satisfied.

The Bad Pair Axiom Let $\partial_g \colon P \to P_2$ be an elementary face map. There is an elementary face map $\partial_f \colon P \to P_1$ in F only if f is a vertex of the form $l_j \otimes w$ for an X-vertex w with an output x, $j \in \{1, ..., m\}$ and g is a top vertex of P_2 with the output $l_j \otimes x$. There is at most one such $\partial_f \in F$. For such a pair (∂_g, ∂_f) we have described while proving the Amalgamation Axiom the edges $f_1, ..., f_r$ in terms of Lemma 4.1.12. Obviously $\partial_{f_1} \notin \mathcal{E}_P$ because it is an inner or top elementary face map and P has $l_j \otimes x$ as a leaf (there is no tree P' for which f_1 would be an inner edge or a top vertex and $\partial_{f_1} P' = P$).

The Existence Axiom Let P be a missing face of $\partial_{l_1 \otimes r_T} R$ and $\mathcal{F}_F(P) = \emptyset$. Since we have assumed that the edge $l_1 \otimes r_T$ is the output of a black vertex in $\partial_{l_1 \otimes r_T} R$, we could perform an inverse percolation at the vertex $v \otimes r_T$ which (by the characterization of missing trees) implies that every missing P must have at least one of the edges $l_1 \otimes r_T, l_2 \otimes r_T, ..., l_m \otimes r_T$. Since P is a face of $\partial_{l_1 \otimes r_T}$, it must have at least one of the edges of the form $\partial_{l_j \otimes r_T}, j \geq 2$. Let us fix one such j. Since $\mathcal{F}_F(P) = \emptyset$, P has no inner edges of the form $l_j \otimes x, x \in X$ and it has no top vertices of the form $l_j \otimes w$ with w being an X-vertex. Hence there is a leaf $l_j \otimes y$ of P such that $y \notin X$. Since (X, r_T) is an operation of T, the set X has the property that for any edge y of T there either exists $x \in X$ such that $y \leq x$ or $x \leq y$ or there exists a stump w of T with an output x such that $x \leq y$.

In the first case, since P has no inner edges $l_j \otimes x, x \in X$, there must exist $x \in X$ such that $x \leq y$. There exists a unique face P' with a top vertex $l_j \otimes w$ such that w is an X-vertex, $l_j \otimes y$ is the output and $l_j \otimes x$ is one of the leaves of $l_j \otimes w$ and such that $\partial_{l_j \otimes w} P' = P$. Hence $\mathcal{E}_F(P) \neq \emptyset$.

Similarly, in the second case, there exists a unique face P' with a top vertex $l_j \otimes w$ with w the stump with the output $l_j \otimes y$ and such that $\partial_{l_j \otimes w} P' = P$. Hence $\mathcal{E}_F(P) \neq \emptyset$.

Step 2. We next show that the inclusion $B'_{i-1} \to B_i$ is anodyne.

Let us define inductively a filtration

$$D_0 = B'_{i-1} \subseteq D_1 \subseteq .... \subseteq D_k \subseteq ... \subseteq D_K = B_i$$

where D_k is the union of B'_{i-1} with the representables of all R, for all possible faces R' with at most k vertices.

It is enough to show that the inclusion $D_{k-1} \cap \Omega[R] \to \Omega[R]$ is anodyne, for every face R' of R'_i such that $\Omega[R] \to B_i$ does not factor through D_{k-1}, since we have a pushout diagram

$$\begin{tikzcd} \coprod D_{k-1} \cap \Omega[R] \arrow[r] \arrow[d] & D_{k-1} \arrow[d] \\ \coprod \Omega[R] \arrow[r] & D_k \end{tikzcd}$$

where the coproduct is taken over all such faces R'.

Let us fix a face R' of R'_i with the root $l_1 \otimes r_T$.

We define Y to be the subset of T such that x is in Y if there is a leaf of R' denoted by $x \otimes s$, $s \in S$. Since Y is a subset of T, it has the induced partial order.

We define X to be the set of minimal elements of Y with respect to that induced partial order.

Let T' be a face of T. We call a top vertex w of T' an X*–vertex* if

- w is non-empty and all inputs of w are elements of X (i.e. w is a non-empty subset of X) or

- w is empty (a stump) and for its output y there is no $x \in X$ such that $y \leq x$ in T.

If $\Omega[R'] \to \Omega[R_i]$ does not factor through B'_{i-1}, consider the tree R'' obtained by grafting R' on the leaf $l_1 \otimes r_T$ of the corolla with the root $r_S \otimes r_T$ and the leaves (other than $l_1 \otimes r_T$) of the form $l_j \otimes x, x \in X, j \in \{2, ..., m\}$. Let us call u the unique vertex of R'' attached to the root. The inclusion $\Lambda^u[R''] \to \Omega[R'']$ is a stable anodyne extension and $\Lambda^u[R'']$ factors through D_{k-1} where k is the number of vertices of R'.

Note that if $\Omega[R'] \to \Omega[R_i]$ does not factor through B'_{i-1}, then $\Omega[R''] \to \Omega[R_i]$ also does not factor through B'_{i-1}.

If $\Omega[R'] \to \Omega[R_i]$ does not factor through B'_{i-1} we denote

$$A_0 := (D_{k-1} \cup \Omega[R'']) \cap \Omega[R]$$

and otherwise we define $A_0 := D_{k-1} \cap \Omega[R]$. We apply the proposition 4.2.12 with the tree R and an extension set F defined in the same way as in the previous case (Case 1.)

Axioms (F1)-(F5) are satisfied because of the same arguments as in the previous case. For the Existence Axiom, we note that any missing tree P of R we must have an edge of the form $l_j \otimes y$ for $y \notin X$, $j \in \{2, ..., m\}$ (since R'' is not missing). □

Remark 4.3.4. Our method also applies to show Proposition 9.2. in [MW09]. Let T and S be two trees and e an inner edge of the tree S. If both S and T are open trees or one of them is linear, then the morphism

$$\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \Omega[T]$$

is an inner anodyne extension.

We use the filtration given by adding one by one shuffle. Let v be the unique vertex of S such that e is the output of v. For a fixed shuffle R_i we define $X = \{x \in T : v \otimes x \text{ is a vertex of } R_i\}$ and the extension set is then given by (inner elementary face maps)

$$F = \{\partial_{x \otimes e} \colon \partial_{x \otimes e} P \to P | P \text{ missing face}, x \in X\}.$$

Chapter 5

Construction of the stable model structure

In this expository chapter we give a construction of a model structure on dendroidal sets in which the cofibrations are normal monomorphisms and the fibrant objects are fully Kan dendroidal sets. The construction uses the combinatorial results that we have obtained in the previous chapter and the standard homotopy theoretical arguments. In fact, most of the arguments that we use in this chapter are well-known to specialists and have appeared in different places. We follow the presentation of [MT10] and [HHM13].

5.1 Homotopy of dendroidal sets

Recall from Chapter 3 that the category of dendroidal set is simplicially enriched and that it is weakly tensored and cotensored. We denote the function complex between objects X and Y by $\underline{\hom}(X, Y)$. Also, recall that the elements of the smallest saturated class containing all horn inclusions are called dendroidal anodyne extensions. An object in dSet is fully Kan if and only if it has the RLP with respect to all dendroidal anodyne extensions. Moreover, the class of normal monomorphisms is the smallest saturated class generated by boundary inclusions.

Proposition 5.1.1. *Let Z be a fully Kan object and $f\colon A \to B$ a normal monomorphism. Then*

$$f^*\colon \underline{\hom}(B, Z) \to \underline{\hom}(A, Z)$$

is a Kan fibration (in simplicial sets).

Proof. It is enough to prove the statement when f is a boundary inclusion $\partial\Omega[T] \to \Omega[T]$. Map f^* has the RLP with respect to horn inclusion $\Lambda^k[n] \to \Delta[n]$ if and only if Z has the RLP with respect to the map

$$\Lambda^k[n] \otimes \Omega[T] \cup \Delta[n] \otimes \partial\Omega[T] \to \Delta[n] \otimes \Omega[T]. \tag{5.1}$$

By Theorem 4.3.1, Theorem 4.3.2 and Remark 4.3.4, map (5.1) is a dendroidal anodyne extension, so Z has the RLP with respect to it. □

Corollary 5.1.2. *If Z is a fully Kan object and B a normal object, then $\underline{\mathrm{hom}}(B, Z)$ is a Kan complex.*

Proof. We apply the previous proposition to the normal monomorphism $f\colon 0 \to B$. □

Let X be a dendroidal set. The following factorization of the fold map

$$\begin{tikzcd} X \sqcup X \arrow[rr, "1_X \sqcup 1_X"] \arrow[dr, "i_0 \sqcup i_1"'] & & X \\ & X \otimes \Delta^1 \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

gives a cylinder object. If X is a normal dendroidal set, then the map $i_0 \sqcup i_1$ is a normal monomorphism.

Definition 5.1.3. Let $f, g\colon X \to Y$ be two morphisms of dendroidal sets. We say that f and g are *homotopic* (and write $f \simeq g$) if there is a morphism $H\colon X \otimes \Delta^1 \to Y$ such that $f = Hi_0, g = Hi_1$. We say that $f\colon X \to Y$ is a *homotopy equivalence* of dendroidal sets if there exists a morphism $g\colon Y \to X$ such that $fg \simeq 1_Y$ and $gf \simeq 1_X$.

Remark 5.1.4. Let X be a dendroidal set and K a simplicial set. Recall from Remark 3.2.10 and Remark 3.2.11 that there is a natural map

$$\alpha\colon X \otimes (K \times \Delta^n) \to (X \otimes K) \otimes \Delta^n.$$

This map induces a map $\tilde{\alpha}\colon \underline{\mathrm{hom}}(X \otimes K, Z) \to \underline{\mathrm{hom}}(X, Z)^K$ for every object Z. Explicitly, this is

$$\begin{aligned} \underline{\mathrm{hom}}(X \otimes K, Z)_n &= \mathrm{hom}((X \otimes K) \otimes \Delta^n, Z) \\ &\to \mathrm{hom}(X \otimes (K \times \Delta^n), Z) = \mathrm{hom}(K \times \Delta^n, \underline{\mathrm{hom}}(X, Z)) \\ &= \mathrm{hom}(\Delta^n, \underline{\mathrm{hom}}(X, Z)^K) \\ &= \underline{\mathrm{hom}}(X, Z)^K_n. \end{aligned}$$

Remark 5.1.5. Let $H\colon X \otimes \Delta^1 \to Y$ be a homotopy from f to g and let Z be a fully Kan dendroidal set. The composition $\tilde{H}$

$$\begin{tikzcd} \underline{\mathrm{hom}}(Y, Z) \arrow[rr, "\tilde{H}"] \arrow[dr, "H^*"'] & & \underline{\mathrm{hom}}(X, Z)^{\Delta^1} \\ & \underline{\mathrm{hom}}(X \otimes \Delta^1, Z) \arrow[ur, "\tilde{\alpha}"'] & \end{tikzcd}$$

is a homotopy from f^* to g^*.

This shows that a homotopy equivalence $f\colon X \to Y$ between normal dendroidal sets induces a homotopy equivalence $f^*\colon \underline{\hom}(Y,Z) \to \underline{\hom}(X,Z)$ between Kan complexes for any fully Kan dendroidal set Z.

Remark 5.1.6. By Remark 3.2.9, a natural transformation $P \otimes_{BV} [1] \to Q$ of maps of operads induces a homotopy $N_d P \otimes \Delta^1 \to N_d(P \otimes_{BV} [1]) \to N_d Q$.

5.2 Stable weak equivalences

Definition 5.2.1. A map $f\colon X \to Y$ is called a *stable weak equivalence* if there exists a normalization $f'\colon X' \to Y'$ of f which induces an equivalence of Kan complexes

$$\underline{\hom}(Y',Z) \to \underline{\hom}(X',Z)$$

for every fully Kan object Z.

Remark 5.2.2. Every homotopy equivalence $f\colon X \to Y$ between normal dendroidal sets is a stable weak equivalence.

Lemma 5.2.3. *Let $f\colon X \to Y$ be a stable weak equivalence and let*

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & Y'' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram where the vertical maps are normalizations. Then $X'' \to Y''$ induces an equivalence of Kan complexes

$$\hom(Y'',Z) \to \hom(X'',Z)$$

for every fully Kan object Z.

Proof. We can take a normalization $X' \to X$ and factor the composite map $X' \to X \to Y$ as a normal monomorphism $X' \to Y'$ followed by a normalization $Y' \to Y$. Since X' and X'' are normal, we have the following lifts

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X'' \arrow[d] \\ X' \arrow[r] \arrow[ur, dotted, "g"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X' \arrow[d] \\ X'' \arrow[r] \arrow[ur, dotted, "f"'] & X \end{tikzcd}$$

Since X' is normal, the map $X' \sqcup X' = X' \otimes \partial\Delta^1 \to X' \otimes \Delta^1$ is a normal monomorphism. Hence there is a lift

$$\begin{tikzcd} X' \sqcup X' \arrow[r, "1 \sqcup fg"] \arrow[d] & X' \arrow[d] \\ X' \otimes \Delta^1 \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

which gives us a homotopy from $1_{X'}$ to fg. Similarly, for X'' we get a homotopy from $1_{X''}$ to gf. Hence, f and g are homotopy equivalences between normal dendroidal sets. Previous considerations imply that f and g are stable weak equivalences. Similarly, we show that there are stable weak equivalences between Y' and Y''.

Since $X' \to Y'$ is a normal monomorphism, the pushout $X'' \to X'' \sqcup_{X'} Y'$ is a normal monomorphism, too. So, there is a lift

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & Y'' \arrow[d] \\ X'' \sqcup_{X'} Y' \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

which gives us the following commutative diagram

$$\begin{tikzcd} X' \arrow[rr] \arrow[dd] \arrow[dr] & & Y' \arrow[dd] \arrow[dr] & \\ & X'' \arrow[rr] \arrow[dl] & & Y'' \arrow[dl] \\ X \arrow[rr] & & Y & \end{tikzcd}$$

This shows that an arbitrary normalization induces an equivalence if and only if the map $X' \to Y'$ induces an equivalence. We know that there exists at least one normalization which induces an equivalence, so we conclude that every normalization induces an equivalence. $\square$

Remark 5.2.4. The stable weak equivalences satisfies the two out of three property.

Proposition 5.2.5. *A morphism of dendroidal sets having the right lifting property with respect to all normal monomorphisms is a stable weak equivalence.*

Proof. Let $f\colon X \to Y$ be a map having RLP with respect to all normal monomorphisms. We consider a normalization $Y' \to Y$. Since Y' is normal there is a lift

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] \arrow[ur, dotted, "s"'] & Y \end{tikzcd}$$

We factor $s\colon Y' \to X$ as a normal monomorphism $i\colon Y' \to X'$ followed by a normalization $t\colon X' \to X$. There is a lift

$$\begin{tikzcd} Y' \arrow[r, equal] \arrow[d, "i"'] & Y' \arrow[d] \\ X' \arrow[r, "ft"'] \arrow[ur, dotted, "f'"] & Y \end{tikzcd}$$

We have $f'i = 1_{Y'}$. Since $Y' \to X'$ is a normal monomorphism, $\partial I \otimes X' \cup I \otimes Y' \to I \otimes X'$ is a normal monomorphism, too. Hence there is a lift in the following diagram

$$\begin{tikzcd} \partial I \otimes X' \cup I \otimes Y' \arrow[r, "(if',1_{X'})\cup i\varepsilon"] \arrow[d] & X' \arrow[d] \\ I \otimes X' \arrow[r, "ft\varepsilon"'] \arrow[ur, dotted] & X \end{tikzcd}$$

Hence there is a homotopy from if' to $1_{X'}$. This shows that the normalization i of f is a homotopy equivalence, i.e. it induces an equivalence of Kan complexes. $\square$

5.3 Stable trivial cofibrations

Definition 5.3.1. A *stable trivial cofibration* of dendroidal sets is a cofibration which is also a stable weak equivalence.

Lemma 5.3.2. *A pushout of a stable trivial cofibration is a stable trivial cofibration.*

Proof. Let $f\colon A \to B$ be a trivial cofibration and let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

be a pushout square. Normal monomorphisms are closed under pushouts, so $C \to D$ is a cofibration and we need to show that it is a stable weak equivalence.

We first consider the case that A and B are normal. For a fully Kan dendroidal set Z, the above pushout square induces a pullback square

$$\begin{tikzcd} \underline{\hom}(D,Z) \arrow[r] \arrow[d] & \underline{\hom}(B,Z) \arrow[d] \\ \underline{\hom}(C,Z) \arrow[r] & \underline{\hom}(A,Z) \end{tikzcd}$$

The right vertical map is a trivial fibration because of the assumption that $A \to B$ is a trivial cofibration between normal objects. Trivial fibrations are closed under pullbacks, so the left vertical map is also a trivial fibration. This show that $C \to D$ is a trivial cofibration in this case.

In the case that A and B are not necessarily normal, we will apply the *cube argument*. Let $D' \to D$ be a normalization of D and consider the commutative diagram

$$\begin{tikzcd}
A' \arrow[rr] \arrow[dr] \arrow[dd] & & C' \arrow[dr] \arrow[dd] & \\
& B' \arrow[rr] \arrow[dd] & & D' \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & C \arrow[dr] & \\
& B \arrow[rr] & & D
\end{tikzcd}$$

constructed so that the vertical squares of the cube are pullbacks. Note that the pullback of a monomorphism in a topos is a monomorphism. As D' is normal, Proposition 3.2.26 implies that the maps $A' \to B'$ and $C' \to D'$ are normal monomorphisms and all vertical maps are normalizations. Also, the top square is a pushout because pullbacks preserve pushouts in a topos. Hence $A' \to B'$ is a trivial cofibration between normal objects and $C' \to D'$ is also by the first part of the proof. This shows that $C \to D$ has a normalization that is a stable weak equivalence, so it is a stable weak equivalence by definition. □

Lemma 5.3.3. *Dendroidal anodyne extensions are stable trivial cofibrations.*

Proof. It is enough to prove that every horn inclusion is a stable weak equivalence. Let $\Lambda^f[T] \to \Omega[T]$ be a horn inclusion and $\partial\Delta[n] \to \Delta[n]$ a boundary inclusion of a simplex. By Theorem 4.3.2, the map

$$\partial\Delta[n] \otimes \Omega[T] \cup \Delta[n] \otimes \Lambda^f\Omega[T] \to \Delta[n] \otimes \Omega[T] \tag{5.2}$$

is a dendroidal anodyne extension, so every fully Kan object Z has the RLP with respect to this map. This implies that the map $\underline{\mathrm{hom}}(\Omega[T], Z) \to \underline{\mathrm{hom}}(\Lambda^f[T], Z)$ is a trivial fibration in simplicial sets for every fully Kan Z. Hence, $\Lambda^f[T] \to \Omega[T]$ is a stable weak equivalence. □

Lemma 5.3.4. *Every stable trivial cofibration is a retract of a pushout of a stable trivial cofibration between normal objects.*

Proof. Let $u\colon A \to B$ be a trivial cofibration and let $A' \to A$ be a normalization of A. We form the commutative square

$$\begin{tikzcd}
A' \arrow[r, "u'"] \arrow[d] & B' \arrow[d] \\
A \arrow[r, "u"'] & B
\end{tikzcd}$$

by factoring $A' \to B$ into a normal monomorphism $A' \to B'$ and a normalization $B' \to B$. We know that normalizations are stable weak equivalences by Proposition 5.2.5. Since stable weak equivalences satisfy the two out of three property, we conclude that $A' \to B'$ is a trivial cofibration between normal objects.

Next, we form a pushout square

$$\begin{tikzcd} A' \arrow[r, "u'"'] \arrow[d] & B' \arrow[d] \\ A \arrow[r, "v"'] & P \end{tikzcd}$$

which gives a canonical map $s\colon P \to B$. It suffices to prove that s has the RLP with respect to all normal monomorphisms, because this would make u a retract of v by lifting in the square

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u"] & P \arrow[d, "s"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

So, consider the lifting problem

$$\begin{tikzcd} \partial\Omega[T] \arrow[r] \arrow[d] & P \arrow[d, "s"'] \\ \Omega[T] \arrow[r] & B \end{tikzcd}$$

We again use the cube argument as in the Proposition 5.3.2. We pull back the above pushout square along $\partial\Omega[T] \to P$ to get a cube

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] \arrow[ddd] & & D \arrow[dr] \arrow[ddd] & \\ & C \arrow[rr] \arrow[ddd] & & \partial\Omega[T] \arrow[ddd] \\ & & & \\ A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\ & A \arrow[rr] & & P \end{tikzcd}$$

in which horizontal faces are pushouts and vertical faces are pullbacks. Then $E \to C$ is a normalization and all objects in the top face are normal. So $E \to C$ has a section and hence so does the pushout $D \to \partial\Omega[T]$. Using this section, we can form a commutative diagram

$$\begin{tikzcd}
\partial\Omega[T] \arrow[r] \arrow[d] & D \arrow[r] & B' \arrow[d] \\
\Omega[T] \arrow[rr] \arrow[urr, dotted] & & B
\end{tikzcd}$$

in which the indicated lift exists and this also gives a solution to our previous lifting problem. □

Lemma 5.3.5. *A stable trivial cofibration between normal fully Kan objects is a deformation retract.*

Proof. Let $f\colon A \to B$ be such a trivial cofibration. Then the map

$$f^*\colon \underline{\mathrm{hom}}(B, A) \to \underline{\mathrm{hom}}(A, A)$$

is a trivial fibration of simplicial sets and therefore surjective on vertices. Hence there exists a map $r\colon B \to A$ such that $rf = 1_A$. We consider the diagram

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r, "{(1_B, fr)}"] \arrow[d, "f^*"] & \underline{\mathrm{hom}}(B, B) \arrow[d] \\
\Delta^1 \arrow[r, "K_f"'] \arrow[ur, dotted] & \underline{\mathrm{hom}}(A, B)
\end{tikzcd}$$

where K_f is the constant homotopy on f. Since $\partial\Delta^1 \to \Delta^1$ is a cofibration and f^* is a trivial fibration there exists a lift as indicated by the dotted arrow which shows that fr is homotopic to 1_B. □

Lemma 5.3.6. *Let $u\colon X \to Y$ be a stable trivial cofibration between normal objects. Let $U \subseteq X_K$ and $V \subseteq Y_K$ be countable subpresheaves. Then there are $A \subseteq X$ and $B \subseteq Y$ such that $U \subseteq A_K$, $V \subseteq B_K$ and $u^{-1}(B) = A$.*

Proof. As in Proposition 3.2.36 (iv), we construct countable $B \subseteq Y$ such that $V \subseteq B_K$. We use the same idea to construct $A \subseteq X$, but in addition in each step of the construction we have to take care that A must be large enough so that $u^{-1}(B) = A$. □

Lemma 5.3.7. *Let $u\colon X \to Y$ be a stable trivial cofibration between normal objects. Then for any countable subpresheaves $A \subseteq X$ and $B \subseteq Y$, there exist intermediate countable subpresheaves $A \subseteq \tilde{A} \subseteq X$ and $B \subseteq \tilde{B} \subseteq Y$ which fit into a pullback diagram*

$$\begin{tikzcd}
\tilde{A} \arrow[r] \arrow[d] & X \arrow[d, "u"] \\
\tilde{B} \arrow[r] & Y.
\end{tikzcd}$$

Proof. As in Proposition 3.2.36 (ii) we construct a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X_K \arrow[r, "u_K"] & Y_K \end{tikzcd}$$

where X_K and Y_K are fully Kan dendroidal sets. Since $X \to Y$ is a normal monomorphism between normal objects, we can construct the above square so that $X_K \to Y_K$ is also a normal monomorphism between normal objects. By two out of three property for stable weak equivalences, we conclude that $X_K \to Y_K$ is a stable weak equivalence.

By Lemma 5.3.5, the trivial cofibration $u_K : X_K \to Y_K$ is a deformation retraction. Let $r: Y_K \to X_K$ be such that $ru_K = 1_{X_K}$ and $u_K r$ is homotopic to the identity under the homotopy $H: \Delta^1 \otimes Y_K \to Y_K$.

First, we want to construct countable subpresheaves $A'_0 \subseteq X_K$ and $B'_0 \subseteq Y_K$ such that A'_0 is a deformation retract of B'_0.

We can directly construct countable $A' \subseteq X_K$ and $B' \subseteq Y_K$ such that $A \subseteq A'$, $B \subseteq B'$, $r(B) \subseteq A$, $u_K(A) \subseteq B$ and $H(\Delta^1 \otimes B) \subseteq B'$. Then we iterate this procedure to get a sequences of inclusions

$$A \subseteq A' \subseteq A'' \subseteq \ldots \subseteq A^{(n)} \subseteq \ldots$$
$$B \subseteq B' \subseteq B'' \subseteq \ldots \subseteq B^{(n)} \subseteq \ldots$$

such that $A^{(n)}$ is a deformation retract of $B^{(n)}$. We define $A'_0 = \bigcup_n A^{(n)}$ and $B'_0 = \bigcup_n B^{(n)}$, so A'_0 is a deformation retract of B'_0 what is exhibited by the restriction of the homotopy H. By applying the Lemma 5.3.6 we obtain countable $A_1 \subset X$ and $B_1 \subset Y$ such that $A'_0 \subseteq (A_1)_K$ and $B'_0 \subseteq (B_1)_K$.

We iterate this procedure to obtain sequences of inclusions

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots \subseteq X$$

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n \subseteq \ldots \subseteq Y$$

and sequences

$$A'_0 \subseteq (A_1)_K \subseteq A'_1 \subseteq (A_2)_K \subseteq A'_2 \subseteq \ldots \subseteq (A_n)_K \subseteq A'_n \subseteq \ldots \subseteq X_K$$

$$B'_0 \subseteq (B_1)_K \subseteq B'_1 \subseteq (B_2)_K \subseteq B'_2 \subseteq \ldots \subseteq (B_n)_K \subseteq B'_n \subseteq \ldots \subseteq Y_K$$

such that A'_n is a deformation retract of B'_n (with the restriction of the map H as the homotopy) and $u^{-1}(B_n) = A_n$. We define $\tilde{A} = \bigcup_n A_n$ and $\tilde{B} = \bigcup_n B_n$. Then $\tilde{A}_K = \bigcup_n (A_n)_K = \bigcup_n A'_n$ and $\tilde{B}_K = \bigcup_n (B_n)_K = \bigcup_n B'_n$. Since each $A'_n \to B'_n$ is a deformation retract, we conclude that $\tilde{A}_K \to \tilde{B}_K$ is a weak equivalence. This implies that $\tilde{A} \to \tilde{B}$ is a weak equivalence. Also, since $u^{-1}(B_n) = A_n$, we have $u^{-1}(\tilde{B}) = \tilde{A}$, i.e. there is a pullback diagram as described in the statement. □

Proposition 5.3.8. *The class of stable trivial cofibrations is generated by the stable trivial cofibrations between countable and normal objects.*

Proof. Because of Lemma 5.3.4, it is enough to show that any trivial cofibration $u\colon X \to Y$ between normal objects is in the saturated class generated by trivial cofibrations between countable and normal objects.

We fix a well-order on the set $\{y_\xi : \xi < \lambda\}$ of non-degenerate dendrices in Y which are not in the image of u.

We will inductively construct factorizations $X \to X_\xi \to Y$ of u into trivial cofibrations, such that for $\xi < \xi'$ there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_{\xi'} \arrow[d, "s"] \\ X_\xi \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

such that $y_\xi \in X_\xi$ and $X \to X_\xi$ is a composition of pushouts of trivial cofibrations between countable and normal objects. We start with $X_0 = X$. If X_ξ is constructed for all $\xi < \zeta$, we define

$$X_\zeta^- = \operatorname*{colim}_{\xi<\zeta} X_\xi.$$

We consider a pullback diagram

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & X_\zeta^- \arrow[d] \\ \tilde{B} \arrow[r] & Y \end{tikzcd}$$

as in the Lemma 5.3.7 with $y_\xi \in \tilde{B}$ and construct the pushout

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & X_\zeta^- \arrow[d] \\ \tilde{B} \arrow[r] & X_\zeta \end{tikzcd}$$

The canonical map $X_\zeta \to Y$ is a monomorphism since $X_\zeta^- \to Y$ is a monomorphism (and pullbacks preserve monomorphisms in a topos). □

5.4 The stable model structure

Definition 5.4.1. A morphism of dendroidal sets is called a *stable fibration* if it has the right lifting property with respect to all stable trivial cofibrations.

Theorem 5.4.2. *There is a model structure on dSet for which cofibrations are normal monomorphisms, weak equivalences are stable weak equivalences and fibrations are stable fibrations. This model structure is cofibrantly generated and left proper.*

We call this structure the *stable model structure*.

Proof. We need to check axioms (CM1)-(CM5). Axiom (CM1) holds since dSet is a presheaf category. Axiom (CM2) follows from the definition of the stable weak equivalences and the fact that weak equivalences in the Kan-Quillen model structure on sSet satsify (CM2). Axiom (CM3) is easy to check, as it is left to the reader.

We first show axiom (CM5). That every map can be factored as a cofibration followed by a trivial fibration follows from the small object argument for the set of boundary inclusions $\partial\Omega[T] \to \Omega[T]$ and Lemma 5.2.5.

The other factorization follows from the small object argument for the set of all trivial cofibrations between countable normal dendroidal sets (note that this is indeed a set and it permits the small object argument) and 5.3.7.

One part of the axiom (CM4) is automatically satisfied by the definition of stable fibrations. We now show the other part. Assume that we have the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"'] \\ B \arrow[r] & Y \end{tikzcd}$$

in which i is a cofibration and p is a trivial fibration. We factor $p\colon X \to Y$ as a cofibration $Y \to Z$ followed by a map $Z \to X$ having the RLP with respect to all cofibrations. Then $Z \to X$ is a weak equivalence and by two out of three property so is $Y \to Z$. We first find a lift in

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[r] & Z \arrow[d] \\ B \arrow[rr] \arrow[urr, dotted] & & Y \end{tikzcd}$$

and then in

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[d, "p"'] \\ Z \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

The composition of these two lifts gives us the required lift $B \to X$.

The generating cofibrations are boundary inclusions and the set of generating trivial cofibrations consists of trivial cofibrations between normal and countable dendroidal sets. The arguments that we used so far show that there exists a described model structure and that it is cofibrantly generated.

It remains to prove that this model structure is left proper. We consider a pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

in which $A \to B$ is a stable weak equivalence and $A \to C$ is a cofibration. We us the cube argument as in Proposition 5.3.2 to reduce to the case where all objects A, B, C, D are normal. Then the diagram

$$\begin{tikzcd} \underline{\hom}(D, Z) \arrow[r] \arrow[d] & \underline{\hom}(B, Z) \arrow[d] \\ \underline{\hom}(C, Z) \arrow[r] & \underline{\hom}(A, Z) \end{tikzcd}$$

is a pullback square for any object Z. In particular, for a fully Kan dendroidal set Z, all the simplicial sets in this diagram are Kan complexes and the right vertical map is an equivalence. Then the left vertical map is also an equivalence. Indeed, trivial fibrations are stable under pullbacks, so by Ken Brown's lemma all weak equivalences between fibrant objects are stable under pullbacks. $\square$

Proposition 5.4.3. *The fibrant objects in the stable model structure are exactly the fully Kan objects.*

Proof. Let Z be a fibrant object in the stable model structure. By Lemma 5.3.3, dendroidal anodyne extensions are trivial cofibrations, so Z has the RLP with respect to all anodyne extensions, i.e. Z is a fully Kan object.

If Z is a fully Kan object and $A \to B$ is a trivial cofibration between normal objects, then the induced map

$$\underline{\hom}(B, Z) \to \underline{\hom}(A, Z)$$

is a trivial fibration in simplicial sets. Since trivial fibrations in sSet are surjective on vertices, we deduce that Z has the RLP with respect to $A \to B$. Lemma 5.3.4 implies that every fully Kan object has the RLP with respect to all trivial cofibrations. $\square$

Proposition 5.4.4. *The stable model structure is a weakly simplicial model structure.*

Proof. We will prove that for trivial cofibration $Y \to Z$ between normal dendroidal sets, a monomorphism $M \to N$ of simplicial sets and a fully Kan dendroidal set F, the map

$$\underline{\hom}(Z \otimes N, F) \to \underline{\hom}(Z \otimes M \cup Y \otimes N, F)$$

is a trivial fibration.

Note that $\underline{\mathrm{hom}}(Z \otimes M \cup Y \otimes N, F) = \underline{\mathrm{hom}}(Z \otimes M, F) \times_{\underline{\mathrm{hom}}(Y \otimes M, F)} \underline{\mathrm{hom}}(Y \otimes N, F)$. By Remark 5.1.4, we have the following diagram

$$\begin{tikzcd}
\underline{\mathrm{hom}}(Z \otimes N, F) \arrow[r] \arrow[d] & \underline{\mathrm{hom}}(Z \otimes M, F) \times_{\underline{\mathrm{hom}}(Y \otimes M, F)} \underline{\mathrm{hom}}(Y \otimes N, F) \arrow[d] \\
\underline{\mathrm{hom}}(Z, F)^N \arrow[r] & \underline{\mathrm{hom}}(Z, F)^M \times_{\underline{\mathrm{hom}}(Y, F)^M} \underline{\mathrm{hom}}(Y, F)^N
\end{tikzcd}$$

and we know that the map at the bottom is a trivial fibration because $\underline{\mathrm{hom}}(Z, F) \to \underline{\mathrm{hom}}(Y, F)$ is one.

The map on the top is a fibration. To check that is is a weak equivalence it is enough to see that the vertical maps are weak equivalences. For the left vertical map, we see that the lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \underline{\mathrm{hom}}(Z \otimes N, F) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \underline{\mathrm{hom}}(Z, F)^N
\end{tikzcd}$$

is equivalent to the problem of finding the dotted arrow in the following diagram

$$\begin{tikzcd}
(\partial\Delta^n \times N) \otimes Z \arrow[r] \arrow[d] & \partial\Delta^n \otimes (N \otimes Z) \arrow[d] \arrow[ddr] & \\
(\Delta^n \times N) \otimes Z \arrow[r] \arrow[drr] & \Delta^n \otimes (N \otimes Z) \arrow[dr, dotted] & \\
& & F.
\end{tikzcd}$$

By the arguments of [HHM13] (cf. Lemma 3.8.1 in loc. cit.), the map

$$(\Delta^n \times N) \otimes Z \cup \partial\Delta^n \otimes (N \otimes Z) \to \Delta^n \otimes (N \otimes Z)$$

is an inner anodyne extension, so the lifting problem has a solution. This shows that the left vertical map is a trivial fibration. The fact that the right vertical map is a trivial fibration now follows since trivial fibrations are closed under pullbacks. □

Proposition 5.4.5. *A map between fibrant objects is a fibration if and only if it has the right lifting property with respect to the dendroidal anodyne extensions.*

Proof. let $X \to Y$ be a map between fibrant objects. If it is a fibration, then it has the RLP with respect to all trivial cofibrations. By Lemma 5.3.3, that means that, in particular, it has the RLP with respect to all dendroidal anodyne extensions.

Conversely, let $X \to Y$ have the RLP with respect to all dendroidal anodyne extensions. We factor it as trivial cofibration $i\colon X \to Z$ followed by a fibration $p\colon Z \to Y$. Since X is fibrant there is a map $r\colon Z \to X$ such that $ri = 1_X$.

Since we have shown that the stable model structure is weakly simplicial model structure, the map

$$X \otimes \Delta^1 \cup Z \otimes \partial\Delta^1 \to Z \otimes \Delta^1$$

is a trivial cofibration. As Y is also fibrant, there is a lift in the following diagram

$$\begin{tikzcd} X \otimes \Delta^1 \cup Z \otimes \partial\Delta^1 \arrow[r] \arrow[d] & Y \\ Z \otimes \Delta^1 \arrow[ur, dotted] & \end{tikzcd}$$

where the top map is $f\varepsilon \cup (fr, p)$. This gives a homotopy H from fr to p relative to X.

Let $(X \otimes \Delta^1) \cup_X Z$ be the pushout along $i_0\colon X \to X \otimes \Delta^1$. We can find a lift in the following diagram

$$\begin{tikzcd} X \otimes \Delta^1 \cup_X Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \otimes \Delta^1 \arrow[r, "h"'] \arrow[ur, dotted, "k"] & Y \end{tikzcd}$$

where the top map is $\varepsilon \cup r$. This is possible because the map on the left is a dendroidal anodyne extension. Then $r' = k_1$ has the property that $fr' = h_1 = p$ and $r'i = \varepsilon i_1 = 1_Y$. So f is a retract of p over X and hence a fibration, since p is. $\square$

5.5 Quillen functors from the stable model structure

Lemma 5.5.1. *Let M be a model category and let $F, G\colon dSet \to M$ be left adjoint functors that send normal monomorphisms to cofibrations. If there is a natural transformation $\alpha\colon F \to G$ such that $\alpha_{\Omega[T]}\colon F(\Omega[T]) \to G(\Omega[T])$ is a weak equivalence for every tree T, then $\alpha_X\colon F(X) \to G(X)$ is a weak equivalence for every normal dendroidal set X.*

Proof. For a non-negative integer n, we say that a dendroidal set X is *n-dimensional* if X has no non-degenerate dendrex of shape T for $|T| > n$.

We prove by induction on n that if X is a normal n-dimensional dendroidal set, then $F(X) \to G(X)$ is a weak equivalence. If X is 0-dimensional, then X is just a coproduct of copies of η. By the assumption, $F(\eta) \to G(\eta)$ is a weak equivalence, so $F(X) \to G(X)$ is a weak equivalence since it is a coproduct of weak equivalences between cofibrant objects (cf. Corollary 2.3.14).

For the inductive step, assume X is an n-dimensional normal dendroidal set and let X' be its $(n-1)$-skeleton. Then $X = X' \sqcup_{\coprod \partial\Omega[T]} \coprod \Omega[T]$, where the coproduct varies over all isomorphism classes of nondegenerate dendrices of X.

Since F and G are left adjoints they preserve colimits, so there is a commutative diagram

$$\begin{tikzcd}
\coprod F(\partial\Omega[T]) \arrow[rr] \arrow[dr, "\sim"] \arrow[dd] & & F(X') \arrow[dr, "\sim"] \arrow[dd] & \\
& \coprod G(\partial\Omega[T]) \arrow[rr] \arrow[dd] & & G(X') \arrow[dd] \\
\coprod F(\Omega[T]) \arrow[rr] \arrow[dr, "\sim"] & & F(X) \arrow[dr] & \\
& \coprod G(\Omega[T]) \arrow[rr] & & G(X)
\end{tikzcd}$$

where all the objects are cofibrant, the back and front sides are pushout squares and the two vertical maps on the left are cofibrations. The two maps in the upper square are weak equivalences by the inductive hypothesis. The map $\coprod F(\Omega[T]) \to \coprod G(\Omega[T])$ is a weak equivalence by Corollary 2.3.14. Hence $F(X) \to G(X)$ is also a weak equivalence.

Finally, for a normal dendroidal set X, consider the skeletal filtration of X:

$$\emptyset = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \to \ldots \to X^{(n)} \subseteq \ldots$$

Since $X^{(n)}$ is n-dimensional, we have shown already that $F(X^{(n)}) \to G(X^{(n)})$ is a weak equivalence between cofibrant objects. Since F (resp. G) preserves colimits, $F(X)$ (resp. $G(X)$) is a filtered colimit of $F(X^{(n)})$ (resp. $G(X^{(n)})$) and hence $F(X) \to G(X)$ is a weak equivalence, too. □

Corollary 5.5.2. *Let $F, G : dSet \to M$ be two left Quillen functors and $\varphi : F \Rightarrow G$ a natural transformation such that $\varphi_\eta : F(\eta) \to G(\eta)$ is an equivalence. Then $\varphi_X : F(X) \to G(X)$ is an equivalence for any normal dendroidal set X. In particular, F and G induce equivalent functors on homotopy categories.*

Proof. For any tree T, the inclusion of leaves $\sqcup_{l(T)}\eta \to \Omega[T]$ is a stable trivial cofibration. Since F and G are left Quillen, it follows that $F(\Omega[T]) \to G(\Omega[T])$ is a weak equivalence. The result follows from 5.5.1. □

Lemma 5.5.3. *Let M be a model category and $F : dSet \to M$ a left adjoint functor. Then F is a left Quillen functor with respect to the stable model structure on dendroidal sets if and only if F sends boundary inclusions to cofibrations and horn inclusions to trivial cofibrations in M.*

Proof. If F is left Quillen, then F clearly sends boundary inclusions to cofibrations and horn inclusions to trivial cofibrations in M.

To prove the converse, let us assume that F sends boundary inclusions to cofibrations and horn inclusions to trivial cofibrations in M. Since cofibrations in the stable model structure are generated as a saturated class by boundary inclusions, it follows that F preserves cofibrations.

Let G be the right adjoint of F. It is enough to show that G preserves fibrations between fibrant objects. Let X be a fibrant object in M. The lifting problem for GX against a horn inclusion $i\colon A \to B$ is equivalent to the lifting problem for X against $F(i)\colon F(A) \to F(B)$. Since we assume that $F(i)$ is a trivial cofibration in M, this problem has a solution. Hence $G(X)$ is fibrant, i.e. G preserves fibrant objects.

Similarly, if $f\colon X \to Y$ is a fibration between fibrant objects in M, objects $G(X)$ and $G(Y)$ are fibrant. The same argument as above shows that $G(f)$ has the right lifting property against all horn inclusions, i.e. it is a fibration between fibrant objects. This finishes the proof. $\square$

Chapter 6

Dendroidal sets as models for connective spectra

In this chapter we will further study the stable model structure on dendroidal sets. We will show that this model structure is Quillen equivalent to a model structure on a category of a more geometric flavour. Our main result is that the stable homotopy theory of dendroidal sets is actually the stable homotopy theory of connective spectra. More precisely, there is a model structure on E_∞-spaces in which the fibrant objects are grouplike E_∞-spaces which is Quillen equivalent to the stable model structure on dendroidal sets. By the recognition principle, the grouplike E_∞-spaces correspond to connective spectra, so the associated ∞-category of dendroidal sets with respect to the stable model structure is equivalent to the ∞-category of connective spectra.

To show this we will give a different construction of the stable model structure on dendroidal sets. Our result is based on certain results of Heuts. In [Heu11a], Heuts established the covariant model structure, which lies between the operadic and the stable model structure. The idea is, as we described in the Introduction, to construct the stable model structure as a Bousfield localization of the covariant model structure. This enables us to directly use another main result of Heuts: there is a Quillen equivalence between the covariant model structure and the model category on E_∞-spaces. Our Quillen equivalence (Theorem 6.3.4) can then be derived by showing that the stable localization on the side of dendroidal sets corresponds to the group-like localization of E_∞-spaces.

One disadvantage of this construction is that showing that the fibrant objects of the Bousfield localization are exactly the fully Kan dendroidal sets (i.e. the fibrant objects of the stable model structure) is technically demanding. Not to disturb the flow of the main ideas we postpone the combinatorial proofs until the later part of the chapter.

This chapter contains the essential parts of the article *Dendroidal sets as models for connective spectra* written jointly with Thomas Nikolaus and published in Journal of K-theory in 2014, [BN14].

6.1 Fully Kan dendroidal sets and Picard groupoids

Recall the definition of a fully Kan dendroidal set.

Definition 6.1.1. A dendroidal set X is called *fully Kan* if it has fillers for all horn inclusions. This means that for each morphism $\Lambda^a[T] \to X$ (where a is an inner edge or an outer vertex) there is a morphism $\Omega[T] \to X$ rendering the diagram

$$\begin{tikzcd} \Lambda^a[T] \arrow[r] \arrow[d] & X \\ \Omega[T] \arrow[ur] & \end{tikzcd}$$

commutative. A dendroidal set X is called *strictly fully Kan* if additionally all fillers for trees T with more than one vertex are unique.

We denote by Sym the category of symmetric monoidal categories together with lax monoidal functors. Recall that a lax monoidal functor $F : C \to D$ is a functor together with morphisms $F(c) \otimes F(c') \to F(c \otimes c')$ for each $c, c' \in C$ and $1 \to F(1)$ which have to satisfy certain coherence conditions but do not have to be isomorphisms. In Example 2.2.36 we have described a construction associating a symmetric coloured operad O_C to any symmetric monoidal category C. That construction gives a fully faithful functor

$$\mathrm{Sym} \to \mathrm{Oper}.$$

By composing with the dendroidal nerve $N_d : \mathrm{Oper} \to \mathrm{dSet}$ for each symmetric monoidal category C we obtain a dendroidal set which we denote by abuse of notation with $N_d(C)$.

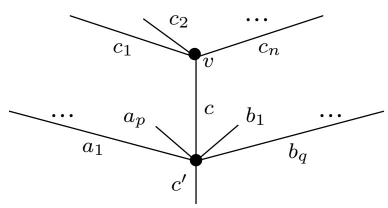
In [MW09] it is shown that a dendroidal set is strictly inner Kan if and only if it is of the form $N_d(P)$ for a coloured operad P. An analogous statement is true for strictly fully Kan dendroidal sets. Note that a symmetric monoidal category is called a *Picard groupoid* if its underlying category is a groupoid and its set of isomorphism classes is a group, i.e. there are 'tensor inverses' for objects.

Proposition 6.1.2. *A dendroidal set X is strictly fully Kan if and only if there is a Picard groupoid C with $X \cong N_d(C)$.*

Proof. First assume that X is strictly fully Kan. Then X is, in particular, a strictly inner Kan dendroidal set and [MW09, Theorem 6.1] shows that there is a coloured operad P with $N_d(P) \cong X$. Let C be the underlying category of P. Since the underlying simplicial set of $N_d(P)$ is a Kan complex we conclude that C is a groupoid.

By [Lei04, Theorem 3.3.4] an operad P comes from a unique symmetric monoidal category as described above if and only if for every sequence $c_1, ..., c_n$ of objects in P there is a universal tensor product, that is an object c together with an operation

$t \in P(c_1,...,c_n;c)$ such that for all objects $a_1,...,a_p,b_1,...,b_q,c'$ and operations $t' \in P(a_1,...,a_p,c_1,...,c_n,b_1,...,b_q;c')$ there is a unique element $s \in P(a_1,...,a_p,c,b_1,...,b_q;c')$ such that the partial composition of s and t in P is equal to t'. A sequence $c_1,...,c_n$ of objects of P determines a map from $\eta_{c_1} \sqcup \cdots \sqcup \eta_{c_n}$ to $N_d(P)$. Since $N_d(P)$ is fully Kan we can fill the horn $\eta_{c_1} \sqcup \cdots \sqcup \eta_{c_n} \to \Omega[C_n]$ and obtain a morphism $\Omega[C_n] \to N_d(P)$. The root colour of this morphism provides an object c in P and the corolla provides an operation $t \in P(c_1,...,c_n;c)$. Assume we have another operation $t' \in P(a_1,...,a_p,c_1,...,c_n,b_1,...,b_q;c')$. Then we consider the tree T which is given by



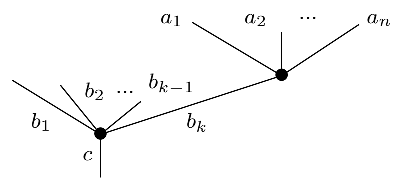
The operations t and t' provide a morphism $\Lambda^v[T] \to N_dP$, where $\Lambda^v[T]$ is the outer horn of $\Omega[T]$ at v. Since D is strictly fully Kan we obtain a unique filler $\Omega[T] \to N_d(P)$, i.e. a unique $s \in P(a_1,...,a_p,c,b_1,...,b_q;c')$ with the sought condition. This shows that c is the desired universal tensor product and that P comes from a symmetric monoidal category.

The last thing to show is that C is group-like. For a and c in C we obtain an object b together with a morphism $t \in P(a,b;c)$ by filling the root horn $\eta_a \sqcup \eta_c \to \Omega[C_2]$. But this is the same as a morphism $a \otimes b \to c$ which is an isomorphism since C is a groupoid. If we let c be the tensor unit in C then b is the necessary inverse for a.

Assume conversely that C is a Picard groupoid. Then the associated dendroidal set $N_d(C)$ admits lifts for corolla horns since tensor products and inverses exist (the proof is essentially the same as above). It remains to show that all higher horns admit unique fillers. To see this let T be a tree with more than one vertex and $\Lambda^a[T]$ be any horn. A morphism $\Omega[T] \to N_d(C)$ is given by labeling the edges of T with objects of C and the vertices with operations in C of higher arity, i.e. morphisms out of the tensor product of the ingoing objects into the outgoing object of the vertex. The same applies for a morphism $\Lambda^a[T] \to N_d(C)$ where the faces in the horn are labeled in the same manner and consistently.

The first observation is that for any labeling of the horn $\Lambda^a[T]$ already all edges of the tree T are labeled, since the horn contains all colours of T (for T with more than one vertex). If the horn is inner then also all vertices of T are already labeled if we label $\Lambda^a[T]$ and thus there is a unique filler. If a is an outer vertex and T has more then two vertices then the same applies as one easily checks. Thus the horn can be uniquely filled. Therefore we only have to deal with outer horns of trees with exactly two vertices. Such trees can all be obtained by grafting an n-corolla C_n for $n \geq 0$ on top of a k-corolla for $k \geq 1$. We call this tree $C_{n,k}$.

$C_{n,k} =$



A morphism from the non-root horn $\Lambda^v[C_{n,k}] \to N_d(C)$ is then given by a pair consisting of a morphism $f : a_1 \otimes ... \otimes a_n \longrightarrow b_k$ and a morphism $g : b_1 \otimes ... \otimes b_{k-1} \otimes a_1 \otimes ... \otimes a_n \longrightarrow c$ in C. Now we find a unique morphism $g \circ (id \otimes f^{-1}) : b_1 \otimes ... \otimes b_k \to c$ which renders the relevant diagram commutative, i.e. provides a filler $\Omega[C_{n,k}] \to N_d(C)$. A similar argument works for the case of the root horn of $C_{n,k}$. This finishes the proof. □

Corollary 6.1.3. *The functor N_d : $Sym \to dSet$ induces an equivalence between the full subcategory of Picard groupoids on the left and the full subcategory of strictly fully Kan dendroidal sets on the right.*

Proof. The functor N_d is fully faithful since both functors Sym $\to$ Oper and Oper $\to$ dSet are. The restriction is essentially surjective by the last proposition. □

One of the main results of this paper shows that a similar statement is valid for fully Kan dendroidal sets that are not strict. They form a model for Picard ∞-groupoids, as we will show in the next sections.

Finally we want to give a characterization of strictly fully Kan dendroidal sets for which the corolla horns also admit unique fillers. Let A be an abelian group, then we can associate to A a symmetric monoidal category A_{dis} which has A as objects and only identity morphisms. The tensor product is given by the group multiplication of A and is symmetric since A is abelian. This construction provides a fully faithful functor from the category AbGr of abelian groups to the category Sym. Composing with the functor Sym $\to$ dSet constructed above we obtain a fully faithful functor

$$i : \text{AbGr} \to \text{dSet}.$$

Now we can characterize the essential image of i.

Proposition 6.1.4. *For a dendroidal set X the following two statements are equivalent*

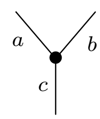
- *X is fully Kan with all fillers unique.*
- *$X \cong i(A)$ for an abelian group A.*

Proof. We already know by Proposition 6.1.2 that strictly fully Kan dendroidal sets are of the form $N_d(C)$ for C a Picard groupoid. We consider the underlying space $i^*X = NC$. This is now a strict Kan complex in the sense that all horn fillers are unique. In particular fillers for the horn $\Lambda^0[1] \to \Delta[1]$ are unique which shows that there are no non-degenerated 1-simplices in NC, hence no non-identity morphisms in C. Thus C is a discrete category. But a discrete category which is a Picard groupoid is clearly of the form A_{dis} for an abelian group A. This shows one direction of the claim. The other is easier and left to the reader. □

6.2 The stable model structure

The idea is to localize at a root horn of the 2-corolla

$C_2 =$



The relevant horn is given by the inclusion of the colours a and c, i.e. by the map

$$s \colon \Lambda^b[C_2] = \eta_a \sqcup \eta_c \longrightarrow \Omega[C_2]. \tag{6.1}$$

Note that there is also the inclusion of the colours b and c, but this is essentially the same map since we deal with symmetric operads.

Definition 6.2.1. The *local* model structure on dendroidal sets is the left Bousfield localization of the covariant model structure at the map s. Hence the local cofibrations are normal monomorphisms between dendroidal sets and the locally fibrant objects are those dendroidal Kan complexes X for which the map

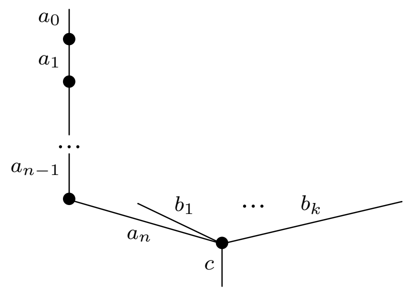
$$s^* : \underline{\mathrm{hom}}(\Omega[C_2], X) \to \underline{\mathrm{hom}}(\eta_a \sqcup \eta_c, X)$$

is a weak equivalence of simplicial sets.

By Proposition 2.3.20, a model structure is uniquely determined by cofibrations and fibrant objects. The following theorem implies that the local and the stable model structure coincide. We will need the following terminology.

Definition 6.2.2. An *extended corolla* is a tree of the form

$EC_{n,k} =$



In particular we have $EC_{0,k} = C_{k+1}$. The trees $EC_{n,1}$ are called *binary extended corollas.* The root horn of the extended corolla is the union of all faces except the face obtained by chopping off the root vertex.

Theorem 6.2.3. *For a dendroidal set X the following statements are equivalent.*

(i) X is fibrant in the local model structure.

(ii) X is dendroidal Kan and admits fillers for all root horns of extended corollas $EC_{n,1}$.

(iii) X is dendroidal Kan and admits fillers for all root horns of extended corollas $EC_{n,k}$.

(iv) X is fully Kan.

We will prove Theorem 6.2.3 at the end of this chapter. More precisely the equivalence of (i) and (ii) is in Proposition 6.4.2. The equivalence of (ii) and (iii) is in Proposition 6.5.1 and the equivalence of (iii) and (iv) is in Proposition 6.6.2.

Corollary 6.2.4. *The stable model structure is the Bousfield localization of the covariant model structure at the map $s\colon \Lambda^b[C_2] = \eta_a \sqcup \eta_c \longrightarrow \Omega[C_2]$.*

Remark 6.2.5. We could as well have localized at bigger collections of maps:

- all corolla root horns
- all outer horns

These localizations would yield the same model structure as the Theorem 6.2.3 implies. We decided to use only the 2-corolla in order to keep the localization (and the proofs) as simple as possible.

The general theory of left Bousfield localization yields the following:

Theorem 6.2.6. *(i) The category of dendroidal sets together with the stable model structure is a left proper, combinatorial, simplicial model category.*

(ii) The adjoint pair

$$i_! : sSet \rightleftarrows dSet : i^*$$

is a Quillen adjunction (for the stable model structure on dendroidal sets and the Kan-Quillen model structure on simplicial sets).

(iii) The functor i^* is homotopy right conservative, that is a morphism $f : D \to D'$ between stably fibrant dendroidal sets D and D' is a stable equivalence if and only if the underlying map $i^*f : i^*D \to i^*D'$ is a homotopy equivalence of Kan complexes.

Proof. The first part follows from the general theory of Bousfield localizations (see e.g. [Lur09, A.3]). For the second statement, note that the corresponding fact for the covariant model structure is true. Since the stable model structure is a left Bousfield localization of the covariant model structure the claim follows by composition with the identity functor. The last assertion is true since a morphism between stably fibrant objects is a stable equivalence if and only if it is a covariant equivalence and covariant equivalences between fibrant objects can be tested on the underlying spaces (see [Heu11b, Proposition 2.2.]). □

Corollary 6.2.7. *Let $f : X \to Y$ be a map of dendroidal sets. Then f is a stable equivalence exactly if $i^*(f_K)$ is a weak equivalence where $f_K : X_K \to Y_K$ is the corresponding map between fully Kan (fibrant) replacements of X and Y.*

6.3 Equivalence to connective spectra

Let $\mathcal{E}_\infty \in$ dSet be a cofibrant resolution of the terminal object in dSet. We furthermore assume that $\mathcal{E}_\infty$ has the property that the underlying space $i^*\mathcal{E}_\infty$ is equal to the terminal object $\Delta[0] \in$ sSet. In the following we denote $E_\infty := hc\tau_d(\mathcal{E}_\infty)$ which is an operad enriched over simplicial sets. Here

$$hc\tau_d : \mathrm{dSet} \to \mathrm{sOper}$$

is the left adjoint to the homotopy coherent nerve functor (this functor was mentioned in the Introduction, for more details see [CM13b]).The operad E_∞ is then cofibrant, has one colour and the property that each space of operations is contractible. Thus it is indeed an E_∞-operad in the classical terminology. Therefore for each E_∞-algebra X in sSet, the set $\pi_0(X)$ inherits the structure of an abelian monoid. Recall that such an algebra X is called group-like if $\pi_0(X)$ is an abelian group, i.e. there exist inverses for each element.

Now denote by E_∞-spaces the category of E_∞-algebras in simplicial sets. In [Heu11b, Section 3] the following adjoint pair is introduced $St : \mathrm{dSet}_{/\mathcal{E}_\infty} \rightleftarrows E_\infty\text{-spaces} : Un$ where $\mathrm{dSet}_{/\mathcal{E}_\infty}$ denotes the category of dendroidal sets over $\mathcal{E}_\infty$. We do not repeat the definition of St here since we need the formula only for a few particular simple cases and for these cases we give the result explicitly.

Example 6.3.1. • The E_∞-algebra $St(\eta \to \mathcal{E}_\infty)$ is the free E_∞-algebra on one generator, which we denote by $Fr(a)$ where a is the generator.

- An object in $\mathrm{dSet}_{/\mathcal{E}_\infty}$ of the form $p : \Omega[C_2] \to \mathcal{E}_\infty$ encodes a binary operation $- \cdot_p -$ in the operad E_∞. Then $St(p)$ is the E_∞-algebra freely generated by two generators a,b and the square $\Delta[1] \times \Delta[1]$ subject to the relation that $a \cdot_p b \sim (1,1) \in \Delta[1] \times \Delta[1]$. We write this as
$$St(\Omega[C_2] \to \mathcal{E}_\infty) = \frac{Fr(a, b, \Delta[1]^2)}{a \cdot_p b \sim (1,1)}.$$
- The three inclusions $\eta \to \Omega[C_2]$ induce maps $St(\eta \to \mathcal{E}_\infty) \to St(p)$. As usual we let a, b be the leaves of the tree C_2 and c the root. The first two maps are simply given by
$$Fr(a) \to Fr(a, b, \Delta[1]^2)/_\sim \quad a \mapsto a \qquad \text{and} \qquad Fr(b) \to Fr(a, b, \Delta[1]^2)/_\sim \quad b \mapsto b$$
The third map $Fr(c) \to Fr(a, b, \Delta[1]^2)/_\sim$ is given by sending c to $(0,0) \in \Delta[1]^2$. Note that this third map is obviously homotopic to the map sending c to $(1,1) = a \cdot_p b$.

The functor $P(X) := X \times \mathcal{E}_\infty$ induces a further adjoint pair $P : \mathrm{dSet} \rightleftarrows \mathrm{dSet}_{/\mathcal{E}_\infty} : \Gamma$. Composing the two pairs (St, Un) and (P, Γ) we obtain an adjunction

$$St_{\times \mathcal{E}_\infty} : \mathrm{dSet} \rightleftarrows E_\infty\text{-spaces} : Un_\Gamma \tag{6.2}$$

Moreover E_∞-spaces carries a left proper, simplicial model structure where weak equivalences and fibrations are just weak equivalences and fibrations of the underlying space of an E_∞-algebra, see [Spi01, Theorem 4.3. and Proposition 5.3] or [BM03]. For this model structure and the covariant model structure on dendroidal sets the above adjunction (6.2) is in fact a Quillen equivalence as shown by Heuts [Heu11b] [1].

Lemma 6.3.2. *Let X be a fibrant E_∞-space. Then X is group-like if and only if $Un_\Gamma(X) \in$ dSet is fully Kan.*

Proof. The condition that $Un_\Gamma(X)$ is fully Kan is by Theorem 6.2.3 equivalent to the map

$$s^* : \underline{\mathrm{hom}}(\Omega[C_2], Un_\Gamma(X)) \to \underline{\mathrm{hom}}(\eta_a \sqcup \eta_c, Un_\Gamma(X))$$

being a weak equivalence of simplicial sets. By the Quillen equivalence (6.2) and the fact that $\Omega[C_2]$ is cofibrant the space $\underline{\mathrm{hom}}(\Omega[C_2], Un_\Gamma(X))$ is homotopy equivalent to the space $\underline{\mathrm{hom}}(St(\Omega[C_2] \times \mathcal{E}_\infty \to \mathcal{E}_\infty), X)$. We can choose a morphism $p : \Omega[C_2] \to \mathcal{E}_\infty$ (and this choice is essentially unique) because $\Omega[C_2]$ is cofibrant and $\mathcal{E}_\infty \to *$ is a trivial fibration. In the covariant model structure on $\mathrm{dSet}_{/\mathcal{E}_\infty}$ (see [Heu11b, Section 2]) the objects $\Omega[C_2] \times \mathcal{E}_\infty \to \mathcal{E}_\infty$ and $\Omega[C_2] \to \mathcal{E}_\infty$ are cofibrant and equivalent. Cofibrancy is immediate

[1] Note that Heuts in fact uses a slightly different variant where P is a right Quillen functor (instead of left Quillen). But if a right Quillen equivalence happens to be a left Quillen functor as well, then this left Quillen functor is also an equivalence. Thus Heuts' results immediately imply the claimed fact.

and the fact that they are equivalent follows since the forgetful functor to dendroidal sets is a left Quillen equivalence and $\Omega[C_2] \simeq \Omega[C_2] \times \mathcal{E}_\infty$ in dSet. Therefore $St(\Omega[C_2] \times \mathcal{E}_\infty \to \mathcal{E}_\infty)$ is weakly equivalent to $St(\Omega[C_2] \to \mathcal{E}_\infty)$ in E_∞-spaces. Together we have the following weak equivalence of spaces

$$\underline{\mathrm{hom}}(\Omega[C_2], Un_\Gamma(X)) \simeq \underline{\mathrm{hom}}(St(\Omega[C_2] \to \mathcal{E}_\infty), X).$$

The same reasoning yields a weak equivalence $\underline{\mathrm{hom}}(\eta_a \sqcup \eta_c, Un_\Gamma(X)) \simeq \underline{\mathrm{hom}}(St(\eta_a \sqcup \eta_c \to \mathcal{E}_\infty), X)$ such that the diagram

$$\begin{tikzcd} \underline{\mathrm{hom}}(\Omega[C_2], Un_\Gamma(X)) \arrow[r, "s^*"] \arrow[d, "\sim"] & \underline{\mathrm{hom}}(\eta_a \sqcup \eta_c, Un_\Gamma(X)) \arrow[d, "\sim"] \\ \underline{\mathrm{hom}}(St(\Omega[C_2] \to \mathcal{E}_\infty), X) \arrow[r, "s^*"] & \underline{\mathrm{hom}}(St(\eta_a \sqcup \eta_c \to \mathcal{E}_\infty), X) \end{tikzcd} \tag{6.3}$$

commutes.

Finally we use the fact that in the covariant model structure over $\mathcal{E}_\infty$ the leaf inclusion $i : \eta_a \sqcup \eta_b \to \Omega[C_2]$ is a weak equivalence. This implies that there is a further weak equivalence $St(\eta_a \sqcup \eta_b \to \mathcal{E}_\infty) \xrightarrow{\sim} St(\Omega[C_2] \to \mathcal{E}_\infty)$. As remarked above the straightening of $\eta \to \mathcal{E}_\infty$ is equal to $Fr(*)$, the free E_∞-algebra on one generator. Thus $St(\eta_a \sqcup \eta_b \to \mathcal{E}_\infty)$ is the coproduct of $Fr(a)$ and $Fr(b)$ which is isomorphic to $Fr(a, b)$ (here we used a and b instead of $*$ to label the generators). Then the above equivalence reads $Fr(a, b) \xrightarrow{\sim} St(\Omega[C_2] \to \mathcal{E}_\infty)$. The root inclusion $r : \eta_c \to \Omega[C_2]$ induces a further map $r^* : Fr(c) = St(\eta_c \to \mathcal{E}_\infty) \to St(\Omega[C_2] \to \mathcal{E}_\infty)$ and using the explicit description of $St(p)$ given above we see that there is a homotopy commutative diagram

$$\begin{tikzcd} St(\Omega[C_2] \to \mathcal{E}_\infty) & & Fr(c) \arrow[ll, "St(r)"'] \arrow[dl, "f"] \\ & Fr(a,b) \arrow[ul, "St(i)"] & \end{tikzcd}$$

where f is defined as the map sending c to the product $a \cdot_p b$. Thus the horn $s : \eta_a \sqcup \eta_c \to C_2$ fits in a homotopy commutative diagram

$$\begin{tikzcd} St(\Omega[C_2] \to \mathcal{E}_\infty) & & Fr(a,c) \arrow[ll, "St(s)"'] \arrow[dl, "sh"] \\ & Fr(a,b) \arrow[ul, "St(i)"] & \end{tikzcd}$$

with the map sh that sends c to the binary product of a and b and a to itself.

Putting the induced diagram together with diagram (6.3) we obtain a big diagram

$$\begin{tikzcd}
\underline{\hom}(\Omega[C_2], Un_\Gamma(X)) \arrow[r, "s^*"] \arrow[d, "\sim"] & \underline{\hom}(\eta_a \sqcup \eta_c, Un_\Gamma(X)) \arrow[d, "\sim"] \\
\underline{\hom}(St(\Omega[C_2] \to \mathcal{E}_\infty), X) \arrow[r, "s^*"] \arrow[d, "\sim"] & \underline{\hom}(Fr(a,c), X) \\
\underline{\hom}(Fr(a,b), X) \arrow[ur, "sh*"] &
\end{tikzcd} \tag{6.4}$$

in which all the vertical arrows are weak equivalences. This shows that $Un_\Gamma(X)$ is fully Kan if and only if $sh^* : \underline{\hom}(Fr(a,b), X) \to \underline{\hom}(Fr(a,c), X)$ is a weak equivalence. But we clearly have that domain and codomain of this map are given by $X \times X$. Thus the map in question is given by the shear map

$$Sh : X \times X \to X \times X \qquad (x, y) \mapsto (x, x \cdot_p y)$$

where $x \cdot_p y$ is the composition of x and y using the binary operation given by $hc\tau_d(p) : \Omega(C_2) \to E_\infty$

It remains to show that a fibrant E_∞-space X is group-like precisely when the shear map $Sh : X \times X \to X \times X$ is a weak homotopy equivalence. This is well known [Whi95, chapter III.4], but we include it for completeness. Assume first that the shear map is a weak equivalence. Then the induced shear map $\pi_0(X) \times \pi_0(X) \to \pi_0(X) \times \pi_0(X)$ is an isomorphism. This shows that $\pi_0(X)$ is a group, thus X is group-like. Assume conversely that X is group-like and $y \in X$ is a point in X. Then there is an inverse $y' \in X$ together with a path connecting $y' \cdot_p y$ to the point 1. This induces a homotopy inverse for the map $R_y : X \to X$ given by right multiplication with y (for the fixed binary operation). Now the shear map is a map of fibre bundles

$$\begin{tikzcd}
X \times X \arrow[rr, "Sh"] \arrow[dr, "pr_1"'] & & X \times X \arrow[dl, "pr_1"] \\
& X &
\end{tikzcd}$$

Thus the fact that it is over each point $y \in X$ a weak equivalence as shown above already implies that the shear map is a weak equivalence. □

The last lemma shows that fully Kan dendroidal sets correspond to group-like E_∞-spaces. We want to turn this into a statement about model structures. Therefore we need a model structure on E_∞-spaces where the fibrant objects are precisely the group-like E_∞-spaces.

Proposition 6.3.3. *There is a left proper, combinatorial model structure on E_∞-spaces* *where the fibrant objects are precisely the fibrant, group-like E_∞-spaces and which is a left Bousfield localization of the standard model structure on E_∞-spaces. We call it the* group-completion *model structure.*

Proof. Since the model category of E_∞-spaces is left proper, simplicial and combinatorial the existence follows from general existence results provided that we can characterize the property of being group-like as a lifting property against a set of morphisms. It was already carried out how to do this in the last lemma, namely let the set consist of one map from the free E_∞-algebra on two generators to itself given by the shear map (actually there is one shear map for each binary operation in E_∞ but we simply pick one out). □

We have already commented in the Introduction that group-like E_∞-spaces model all connective spectra by the use of a delooping machine. (see also [May74]). More precisely the ∞-category of group-like E_∞-spaces obtained from the group-completion model structure is equivalent as an ∞-category to the ∞-category of connective spectra, see e.g. [Lur11, Remark 5.1.3.17].

Theorem 6.3.4. *The stable model structure on dendroidal sets is Quillen equivalent to the group-completion model structure on E_∞-spaces by the adjunction* (6.2)*. Thus the stable model structure on dendroidal sets is a model for connective spectra in the sense that there is an equivalence of ∞-categories.*

The theorem follows from Lemma 6.3.2 and the following more general statement about left Bousfield localization and Quillen equivalences. Therefore recall from [Bar07, Definition 1.3.] that a combinatorial model category is called *tractable* if it admits a set of generating cofibrations and generating trivial cofibrations with cofibrant domain and codomain. It turns out that it suffices to check this for generating cofibrations [Bar07, Corollary 1.12.]. Thus all model structures on dendroidal sets are clearly tractable.

Lemma 6.3.5. *Let C and D be simplicial model categories with C tractable and a (not necessarily simplicial) Quillen equivalence*

$$L : C \rightleftarrows D : R$$

Moreover let C' and D' be left Bousfield localizations of C and D. Assume R has the property that a fibrant object $d \in D$ is fibrant in D' if and only if $R(d)$ is fibrant in C'.

Then $(L \dashv R)$ is also a Quillen equivalence between C' and D'.

Proof. For simplicity we will refer to the model structures on C and D as the global model structures and to the model structures corresponding to C' and D' as the local model structures. First we have to show that the pair (L, R) induces a Quillen adjunction in the local model structures. We will show that L preserves local cofibrations and trivial cofibrations. Since local and global cofibrations are the same this is true for cofibrations. Thus we need to show it for trivial cofibrations and it follows by standard arguments if we can show it for generating trivial cofibrations. Thus let $i : a \to b$ be generating locally trivial cofibration in C. Now we can assume that a and b are cofibrant since C is tractable. Then the induced morphism $\underline{\hom}(b, c) \to \underline{\hom}(a, c)$ on mapping spaces is a weak equivalence for each locally fibrant object $c \in C$. In particular for $c = R(d)$ with $d \in D$ locally fibrant. Now we use that there are weak equivalences $\underline{\hom}(b, R(d)) \cong \underline{\hom}(Lb, d)$ and $\underline{\hom}(a, R(d)) \cong \underline{\hom}(La, d)$ of simplicial sets which stem from the fact that the pair (L, R) induces an adjunction of ∞-categories. This shows that the induced morphism $\underline{\hom}(Lb, d) \to \underline{\hom}(La, d)$ is a weak equivalence for every locally fibrant objects $d \in D$. This shows that $La \to Lb$ is a local weak equivalence.

It remains to show that (L, R) is a Quillen equivalence in the local model structures. Therefore it suffices to show that the right derived functor

$$R' : Ho(D') \to Ho(C')$$

is an equivalence of categories. Since D' and C' are Bousfield localizations $Ho(C')$ is a full reflective subcategory of $Ho(C)$ and correspondingly for D and D'. Moreover there is a commuting square

$$\begin{tikzcd} Ho(D') \arrow[r, "R'"] \arrow[d, hook] & Ho(C') \arrow[d, hook] \\ Ho(D) \arrow[r, "R"] & Ho(C) \end{tikzcd}$$

Since R is an equivalence it follows that R' is fully faithful. In order to show that R' is essentially surjective pick an object c in $Ho(C')$ represented by a locally fibrant object c of C. Since R is essentially surjective we find an element $d \in D$ which is globally fibrant such that $R(d)$ is equivalent to c in $Ho(C)$. But this implies that $R(d)$ is also locally fibrant (i.e. lies in $Ho(C')$) since this is a property that is invariant under weak equivalence in Bousfield localizations. Therefore we conclude that d is locally fibrant from the assumption on R. This shows that R' is essentially surjective, hence an equivalence of categories. $\square$

The fact that the stable model structure is equivalent to connective spectra has the important consequence that a cofibre sequence in this model structure is also a fibre sequence, which is well-known for connective spectra (note that the converse is not true in connective spectra, but in spectra).

Corollary 6.3.6. *Let $X \to Y \to Z$ be a cofibre sequence of dendroidal sets in any of the considered model structures. Then*

$$i^*X_K \to i^*Y_K \to i^*Z_K$$

is a fibre sequence of simplicial sets. Here $(-)_K$ denotes a fully Kan (fibrant) replacement.

Proof. Since the stable model structure on dendroidal sets is a Bousfield localization of the other model structures we see that a cofibre sequence in any model structure is also a cofibre sequence in the stable model structure. But then it is also a fibre sequence as remarked above. The functor i^* is right Quillen, as shown in Theorem 6.2.6. Thus it sends fibre sequences in dSet to fibre sequences in sSet, which concludes the proof. □

6.4 Proof of Theorem 6.2.3, part I

Recall from Definition 6.2.2 the notion of binary extended corollas. Also recall from [Heu11a] that the weakly saturated class generated by non-root horns of arbitrary trees is called the class of left anodynes. The weakly saturated class generated by inner horn inclusions of arbitrary trees is called the class of inner anodynes. Analogously we set:

Definition 6.4.1. The weakly saturated class generated by non-root horns of all trees and root horns of binary extended corollas is called the class of *binary extended left anodynes*.

Proposition 6.4.2. *A dendroidal set X is locally fibrant if and only if X is a dendroidal Kan complex and it admits fillers for all root horns of binary extended corollas $EC_{n,1}$.*

Proof. We will show in Lemma 6.4.3 that a locally fibrant dendroidal set X admits a filler for the root horn inclusion of $EC_{n,1}$.

Conversely, assume that X is a dendroidal Kan complex and admits lifts against the root horn inclusions of $EC_{n,1}$. Then X clearly admits lifts against all binary extended left anodyne morphisms. In Lemma 6.4.4 we show that the inclusion

$$\Big(\Lambda^b[C_2] \otimes \Omega[L_n]\Big) \;\cup\; \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big) \longrightarrow \Omega[C_2] \otimes \Omega[L_n])$$

is binary extended left anodyne. This implies that X is stably fibrant. □

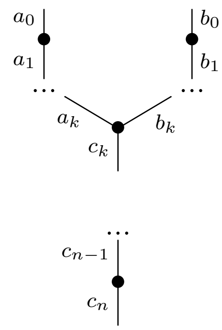
In the rest of the paper we prove some technical lemmas and for this we fix some terminology. We denote the leaves of the corolla C_2 by a and b and its root edge by c. We denote the edges of the linear tree L_n by $0, 1, ..., n$ as indicated in the picture

$L_n =$

We denote the edges in the tensor product $\Omega[C_2] \otimes \Omega[L_n]$ by a_i, b_i, c_i instead of $(a, i), (b, i), (c, i)$ and we let T_k for $k = 0, 1, ..., n$ be the unique shuffle of $\Omega[C_2] \otimes \Omega[L_n]$

that has the edges a_k, b_k and c_k:

$T_k =$

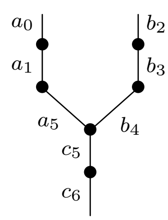


We also use the notation

$$D_i T_j = \begin{cases} \partial_{a_i}\partial_{b_i} T_j, & i < j, \\ \partial_{c_i} T_j, & i > j. \end{cases}$$

We denote the subtrees of a shuffle as sequences of its edges with indices in the ascending order (since there is no danger of ambiguity). For example we denote the following tree

(6.5)

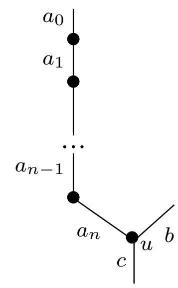


by $(a_0, a_1, a_5, b_2, b_3, b_4, c_5, c_6)$.

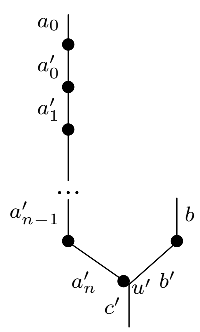
We denote

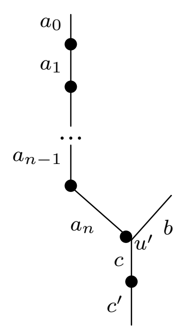
- by π_i the unique dendrex of $\Omega[T_n]$ represented by a subtree with edges b_n, c_n and a_j for all $j \neq i$, for $i = 0, ..., n-1$;
- by π_n the unique dendrex represented by $(a_0, ..., a_{n-1}, b_{n-1}, c_{n-1})$ of $\Omega[T_{n-1}]$;
- by α_n the unique dendrex represented by $(a_0, ..., a_{n-1}, b_{n-1}, b_n, c_n)$ of $\Omega[T_n]$;
- by $\sigma_j\alpha_n$ the degeneracy of α with respect to a_j, for $j = 0, 1, ..., n-1$;
- by β_n the unique dendrex represented by $(a_0, ..., a_{n-1}, b_{n-1}, c_{n-1}, c_n)$ of $\Omega[T_{n-1}]$;
- by γ_n the unique dendrex represented by $(a_0, ..., a_n, b_n, c_n)$ of $\Omega[T_n]$.

We denote the edges of the binary extended corolla as in the following picture:

$EC_{n,1} =$ 

The colours of the tensor product $\Omega[EC_{n,1}] \otimes \Omega[L_1]$ will be denoted by $a_0, ..., a_n, b, c, a'_0, ..., a'_n, b', c'$ and the operations are denoted accordingly. There are $n+1$ shuffles $E_0, E_1, ..., E_n$ where E_i is the unique shuffle that has a_i and a'_i for $i = 0, ..., n$ and one more shuffle F which has c and c'. For example we have the following shuffles

$E_0 =$ 

$F =$ 

Lemma 6.4.3. *A locally fibrant dendroidal set X admits a filler for the root horn inclusion $i\colon \Lambda^u[EC_{n,1}] \to \Omega[EC_{n,1}]$ of the binary extended corolla.*

Proof. Let X be a locally fibrant dendroidal set. By definition X is a dendroidal Kan complex and admits lifts against the maps

$$\Big(\Lambda^b[C_2] \otimes \Omega[L_n]\Big) \,\cup\, \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big) \longrightarrow \Omega[C_2] \otimes \Omega[L_n]$$

for all $n \geq 0$. Note that the inclusion $\Lambda^a[C_2] \to \Omega[C_2]$ is isomorphic to the inclusion $\Lambda^b[C_2] \to \Omega[C_2]$. Hence X also admits lifts against the maps

$$\Big(\Lambda^a[C_2] \otimes \Omega[L_n]\Big) \,\cup\, \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big) \longrightarrow \Omega[C_2] \otimes \Omega[L_n]$$

for all $n \geq 0$.

Consider the following pushout square

$$\begin{tikzcd} \Big(\Lambda^a[C_2] \otimes \Omega[L_n]\Big) \,\cup\, \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big) \arrow[r] \arrow[d] & \Lambda^u[EC_{n,1}] \arrow[d, "k"] \\ \Omega[C_2] \otimes \Omega[L_n] \arrow[r, "l"] & P \end{tikzcd}$$

where the left vertical map is the inclusion and the top horizontal map is the unique map which maps a_i to a_i, b_i to b and c_i to c for $i = 0, 1, ..., n$. It follows that X also admits a lift against the map $u \colon \Lambda^u[EC_{n,1}] \to P$.

We can factor k as a composition $k = pj$ of the inclusion

$$j \colon \Lambda^u EC_{n,1} \cong \Lambda^u EC_{n,1} \otimes \{1\} \to \Big(\Lambda^u EC_{n,1} \otimes \Omega[L_1]\Big) \cup \Big(EC_{n,1} \otimes \{0\}\Big)$$

and the map

$$p \colon \Big(\Lambda^u EC_{n,1} \otimes \Omega[L_1]\Big) \cup \Big(EC_{n,1} \otimes \{0\}\Big) \to P$$

which we now describe explicitly.

The colours of P can be identified with $a_0, ..., a_n, b$ and c. The map p is determined by the image of $EC_{n,1} \otimes \{0\}$ and compatibly chosen images of all the shuffles of $\Lambda^u EC_{n,1} \otimes \Omega[L_1]$, i.e. of $\partial_{a_i} F, i = 0, 1..., n$, $\partial_{a_i} E_j, i = 0, ..., j$ and $\partial_{a'_i} E_j, i = j, ..., n$ for all $j = 0, 1, ..., n$.

Concretely, we send

- $EC_{n,1} \otimes \{0\}$ to $l(\gamma_n)$,
- $\partial_{a_n} F$ to $l(\beta_n)$,
- $\partial_{a'_n} E_j$ to $l(\sigma_j \alpha_n)$ for $j = 0, 1, ..., n-1$,
- and all other shuffles to the corresponding degeneracy of π_i.

One can easily verify that these conditions are compatible in P and hence p is well-defined.

Now we can prove the statement of the lemma. So let us assume a map $f \colon \Lambda^u EC_{n,1} \to X$ is given. We want to prove that there is a lift $\bar{f} \colon EC_{n,1} \to X$ such that $f = \bar{f} i$.

By the above considerations we know that X admits a lift $g : P \to X$ such that $f = gk$ and hence f factors also through $\Big(\Lambda^u EC_{n,1} \otimes \Omega[L_1]\Big) \cup \Big(EC_{n,1} \otimes \{0\}\Big)$ as a composition of j and gp. We get the following commutative diagram

$$\begin{tikzcd} \Lambda^u EC_{n,1} \otimes \{1\} \arrow[r] \arrow[d, "i"'] & \Big(\Lambda^u EC_{n,1} \otimes \Omega[L_1]\Big) \cup \Big(EC_{n,1} \otimes \{0\}\Big) \arrow[r] \arrow[d] & X \\ EC_{n,1} \otimes \{1\} \arrow[r] & EC_{n,1} \otimes \Omega[L_1] & \end{tikzcd}$$

where the top horizontal maps are j and gp respectively and all other maps are obvious inclusions. Since X is a dendroidal Kan complex it admits a lift against left anodynes and the right vertical inclusion $\Big(\Lambda^u EC_{n,1} \otimes \Omega[L_1]\Big) \cup \Big(EC_{n,1} \otimes \{0\}\Big) \to EC_{n,1} \otimes \Omega[L_1]$ is left anodyne because the covariant model structure is simplicial. Hence there is a lift $EC_{n,1} \otimes \Omega[L_1] \to X$ which precomposed with the inclusion $EC_{n,1} \otimes \{1\} \to EC_{n,1} \otimes \Omega[L_1]$ gives the desired lift $\bar{f}$. This finishes the proof. □

Lemma 6.4.4. *The pushout product of the map $s : \Lambda^b[C_2] \to \Omega[C_2]$ with a simplex boundary inclusion*

$$\Big(\Lambda^b[C_2] \otimes \Omega[L_n]\Big) \;\cup\; \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big) \longrightarrow \Omega[C_2] \otimes \Omega[L_n]$$

is a binary extended left anodyne map.

Proof. The case $n = 0$ is just the case of the inclusion $\Lambda^b[C_2] \to \Omega[C_2]$.

Fix $n \geq 1$. We set $A_0 := \Lambda^b[C_2] \otimes \Omega[L_n] \coprod_{\Lambda^b[C_2]\otimes\partial\Omega[L_n]} \Omega[C_2] \otimes \partial\Omega[L_n]$. Note that A_0 is the union of all $\Omega[D_iT_j]$ and of chains $\eta_a \otimes \Omega[L_n]$ and $\eta_c \otimes \Omega[L_n]$. We define dendroidal sets $A_k = A_{k-1} \cup \Omega[T_{k-1}]$ for $k = 1, ..., n+1$. So we have decomposed the map from the lemma into a composition of inclusions

$$A_0 \subset A_1 \subset ... \subset A_{n-1} \subset A_n \subset A_{n+1}.$$

We will show that $A_k \to A_{k+1}$ is inner anodyne for $k = 0, ..., n-1$ and binary extended left anodyne for $k = n$. Note that $A_{n+1} = \Omega[C_2] \otimes \Omega[L_n]$, so the inclusion $A_0 \to \Omega[C_2] \otimes \Omega[L_n]$ is binary extended left anodyne as a composition of such maps.

Case $k = 0$. The faces $\partial_{c_i}\Omega[T_0]$ of T_0 are equal to $\Omega[D_iT_0]$ for all $i > 0$. The outer leaf face of T_0 is equal to $\eta_c \otimes \Omega[L_n]$. The remaining face $\partial_{c_0}\Omega[T_0]$ is in A_1, but not in A_0 so we have a pushout diagram

$$\begin{tikzcd} \Lambda^{c_0}[T_0] \arrow[r] \arrow[d] & A_0 \arrow[d] \\ \Omega[T_0] \arrow[r] & A_1. \end{tikzcd}$$

Since inner anodyne extensions are closed under pushouts it follows that $A_0 \to A_1$ is inner anodyne.

Case $0 < k < n$. We now construct a further filtration

$$A_k = B_0^k \subset B_1^k \subset \cdots \subset B_{k+2}^k = A_{k+1}$$

as follows: informally speaking, we add representables of subtrees of T_k by the number of vertices starting from the minimal ones which are not contained in A_k. More precisely, set $B_0^k := A_k$ and for $l = 1, ..., k+2$ let B_l^k be the union of B_{l-1}^k and all the representables of trees $(a_{j_1}, ..., a_{j_q}, b_{i_1}, ..., b_{i_p}, c_k, ..., c_n)$ with $q+p = l+k$ and $\{j_1, ..., j_q, i_1, ..., i_p\} = \{0, 1, ..., k\}$.

An an example of such a tree for $k=5, l=1, p=q=3$ and $n=6$ is given on the previous page.

For $p+q=k+1$ and the tree $U=(a_{j_1},...,a_{j_q},b_{i_1},...,b_{i_p},c_k,...,c_n)$ we have an inclusion $\Lambda^{c_k}[U]\subset A_0=B_0^k$ because $\partial_{c_i}\Omega[U]\subset\Omega[D_iT_k]$ for $i>k$, $\partial_{a_j}\Omega[U]\subset\Omega[D_jT_k]$ for $j\in\{j_1,...,j_q\}$ and $\partial_{b_i}\Omega[U]\subset\Omega[D_iT_k]$ for $i\in\{i_1,...,i_p\}$. Also note that $\partial_{c_k}\Omega[U]$ is not contained in A_0.

For $p+q=k+l, l\geq 2$ and the tree $U=(a_{j_1},...,a_{j_q},b_{i_1},...,b_{i_p},c_k,...,c_n)$ we have an inclusion $\Lambda^{c_k}[U]\subset B_{l-1}^k$. Indeed, for $j\in\{j_1,...,j_q\}$, $\partial_{a_j}\Omega[U]\subset B_{l-1}^k$ by definition if $j\in\{i_1,...,i_p\}$ and $\partial_{a_j}\Omega[U]\subset A_{k-1}\subset B_{l-1}^k$ if $j\notin\{i_1,...,i_p\}$. Similarly, $\partial_{b_i}\Omega[U]\subset B_{l-1}^k$ for $i\in\{i_1,...,i_p\}$ and $\partial_{c_i}\Omega[U]\subset\Omega[D_iT_k]\subset A_0$ for $i>k$. The remaining face $\partial_{c_k}\Omega[U]$ is not contained B_{l-1}^k.

We conclude that the map $B_{l-1}^k\to B_l^k$ is inner anodyne for $l=1,...,k+2$ because it is the pushout of the inner anodyne map

$$\coprod_{q+p=k+l}\Lambda^{c_k}[U]\to\coprod_{q+p=k+l}\Omega[U]$$

where the coproduct is taken over all subtrees $U=(a_{j_1},a_{j_2},...,a_{j_q},b_{i_1},...,b_{i_p},c_k,...,c_n)$ of T_k such that $q+p=k+l$ and $\{j_1,...,j_q,i_1,...,i_p\}=\{0,1,...,k\}$.

Case $k=n$**.** Note that faces of the shuffle T_n are

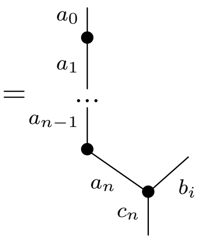
- $\partial_{b_i}T_n=(a_0,...,a_n,b_0,...,\widehat{b_i},...,b_n,c_n)$, $i=0,...,n$;
- $\partial_{a_j}T_n=(a_0,...,\widehat{a_j},...,a_n,b_0,...,b_n,c_n)$, $j=0,...,n$.

Our strategy goes as follows. First, we form the union of A_{n-1} with all $\partial_{b_i}\Omega[T_n], i=0,...n-1$. Second, we consider the union with all proper subsets of $\partial_{b_n}\Omega[T_n]$ that contain edges a_0 and a_n. Third, we consider the union with $\partial_{a_j}\Omega[T_n], j=1,...n$ and then with $\partial_{a_0}\Omega[T_n]$. In the last step we use the horn inclusion $\Lambda^{b_n}[T_n]\subset\Omega[T_n]$. Thus we start with a filtration

$$A_n=P_0\subset\cdots\subset P_{p-1}\subset P_p\subset\cdots\subset P_n,$$

where P_p is the union of P_{p-1} with the representables of the trees of the form $(a_0,...,a_n, b_{i_1},...,b_{i_p},c_n)$ for $p=1,...,n-1$. Also, we define P_n as the union of P_{n-1} with $\partial_{b_i}\Omega[T_n]$ for all $i=1,2,...,n-1$ (but not for $i=n$). Let us show that the maps $P_{p-1}\to P_p$ are left anodyne for $p=1,2,...,n$.

- **Case** $p=1$. For $i\in\{0,1,...,n\}$ and $V_i=(a_0,...,a_n,b_i,c_n)$ all the faces of $\Omega[V_i]$, except $\partial_{a_i}\Omega[V_i]$, are in $P_0=A_n$. The map $P_0\to P_1$ is left anodyne as a pushout of the map $\coprod_{i=0}^{n} \Lambda^{a_i}[V_i] \to \coprod_{i=0}^{n} \Omega[V_i]$.

$V_i =$ 

- **Case** $p \leq n-1$. We give a further filtration

$$P_{p-1} = Q_0^p \subset Q_1^p \subset \cdots \subset Q_m^p \subset \cdots \subset Q_p^p = P_p$$

Let Q_m^p be the union of Q_{m-1}^p with $\Omega[U]$ for all the trees of the form

$$U = (a_{j_1}, ..., a_{j_q}, b_{i_1}, ..., b_{i_p}, c_n),\ q + p = n + m$$

such that there is a subset $I \subseteq \{i_1, ..., i_{p-1}\}$ with $\{j_1, ..., j_q\} = \{0, 1, ..., n\} \setminus I$. Note that $i_p \in \{j_1, ..., j_q\}$. We show that the inclusions $Q_{m-1}^p \to Q_m^p$ are left anodyne for all $m = 1, 2, ..., p-1$. For a fixed m and such a tree U the faces of $\Omega[U]$ are all in Q_{m-1}^p except $\partial_{a_{i_p}}\Omega[U]$. More precisely, the faces $\partial_{b_i}\Omega[U]$ are all in P_{p-1}, the faces $\partial_{a_j}\Omega[U]$ are in A_0 if $j \notin \{i_1, ..., i_p\}$ and in Q_{m-1}^p by definition if $j \in \{i_1, ..., i_p\}$.

We conclude that $Q_{m-1}^p \to Q_m^p$ is left anodyne as a pushout of the left anodyne map $\coprod \Lambda^{a_{i_p}}[U] \to \coprod \Omega[U]$, where the coproduct is taken over trees U described above. We have $P_p = Q_p^p$, so $P_{p-1} \to P_p$ is also left anodyne.

- **Case** $p = n$. Here we do a slight modification of the previous argument. Let $Q_0^n := P_{n-1}$ and for $m = 1, ..., n-1$ let Q_m^n be the union of Q_{m-1}^n with $\Omega[U_i]$ for the trees of the form

$$U_i = (a_{i_1}, ..., a_{i_m}, a_n, b_0, ..., \hat{b_i}, ..., b_n, c_n), i \neq n$$

or of the form

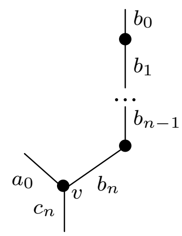
$$U_n = (a_0, a_{i_1}, ..., a_{i_{m-1}}, a_n, b_0, ..., b_{n-1}, c_n).$$

Let Q_n^n be the union of Q_{n-1}^n with $\partial_{b_i}\Omega[T_n]$ for all $i = 1, 2, ..., n-1$ (but not for $i = n$).

Similar argument (using horns $\Lambda^{a_n}[U_i], i \neq n$ and $\Lambda^{a_0}[U_n]$) shows that maps $Q_{m-1}^n \to Q_m^n$ are left anodyne for all $m = 1, ..., n$. Since $P_n = Q_n^n$ we have proven that $P_{n-1} \to P_n$ is left anodyne and hence $A_n \to P_n$ is left anodyne.

Next, we add $\partial_{a_i}\Omega[T_n]$ for $i = 1, 2, ..., n$ to the union. Let us denote the only binary vertex of the tree $W = (a_0, b_0, ..., b_n, c_n)$ by v. Let $P_{n+1} = P_n \cup \Omega[W]$. Then the map $P_n \to P_{n+1}$ is *binary extended left anodyne* because it is a pushout of the map $\Lambda^v[W] \to \Omega[W]$.

$W =$



For each $q = 2, ..., n$ we define P_{n+q} as the union of P_{n+q-1} and the representables of the trees of the form $Z_q = (a_0, a_{i_1}, ..., a_{i_q}, b_0, ..., b_n, c_n)$. The inclusion $P_{n+q-1} \to P_{n+q}$ is left anodyne as the pushout of $\coprod \Lambda^{a_0}[Z_q] \to \coprod \Omega[Z_q]$.

The dendroidal set P_{2n} contains $\partial_{a_i}\Omega[T_n], i = 1, ..., n$. Furthermore, all faces of $\partial_{a_0}\Omega[T_n]$ except $\partial_{b_n}\partial_{a_0}\Omega[T_n]$ are in P_{2n}. Let $P_{2n+1} = P_{2n} \cup \partial_{a_0}\Omega[T_n]$. Then $P_{2n} \to P_{2n+1}$ is inner anodyne as the pushout of $\Lambda^{b_n}\partial_{a_0}[T_n] \to \partial_{a_0}\Omega[T_n]$. From this we conclude that $A_n \to P_{2n+1}$ is binary extended left anodyne. All the faces of $\Omega[T_n]$ except $\partial_{b_n}\Omega[T_n]$ are in P_{2n+1}, so $P_{2n+1} \to A_{n+1}$ is left anodyne as the pushout of the map $\Lambda^{b_n}[T_n] \to \Omega[T_n]$. Hence $A_n \to A_{n+1}$ is binary extended left anodyne, which finishes the proof. $\square$

6.5 Proof of Theorem 6.2.3, part II

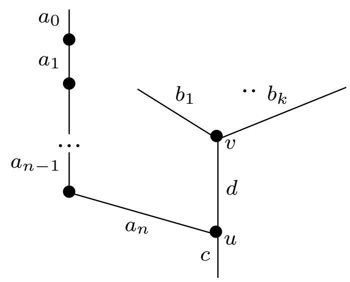
In this section we compare lifts against binary extended corollas and all extended corollas.

Proposition 6.5.1. *Let $X \in dSet$ be a dendroidal Kan complex. Then X admits fillers for all root horns of binary extended corollas $EC_{n,1}$ if and only if X admits fillers for all root horns of arbitrary extended corollas $EC_{n,k}$.*

Proof. One direction is a special case and thus trivial. Hence assume X admits fillers for all root horns of extended corollas $EC_{n,1}$. Then X admits lifts against all binary extended left anodynes (see Definition 6.4.1). We need to show that X admits lifts against the root horn inclusion $\Lambda^u[EC_{n,k}] \to \Omega[EC_{n,k}]$. By Lemma 6.5.2 we find a tree T and a morphism $\Omega[EC_{n,k}] \to \Omega[T]$ such that the composition $\Lambda^u[EC_{n,k}] \to \Omega[T]$ is binary extended left anodyne. Thus given a morphism $\Lambda^u[EC_{n,k}] \to X$ we can find a filler $\Omega[T] \to X$. But the composition $\Omega[EC_{n,k}] \to \Omega[T] \to X$ is then the desired lift. $\square$

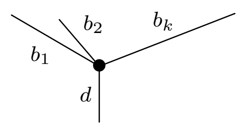
Lemma 6.5.2. *Consider the inclusion of the root horn of the extended corolla $\Lambda^u[EC_{n,k}] \to \Omega[EC_{n,k}]$. There is a tree T and a morphism $\Omega[EC_{n,k}] \to \Omega[T]$ such that the composition $\Lambda^u[EC_{n,k}] \to \Omega[T]$ is a binary extended left anodyne map.*

Proof. We use the labels for edges of the extended corolla $EC_{n,k}$ as given in the definition 6.2.2 and in addition we denote its root vertex by u. Now consider the tree T

$T =$ 

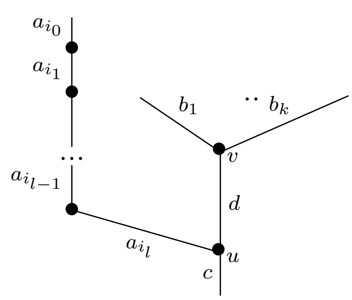
There is an obvious morphism $\Omega[EC_{n,k}] \to \Omega[T]$. We will show that the composition $\Lambda^u[EC_{n,k}] \to \Omega[T]$ is binary extended left anodyne.

We set $E_0 := \Lambda^u[EC_{n,k}]$. Let C_k be a corolla with root d and leaves $b_1, ..., b_k$

$C_k =$ 

Set $E_1 := E_0 \cup \Omega[C_k]$ which is a subobject of $\Omega[T]$. The map $E_0 \to E_1$ is a pushout of the map $\coprod_{i=1}^{k} \eta_{b_i} \to \Omega[C_k]$, so it is left anodyne by definition.

As a next step consider subtrees of T which are of the form

$T_{i_0,...,i_l} =$ 

for $\{i_0, ...i_l\} \subset \{0, 1, ..., n\}$ and $l \leq n-1$. We define dendroidal sets E_{l+2} as the union of E_{l+1} and all representables $\Omega[T_{i_0,...,i_l}]$ for $\{i_0, ...i_l\} \subset \{0, 1, ..., n\}$ and $0 \leq l \leq n-1$. Thus we get a filtration

$$\Lambda^u[EC_{n,k}] = E_0 \subset E_1 \subset E_2 \subset ... \subset E_{n+1} \subset \Omega[T] \tag{6.6}$$

For a fixed $l \leq n-1$ and a subset $\{i_0, ..., i_l\}$ the inner face $\partial_d \Omega[T_{i_0,...,i_l}]$ is contained in E_0

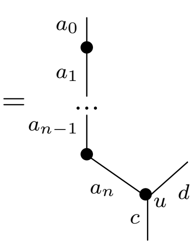
and the faces $\partial_{a_j}\Omega[T_{i_0,\ldots,i_l}]$ are contained in E_{l+1} for every $j \in \{i_0, \ldots, i_n\}$ (and for $l = 0$ the face $\partial_u\Omega[T_{i_0}]$ is in E_1).

Since $\partial_v\Omega[T_{i_0,\ldots,i_l}]$ is not in E_{l+1} we have the following pushout diagram

$$\begin{tikzcd}
\coprod \Lambda^v[T_{i_0,\ldots,i_l}] \arrow[r] \arrow[d] & E_{l+1} \arrow[d] \\
\coprod \Omega[T_{i_0,\ldots,i_l}] \arrow[r] & E_{l+2}
\end{tikzcd}$$

where the coproduct varies over all possible $(i_0, \ldots, i_l)$. This shows that $E_{l+1} \to E_{l+2}$ is left anodyne. From this we conclude that all maps in the above filtration (6.6) except for the last inclusion are left anodyne and therefore also the map $E_0 \to E_{n+1}$ is left anodyne.

We proceed by observing that for the tree

$V =$ 

all faces of $\Omega[V]$ are in E_{n+1} except $\partial_u\Omega[V]$. Notice that $E_{n+1} \cup \Omega[V] = \Lambda^d[T]$. The map $E_{n+1} \to \Lambda^d[T]$ is the pushout of the binary extended left anodyne map $\Lambda^u[V] \to \Omega[V]$, so it is binary extended left anodyne. Finally, since $\Lambda^d\Omega[T] \to \Omega[T]$ is inner anodyne, we conclude that $E_0 \to \Omega[T]$ is binary extended left anodyne. $\square$

6.6 Proof of Theorem 6.2.3, part III

Similarly to Definition 6.4.1 of binary extended left anodynes we define two more classes.

Definition 6.6.1. The weakly saturated class generated by non-root horns of all trees and root horns of extended corollas is called the class of *extended left anodynes*. The weakly saturated class generated by all horn inclusions of trees is called the class of *outer anodynes*.

It would be more logical to call outer anodynes simply anodynes since it also includes the inner anodynes. But in order to distinguish it more clearly we call it outer anodynes here. By definition we have inclusions

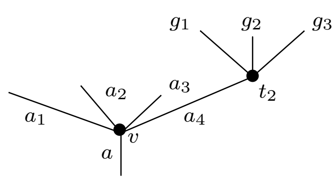
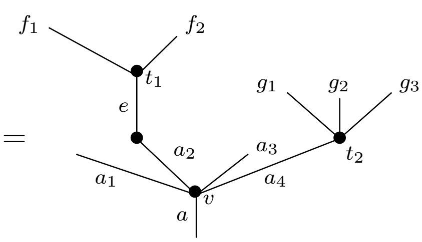
$$\begin{aligned}\{\text{inner anodynes}\} \subset \{\text{left anodynes}\} &\subset \{\text{binary ext. left anodynes}\} \\ &\subset \{\text{ext. left anodynes}\} \subset \{\text{outer anodynes}\}\end{aligned}$$

All of these inclusions are proper, except for the last. In the following proposition we show that the last inclusion is actually an equality.

Proposition 6.6.2. *The class of extended left anodynes and the class of outer anodynes coincide. In particular, a dendroidal set X admits lifts against all non-root horns and root horns of extended corollas if and only if it is fully Kan.*

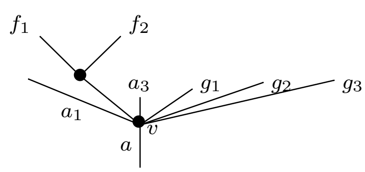
Proof. By the above inclusion of saturated classes it suffices to show that every root horn inclusion is contained in the class of extended left anodynes. A root horn for a tree exists only if this tree is obtained by grafting a smaller tree on a corolla. We give the proof of this technical statement in Lemma 6.6.6. □

Before we can prove the crucial lemma we need to introduce some terminology. Recall from [MW09] that *a top face map* is an outer face map with respect to a top vertex and *an initial segment* of a tree is a subtree obtained by composition of top face maps. For example, the tree V is an initial segment of the tree T in the following picture.

$V =$  $T =$ 

Definition 6.6.3. A subtree which is a composition of an initial segment followed by exactly k inner face maps is called an *initial subtree of codimension k*.

By definition, every initial segment is an initial subtree of codimension zero. An example of an initial subtree of codimension 2 of the above tree T is



Lemma 6.6.4 (Codimension argument). *Let T be a tree and v a vertex of T. Let V be the maximal initial segment of T for which the input edges $d_1, ..., d_p$ of v are leaves. Let X_T be a subobject of $\Omega[T]$ defined in the following way: If V has at least two vertices, then X_T is the union of the following dendroidal sets*

- *the representable $\Omega[V]$,*
- *the inner faces $\partial_e\Omega[T]$ for all inner edges e of V,*

- *the outer faces $\partial_u\Omega[T]$ for vertices u of V, $u \neq v$.*

If V has exactly one vertex, then X_T is the union of the following dendroidal sets

- *the representable $\Omega[V]$,*
- *the representable of the maximal subtree of T having d_i as root for $i = 1, .., p$.*

Then the inclusion $X_T \to \Omega[T]$ is inner anodyne.

Proof. Let $|V|$ and $|T|$ denote the number of vertices of V and T, respectively. Let $N = |T| - |V| + 1$. We say that an initial subtree S of codimension k of T containing V is a (n,k)-*subtree* if it has exactly $|V| - 1 + n$. Note that V is a $(1,0)$-subtree.

Denote $X_{1,0} := X_T$. The strategy is to form an inner anodyne filtration consisting of dendroidal sets $X_{(n,k)}$ by considering unions of $X_{1,0}$ with some (n,k)-subtrees of T for all $n, 1 \leq n \leq N$ and $k, 0 \leq k \leq N - n$.

Before constructing this filtration, we form a set $\mathcal{F}_{n,k}$ of *chosen* (n,k)-subtrees which we do not include in $X_{n,k}$ for each pair (n,k). We start with the tree T which is a $(N,0)$-subtree and we choose $\partial_{d_i}T$ for i being minimal such that d_i is an inner edge of T. The set $\mathcal{F}_{N-1,1}$ has only one element $\partial_{d_i}T$ and $\mathcal{F}_{N-1,0}$ is empty. We proceed inductively by decreasing n from N to 1. Each $(n+1, k-1)$-subtree S which is not in $\mathcal{F}_{n+1,k-1}$ contains at least one inner edge d_j, $j \in \{1, ..., p\}$ and we choose $\partial_{d_i}S$ for minimal such i and put this (n,k)-subtree $\partial_{d_i}S$ in $\mathcal{F}_{n,k}$.

Note that for $n = 2, k \geq 1$ such a subtree S has exactly $|V| + 1$ vertices and only one inner face $\partial_{d_i}S$ and that face belongs to $\mathcal{F}_{1,k}$. Hence $X_{1,0} = X_T$. We define $X_1 = X_T$ and for $2 \leq n \leq N$ we inductively define $X_{n,0}$ as the union of X_{n-1} and the representables of all $(n,0)$-subtrees, $X_{n,k}$ as the union of $X_{n,k-1}$ and the representables of all (n,k)-subtrees that are not in $\mathcal{F}_{n,k}$ and dendroidal sets X_n as the union $\displaystyle\bigcup_{k=0}^{N-n} X_{n,k}$.

The inclusions $X_{n-1} \to X_{n,0}$ are all inner anodyne because each of them is a pushout of the coproduct of the inner horn inclusions. More precisely, each $(n,0)$-subtree S has faces which are in X by definition, outer faces that are $(n-1,0)$-subtrees and hence are all in X_{n-1}, inner faces which are $(n-1,1)$-subtrees and by definition exactly one of them was chosen to be in $F_{n-1,1}$, so is not in X_{n-1}. Denote this inner face by $\partial_s S$. We have the pushout diagram (where the coproduct is taken over all $(n,0)$-subtrees)

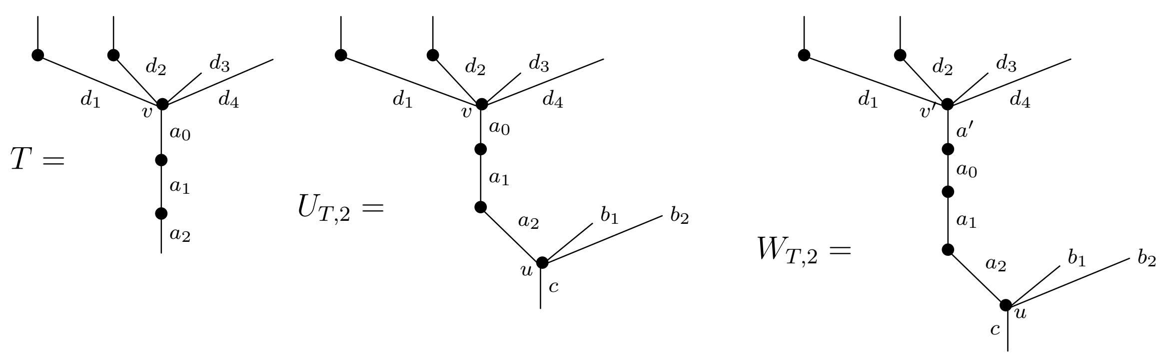
$$\begin{tikzcd}
\coprod \Lambda^s[S] \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod \Omega[S] \arrow[r] & X_{n,0}
\end{tikzcd}$$

Note that the union of representables of $(n+1, k-1)$-subtrees and $X_{n+1,k-1}$ will also contain the representables of elements of $\mathcal{F}_{n,k}$ (since the elements of $\mathcal{F}_{n,k}$ will be faces of the $(n+1, k-1)$-subtrees). So $X_{n+1,k-1}$ will contain representables of all (n, k)-subtrees. The inclusions $X_{n+1,k-1} \to X_{n+1,k}$ are similarly shown to be inner anodyne. Faces of an $(n+1, k)$-subtree are in X or (n, k)-subtrees (and hence all in $X_{n+1,k-1}$ by the previous sentence) or $(n, k+1)$-subtrees (and hence all but one in $X_{n,k+1} \subset X_n \subset X_{n+1,k-1}$ by construction). We again have a horn inclusion with respect to the excluded face, and $X_{n+1,k-1} \to X_{n+1,k}$ is the pushout of the coproduct of these horn inclusions. Finally, we have shown that the inclusion $X_T = X_1 \subset X_2 \subset ... \subset X_N = \Omega[T]$ is left anodyne. $\square$

Definition 6.6.5. For a non-linear tree T the maximal subtree having non-unary root is unique and we call it *the tree top* of T. For a linear tree we say that its tree top is given by its unique leaf (i.e. it is isomorphic to η). The maximal initial segment of T which is a linear tree is also unique and we call it *the stem* of T. Note that T is obtained by grafting the tree top of T to the stem of T and conversely the tree top is obtained from T by chopping off the stem.

For a fixed tree T with the root r we define the tree $U_{T,q}$ obtained by grafting T to the $(q+1)$-corolla with leaves $r, b_1, ..., b_q$, the root c and the root vertex u. Let T' be the tree that has one edge more than T such that this edge, called a', is the leaf of the stem of T' (and root of the tree top of T'). Let $W = W_{T,q}$ be the tree obtained by grafting T' to the $(q+1)$-corolla with leaves $r, b_1, ..., b_q$, the root c and the root vertex u.

We will usually denote by v the root vertex of the tree top of T and the input edges of v by $d_1, ..., d_p$. We will denote by v' the vertex in W having the output a'. The edges of the stem of T will be denoted $a_0, ..., a_l$ with a_i and a_{i+1} being the input and the output of the same vertex for all $i = 0, ..., l-1$ (so a_l is the root). Here is one example.



For a subset $J \subset \{0, 1, ..., l\}$ we denote by

- U^0_J the unique subtree of W containing the edges $d_1, ..., d_p, a', b_1, ..., b_q, c$ and $a_j, j \in J$.
- U'_J the maximal subtree of W not containing the edges $a_j, j \in \{0, 1, ..., l\} \setminus J$.

- T^0_J and T'_J the root face of U^0_J and U'_J, respectively.

Note that T'_J contains the whole tree top of T, while T^0_J only the non-unary root vertex of the tree top of T.

Lemma 6.6.6. *Let U be a tree whose root vertex u is attached to exactly one inner edge. The inclusion $\Lambda^u[U] \to \Omega[U]$ is extended left anodyne.*

Proof. There is a tree T and a natural number $q \geq 0$ such that $U = U_{T,q}$. Let N be the number of vertices of the tree top S of T and let l be the number of vertices of the stem of T. We show the claim by induction on N.

If $N = 0$ tree T is linear and the claim holds by definition of extended left anodynes.

Fix a tree top S with N vertices, $N \geq 1$, and assume that the claim holds for every tree such that the corresponding tree top has less than N vertices. We will prove that for fixed S and for every l, the inclusion $\Lambda^u[U] \to \Omega[U]$ is extended left anodyne. Since $\Lambda^u[U] \to \Omega[U]$ is a retract of $\Lambda^u[U] \to \Omega[W]$, it is enough to show that $\Lambda^u[U] \to \Omega[W]$ is extended left anodyne. We divide the proof in four parts.

Step 1. We show that the inclusion $\Lambda^u[U] \to \cup_{j=0}^{l} \Omega[\partial_{a_j} W]$ is left anodyne.

We denote $B_0 := \Lambda^u[U]$. Inductively, for all $1 \leq k \leq l+1$, we define

$$A'_{k-1} := B_{k-1} \cup \bigcup_{|J|=k-1} \Omega[T^0_J], \quad A_k := A'_{k-1} \cup \bigcup_{|J|=k-1} \Omega[T'_J],$$
$$B'_{k-1} := A_k \cup \bigcup_{|J|=k-1} \Omega[U^0_J], \quad B_k := B'_{k-1} \cup \bigcup_{|J|=k-1} \Omega[U'_J].$$

Since $A'_0 = A_0 \cup \Omega[T^0_\emptyset]$ and $T^0_\emptyset$ is the p-corolla with inputs $d_1, ..., d_p$ and root a', the inclusion $A_0 \to A'_0$ is the pushout of $\eta_{d_1} \cup ... \cup \eta_{d_p} \to \Omega[T^0_\emptyset]$ and hence left anodyne.
Let k be such that $1 \leq k \leq l$. The inclusion $A_k \to B'_{k-1}$ is left anodyne because it is the pushout of the coproduct of leaf horn inclusions $\coprod_{|J|=k} \Lambda^{v'}[U^0_J] \to \coprod_{|J|=k} \Omega[U^0_J]$. The inclusion $B_k \to A'_k$ is left anodyne because it is the pushout of the coproduct of leaf horn inclusions $\coprod_{|J|=k} \Lambda^{v'}[T^0_J] \to \coprod_{|J|=k} \Omega[T^0_J]$.
For all trees $T'_J, |J| = k-1$, and vertex v' the codimension argument gives an inner anodyne $X_{T'_J} \to \Omega[T'_J]$. Since $X_{T'_J}$ is exactly the intersection of of A'_{k-1} and $\Omega[T'_J]$, the inclusion $A'_{k-1} \to A_k$ is inner anodyne as the pushout of the coproduct $\coprod_{|J|=k-1} X_{T'_J} \to \coprod_{|J|=k-1} \Omega[T'_J]$. Similarly, we use the codimension argument to show that $X_{U'_J} \to \Omega[U'_J]$ is inner anodyne. As $X_{U'_J}$ is the intersection of $\Omega[U'_J]$ and B'_{k-1} the inclusion $B'_{k-1} \to B_k$ is inner anodyne as the pushout of the coproduct $\coprod_{|J|=k-1} X_{U'_J} \to \coprod_{|J|=k-1} \Omega[U'_J]$. Note that $B_{l+1} = \bigcup_{j=0}^{l} \Omega[\partial_{a_j} W]$, so this completes the first step.

Step 2. Let V_0 be the unique initial segment of W for which a' is a leaf. We define $D_0 := \Omega[V_0] \cup \bigcup_{j=0}^{l} \Omega[\partial_{a_j} W]$. The map $\bigcup_{j=0}^{l} \Omega[\partial_{a_j} W] \to D_0$ is *extended left anodyne* because it is the pushout of outer root inclusion of $\Omega[V_0]$.

Step 3. For $1 \leq n \leq N-1$, we define the set $\mathcal{V}_n$ of all the initial segments of W with exactly $n+l+2$ vertices. Furthermore we inductively define dendroidal sets $D_n = D_{n-1} \cup \bigcup_{V \in \mathcal{V}_n} \Omega[V]$. Note that all such subtrees $V \in \mathcal{V}_n$ contain $a', a_0, ..., a_l, b_1, ..., b_q, c$ since they are initial and they have exactly n vertices more than V_0. The outer root horn inclusion for $V \in \mathcal{V}_n$ is extended left anodyne by the inductive hypothesis. The intersection of $\Omega[V]$ and D_{n-1} is the horn $\Lambda^u[V]$ because the faces $\partial_{a_j}\Omega[V], j = 0, 1, ..., l$ are in $B_{l+1} \subset D_0$ by the previous arguments, the face $\partial_{a'}V$ is in A_0, and other inner and outer leaf faces are in D_{n-1} by definition. We conclude that the inclusion $D_{n-1} \to D_n$ is also extended left anodyne because it is the pushout of $\bigcup_{V \in \mathcal{V}_n} \Lambda^u(V) \to \bigcup_{V \in \mathcal{V}_n} \Omega[V]$.

Note that D_{N-1} contains all the faces of W except the outer root face T' and $\partial_{a'}W = U$. We have so far proven that $\Lambda^u[U] \to D_{N-1}$ is extended left anodyne.

Step 4. We show that $D_{N-1} \to \Omega[W]$ is inner anodyne. The intersection of $\Omega[T']$ and D_{n-1} is the inner horn $\Lambda^{a'}[T']$ because the inner face $\partial_{a'}T' = T$ is not in D_{N-1} and

- $\partial_{a_j}T'$ is already in A_l;
- $\partial_e T'$ for inner edges e of the tree top S are in D_{N-1} because D_{N-1} contains $\partial_e W$;
- $\partial_t T'$ for top vertices t of the tree top S are in D_{N-1} because D_{N-1} contains $\partial_t W$.

So the map $D_{N-1} \to D_{N-1} \cup \Omega[T'] = \Lambda^{a'}[W]$ is inner anodyne because it is a pushout of an inner horn inclusion. Finally, $\Lambda^{a'}[W] \to \Omega[W]$ is inner anodyne and we have shown that the inclusion $A_0 \to \Omega[W]$ is extended left anodyne. $\square$

Chapter 7

Homology of dendroidal sets

Dendroidal sets generalize simplicial sets, so it is natural to ask which of the results concerning the Kan-Quillen model structure on simplicial sets have an analog in the context of the stable model structure on dendroidal sets. One such problem is generalizing the simplicial homology and the Dold-Kan correspondence between simplicial abelian groups and chain complexes. One approach to this question was addressed in [GLW11], but only for planar dendroidal sets.

In this chapter we construct a functor from dendroidal sets to chain complexes. Using this we can define homology groups associated to dendroidal sets. We show that this functor from dendroidal sets to chain complexes is a left Quillen functor, so homology is a new stable homotopy invariant of dendroidal sets. In the previous chapter we have shown that dendroidal sets model connective spectra, so it is natural to compare this new invariant to the standard homology of the connective spectrum corresponding to a dendroidal set. Our main result is that these two invariants coincide. An immediate consequence is that the stable weak equivalences between cofibrant dendroidal sets can be detected as the morphisms that induce isomorphisms in homology. All the results of this chapter are joint work with Thomas Nikolaus and will appear in [BN].

7.1 The sign conventions

In this section we describe a labelling of vertices of a planar tree and two sign conventions. These labels and signs will be used in the definition of the homology of a dendroidal set. One of these two sign conventions is taken from [GLW11, Section 4.5].

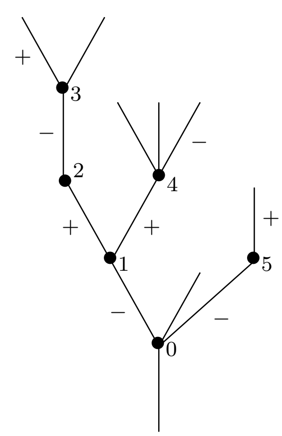
Recall that planar trees are trees with extra structure - the set of inputs of each vertex carries a total order. We depict planar trees by drawing the inputs from left to right in the increasing order. There is a dendroidal set $P : \Omega^{op} \to \mathrm{Set}$ such that $P(T)$ is the set of planar structures of the tree T. We call it also the presheaf of planar structures. Note that $P = A_\infty = N_d(Ass)$ is the dendroidal nerve of the operad for associative algebras and it is a normal dendroidal set.

Let (T, p) be a planar tree, i.e. T is a non-planar tree and p a planar structure on T. For every face map $f\colon S \to T$ there is a planar structure on S given as $P(f)(p)$, so that f is a map of planar trees with these planar structures.

We define a labelling of the vertices of a planar tree with n vertices with numbers 0,1,..., $n-1$. We label the vertex above the root edge with 0 and then proceed recursively. Whenever we label a vertex we continue labelling the vertices of its leftmost branch (until we reach a top vertex), then we label the vertices of the second branch from the left and so on until we label all the vertices. An example of such a labelling is given below.

Definition 7.1.1. We assign a sign $\mathrm{sgn}_p(\partial_a) \in \{-1, +1\}$ to each elementary face map $\partial_a : T/a \to T$ using the labelling of the planar tree (T, p) as follows: If T is a corolla, we assign -1 to the inclusion of the root edge and $+1$ to the inclusion of a leaf. If T has at least two vertices, we assign $+1$ to the root face, which is the face obtained by chopping off the root vertex (and which only exists if the root vertex has only one inner edge assigned to it). We assign $(-1)^k$ to the the face $\partial_e T \to T$ if e is an inner edge which is attached to vertices labelled with k and $k-1$ and we assign $(-1)^{k+1}$ to ∂_v if v is a top vertex labelled with k.

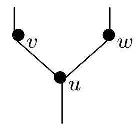
Here is an example of such a labelling of the vertices. The signs associated to the inner faces are shown next to the corresponding inner edge and the signs associated to the top faces are shown next to one of the leaves.



Next we define a sign convention that will be used when we consider different planar structures.

Definition 7.1.2. Let T be a tree and let $p', p \in P_T$ be two planar structures. Each of these planar structures gives a labelling of vertices of T as described above. Thus there is a permutation on the set of labels $\{1, ..., n-1\}$ which sends the labelling induced by p' to the labelling induced by p (we omit the label 0 since the root vertex must be fixed). We define $\mathrm{sgn}(p', p) \in \{-1, +1\}$ to be the sign of that permutation.

Example 7.1.3. Here is a simple example. Let (T, p) be the following planar tree



The same tree T has one more planar structure p'. The vertices v and w, respectively, have labels 1 and 2 in p, but labels 2 and 1 in p', so $\mathrm{sgn}(p', p) = -1$.

Let $\partial_e : T/e \to T$ be an elementary face map. If $p \in P_T$ is a planar structure on T, then we denote $p_e = P(\partial_e)(p) \in P_{T/e}$.

Lemma 7.1.4. *Let $\partial_e : T/e \to T$ be an elementary face map. For any two planar structures $p', p \in P_T$ we have*

$$\mathrm{sgn}(p', p) \cdot \mathrm{sgn}_p(\partial_e) = \mathrm{sgn}(p'_e, p_e) \cdot \mathrm{sgn}_{p'}(\partial_e).$$

Proof. If T is a corolla, the statement is true since all the terms are $+1$. Let $|T| = n+1$ be the number of vertices T and $\tau \in \Sigma_n$ the assigned permutation for the planar structures p' and p. Suppose first that e is an inner edge of T. Let k be the label given to the vertex above e and $\tau(k) = l$. Then $\mathrm{sgn}_p(\partial_e) = (-1)^k$ and $\mathrm{sgn}_{p'}(\partial_e) = (-1)^l$.

We denote by $\tau_e \in \Sigma_{n-1}$ the assigned permutation for the planar structures p'_e and p_e. Observe that the permutation $\tau_e : \{1, 2, ..., n-1\} \to \{1, 2, ..., n-1\}$ is obtained from the permutation $\tau : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ in the following way. We delete k in the domain of τ and relabel the elements greater than k by decreasing them by 1. Also we delete l in the codomain of τ and relabel the elements greater than l by decreasing them by 1. Now we compare the number of inversions of τ (i.e. the instances of pairs (a, b) such that $a, b \in \{1, 2, ..., n\}$, $a < b$ and $\tau(a) > \tau(b)$) to the number of inversions of τ_e. Actually the inversions in τ_e are in bijection with the inversions (a, b) of τ such that a and b are different than k (if $a, b \neq k$ then the mentioned relabelling does not affect the relative order of $\tau(a)$ and $\tau(b)$ when considered in the codomain of τ_e). So we need to calculate the number of elements of the set

$$\{(a, k) : 1 \leq a < k, \tau(a) > l\} \cup \{(k, b) : k < b \leq n, \tau(b) < l\}. \tag{7.1}$$

Denote by p the number of elements of the set $\{a : 1 \leq a < k, \tau(a) > l\}$. Then the number of elements of the set $\{a : 1 \leq a < k, \tau(a) < l\}$ is $k - p - 1$. But the elements of the latter set are in bijection with the elements of $\{c : 1 \leq c < l, \tau^{-1}(c) < k\}$. This implies that the number of elements of the set $\{c : 1 \leq c < l, \tau^{-1}(c) > k\}$ is $l - (k - p - 1) - 1 = l - k + p$, and this set is in bijection with $\{b : k < b \leq n, \tau(b) < l\}$. So the number of the elements of the set 7.1 is $l - k + p + p = l - k + 2p$ and we conclude $\mathrm{sgn}(\tau) = (-1)^{l-k+2p}\mathrm{sgn}(\tau_e)$.

If we suppose ∂_e is a face map corresponding to a top vertex of T' labelled by k and $\tau(k) = l$, then in the same way we conclude $\mathrm{sgn}(\tau) = (-1)^{l-k+2p}\mathrm{sgn}(\tau_e)$. Since in this case $\mathrm{sgn}_p(\partial_e) = (-1)^{k+1}$ and $\mathrm{sgn}_{p'}(\partial_e) = (-1)^{l+1}$, the statement of the lemma holds.

If ∂_e is a face map corresponding to a root vertex, then $\mathrm{sgn}(\tau) = \mathrm{sgn}(\tau_e)$ (because in this case $\tau(1) = 1$ and τ_e is obtained by deleting 1 in domain and codomain of τ) and $\mathrm{sgn}_p(\partial_e) = \mathrm{sgn}_{p'}(\partial_e) = 1$ by definition. $\square$

7.2 The unnormalized chain complex

In this section we define a chain complex associated to a dendroidal set such that the definition extends the construction of a singular chain complex of a simplicial set.

Definition 7.2.1. If T is a tree we denote by $|T|$ the number of vertices of T.

Let X be a dendroidal set and $n \in \mathbb{N}_0$. We consider the free abelian group

$$C(X)_n := \bigoplus_{T\in\Omega, |T|=n} \bigoplus_{p\in P_T} \mathbb{Z}\langle X_T\rangle \tag{7.2}$$

generated by triples (T, p, x) where (T, p) is a planar tree and $x \in X_T$. For trees T and T', planar structures $p \in P_T$ and $p' \in P_{T'}$, an isomorphism $\tau : T' \to T$ and a dendrex $x \in X_T$ we consider the free subgroup $A(X)_n$ generated by

$$(T, p, x) - \mathrm{sgn}(p', \tau^* p)(T', p', \tau^*(x)). \tag{7.3}$$

Here $\tau^*(x)$ denotes $X(\tau)(x)$ for $X(\tau)\colon X_T \to X_{T'}$ and $\tau^*(p)$ denotes $P(\tau)(p)$ for $P(\tau)\colon P_T \to P_{T'}$.

Definition 7.2.2. Let X be a dendroidal set. For each $n \in \mathbb{N}_0$ we define an abelian group $\mathrm{Ch}^{un}(X)_n$ as the quotient

$$\mathrm{Ch}^{un}(X)_n := C(X)_n/A(X)_n$$

or more suggestively

$$\mathrm{Ch}^{un}(X)_n := \frac{\left(\bigoplus_{T\in\Omega,|T|=n} \bigoplus_{p\in P_T} \mathbb{Z}\langle X_T\rangle\right)}{(T, p, x) \sim \mathrm{sgn}(p', \tau^* p)(T', p', \tau^*(x))}.$$

Note that $\mathrm{Ch}^{un}(X)_n$ is a free abelian group since we identified generators of a free abelian group $C(X)_n$. The generators of $\mathrm{Ch}^{un}(X)_n$ are in bijection with the isomorphism classes of dendrices of X. Each representative carries additional information, a planar structure, which is used only for the definition of the differential. As we will show, it does not matter which planar structure we use. We write $[T, p, x]$ for the generator represented by the triple (T, p, x).

Definition 7.2.3. Let X be a dendroidal set. For every positive integer n, we define a map $d\colon \mathrm{Ch}^{un}(X)_n \to \mathrm{Ch}^{un}(X)_{n-1}$ on generators by

$$d([T,p,x]) := \sum_{\partial_e\colon T/e\to T} \mathrm{sgn}_p(\partial_e)[T/e, p_e, \partial_e^* x],$$

and extend it additively. The sum is taken over the set of elementary face maps of T.

Lemma 7.2.4. *The map $d\colon \mathrm{Ch}^{un}(X)_n \to \mathrm{Ch}^{un}(X)_{n-1}$ is well-defined.*

Proof. Let $x \in X_T$ and $x' = \tau^* x \in X_{T'}$ for some isomorphism $\tau\colon T' \to T$. If $p \in P_T$ and $p' \in P_{T'}$ are two planar structures, we have $[T,p,x] = \mathrm{sgn}(p', \tau^* p)[T', p', \tau^* x]$. So, we need to prove that

$$\sum_{\partial_e\colon T'/e\to T'} \mathrm{sgn}_{p'}(\partial_e)[T'/e, p'_e, \partial_e^*(x')] = \mathrm{sgn}(p', \tau^* p) \sum_{\partial_f\colon T/f\to T} \mathrm{sgn}_p(\partial_f)[T/f, p_f, \partial_f^*(x)],$$

where the sums are taken over the set of elementary face maps of T' and T, respectively. There is a unique isomorphism $\tau_e\colon T'/e \to T/\tau(e)$ such that $\tau\partial_e = \partial_{\tau(e)}\tau_e$. Note that

$$(\tau^* p)_e = P(\partial_e)P(\tau)(p) = P(\tau_e)P(\partial_{\tau(e)})(p) = \tau_e^* p_{\tau(e)}.$$

Hence, lemma 7.1.4 implies

$$\begin{aligned}
\mathrm{sgn}_{p'}(\partial_e)[T'/e, p'_e, \partial_e^* x'] &= \mathrm{sgn}_{p'}(\partial_e)[T'/e, p'_e, \partial_e^* \tau^* x] \\
&= \mathrm{sgn}_{p'}(\partial_e)[T'/e, p'_e, \tau_e^* \partial_{\tau(e)}^* x] \\
&= \mathrm{sgn}_{p'}(\partial_e)\mathrm{sgn}(p'_e, \tau_e^* p_{\tau(e)})[T/\tau(e), p_{\tau(e)}, \partial_{\tau(e)}^* x] \\
&= \mathrm{sgn}(p', \tau^* p)\mathrm{sgn}_p(\partial_{\tau(e)})[T/\tau(e), p_{\tau(e)}, \partial_{\tau(e)}^* x]
\end{aligned}$$

The set of elementary face maps $\partial_e\colon T'/e \to T'$ is in bijection with the set of elementary face maps $\partial_f\colon T/f \to T$ by $e \mapsto f = \tau(e)$, so collecting these terms gives the desired statement. □

Proposition 7.2.5. *The graded abelian group $(Ch^{un}(X), d)$ is a chain complex.*

Proof. We need to prove that $d^2 = 0$. Consider $x \in X_T$ and a planar structure p. We write $[x]$ instead of $[T,p,x]$ as the planar structure is clear from the context. We have the following calculation

$$d^2([x]) = d\left(\sum_{\partial_e^*\colon T/e\to T} \mathrm{sgn}_p(\partial_e)[\partial_e^* x]\right)$$
$$= \sum_{\partial_e\colon T/e\to T}\ \sum_{\partial_f\colon (T/e)/f\to T/e} \mathrm{sgn}_p(\partial_e)\mathrm{sgn}_{p_e}(\partial_f)[\partial_f^*\partial_e^* x]$$

For every two elementary face maps $\partial_e\colon T/e \to T$ and $\partial_f\colon (T/e)/f \to T/e$ there are elementary face maps $\partial_{f'}\colon T/f' \to T$ and $\partial_{e'}\colon (T/f')/e' \to T/f'$ such that $\partial_e\partial_f = \partial_{f'}\partial_{e'}$. The sign convention for faces of a planar tree is defined exactly so that the following holds

$$\mathrm{sgn}_p(\partial_e)\mathrm{sgn}_{p_e}(\partial_f) = -\mathrm{sgn}_p(\partial_{f'})\mathrm{sgn}_{p_{f'}}(\partial_{e'}).$$

This follows easily from the sign convention by inspection and it is also stated in [GLW11] as Lemma 4.3. Hence every term in the above sum appears exactly twice, each time with a different sign. This proves that the above sum is zero, i.e. $d^2 = 0$. □

Finally, for a morphism $f : X \to Y$ of dendroidal sets, we define

$$\mathrm{Ch}^{un}(f)_n([T, p, x]) = [T, p, f(x)], \quad x \in X_T.$$

Since f is a morphism of dendroidal sets it follows that $\mathrm{Ch}^{un}(f)_n$ is a well-defined morphism of chain complexes. In this way we obtain a functor $\mathrm{Ch}^{un}\colon \mathrm{dSet} \to \mathrm{Ch}_{\geq 0}$.

Definition 7.2.6. For a dendroidal set X define the homology and cohomology groups with values in an abelian group A as

$$H_n(X, A) := H_n(\mathrm{Ch}^{un}X \otimes A) \qquad \text{and} \qquad H^n(X, A) := H_n(\mathrm{Hom}(\mathrm{Ch}^{un}X, A)).$$

We will write $H_n(X)$ for $H_n(X, \mathbb{Z})$.

Remark 7.2.7. For a dendroidal set of the form $i_!S$ where S is a simplicial set the chain complex $Ch^{un}(i_!S)$ agrees with the unnormalized chain complex of S. Thus we have

$$H_n(i_!S, A) = H_n(S, A) \qquad \text{and} \qquad H^n(i_!S, A) = H^n(S, A)\ .$$

In the next sections we will introduce a normalized version of the chain complex for which it is easier to compute the homology and we will show that the homology for the normalized and the unnormalized complex coincide for normal dendroidal sets. For this reason we postpone giving the examples until we develop the normalized version.

7.3 The normalized chain complex

In this section we will define the normalized chain complex as a quotient of the unnormalized chain complex by the subcomplex generated by degenerated dendrices.

Proposition 7.3.1. *Let X be a dendroidal set. Consider the subgroups $D(X)_n \subset Ch^{un}(X)_n$ generated by the classes of degenerate dendrices. Then $D(X)$ is a subcomplex, i.e. it is closed under taking differentials.*

Proof. We need to check that the differential restricts to classes represented by degeneracies. Let $\sigma : f \setminus T \to T$ be a degeneracy map, so that the tree T has two face maps ∂_f and $\partial_{f'}$ which are equal up to an isomorphism of T/f and T/f'. Let $x \in X_T$. Then

$$d[\sigma^* x] = \sum_{\partial_e \colon T/e \to T} \mathrm{sgn}_d(\partial_e)[\partial_e^* \sigma^* x].$$

By the dendroidal identities σ commutes with all face maps ∂_e except ∂_f and $\partial_{f'}$, but $\mathrm{sgn}_d(\partial_f)[\partial_f^* \sigma^* x] = -\mathrm{sgn}_d(\partial_{f'})[\partial_{f'}^* \sigma^* x]$. We conclude that the above sum is the sum of classes represented by degeneracies. □

Lemma 7.3.2. *Let X be a dendroidal sets such that for every nondegenerate dendrex $x \in X_T$ the associated map $x \colon \Omega[T] \to X$ is a monomorphism. The subcomplex $D(X)$ is acyclic.*

Proof. Let us fix a linear order on the set X_η. Let $x \in X_S$ be a degenerate dendrex of shape S, let a be the smallest element of X_η with respect to the fixed linear order such that x factors through a degeneracy on a. We say that a is the smallest degenerate colour of x. Dendrex x factors through a unique non-degenerate dendrex $x^\#$. By the assumption all the edges of $x^\#$ are distinct, so a appears exactly once in $x^\#$. Let k be the maximal integer such that the $(k-1)$-fold degeneracy of a factors through x. In other words, colour a appears k times as an edge of x. We say that x is canonical if and only if k is an odd number (it must be at least 3 by the definition).

A generating class in $D(X)_n$ has at least one canonical representative if and only if all its representatives are canonical. We call such a generator canonical. We define A_n as the set of all canonical generators of $D(X)_n$ and B_n as the set of all non-canonical generators of $D(X)_n$.

A bijection between B_n and A_{n+1} is established by degenerating x at the smallest degenerate colour a. Note that A_0, B_0 and A_1 are empty sets and $d(x) = 0$ for all $x \in B_1$.

Let $C_n = D(X)_n$ and $C_{n,can} = Z\langle A_n \rangle$. If we define $w : C_n \to \mathbb{N}_0$ to be

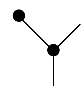
$$w(x) = \begin{cases} 0, & \text{if } x \in C_{n,can} \\ 1, & \text{otherwise} \end{cases}$$

then all assumptions of Proposition 7.7.1 obviously hold. So all homology groups of $D(X)$ vanish. □

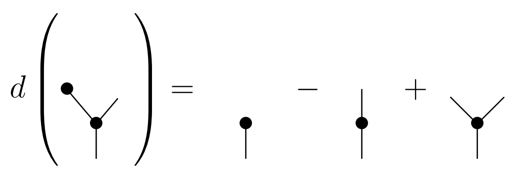
Definition 7.3.3. We define the normalized chain complex as the quotient

$$\mathrm{Ch}(X)_\bullet := \mathrm{Ch}^{un}(X)_\bullet / D(X)_\bullet$$

Remark 7.3.4. Let x be a dendrex of some dendroidal set X of the following shape



such that the inner face of x is degenerate. Then x is nondegenerate, but the differential



calculated in $\mathrm{Ch}^{un}(X)$ is not a linear combination of non-degenerate dendrices.

Nevertheless, the chain complex $\mathrm{Ch}(X)_n$ can be also obtained as a quotient of the free abelian group generated by non-degenerate dendrices of X using the relation (7.3), as we did for $\mathrm{Ch}^{un}(X)$. Although the differential of a nondegenerate dendrex for $\mathrm{Ch}^{un}(X)$ is not necessarily a linear combination of non-degenerate dendrices, the differential of $Ch(X)$ is just a modification where we disregard degenerate dendrices. In other words, there is a canonical inclusion of graded groups $\mathrm{Ch}(X) \to \mathrm{Ch}^{un}(X)$ which splits the quotient map $\mathrm{Ch}^{un}(X) \to \mathrm{Ch}(X)$ but which is not compatible with the differentials.

Example 7.3.5. Let T be a tree with n vertices. The chain complex $\mathrm{Ch}(\Omega[T]/\partial\Omega[T])$ is concentrated in degrees 0 and n, hence we have

$$H_k(\Omega[T]/\partial\Omega[T], A) = \begin{cases} A, & \text{if } k = 0, n \\ 0, & \text{if otherwise.} \end{cases}$$

7.4 The equivalence of the unnormalized and the normalized chain complex

Proposition 7.4.1. *Let $\Gamma_d^{un} : \mathrm{Ch}_{\geq 0} \to dSet$ be defined by the formula*

$$\Gamma_d^{un}(C)_T = \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}}(\mathrm{Ch}^{un}(\Omega[T]), C).$$

The pair $(\mathrm{Ch}^{un}, \Gamma_d^{un})$ forms an adjunction.

Proof. It is well known that a functor from a presheaf category to a cocomplete category is left adjoint if and only if it preserves colimits. We want to show that Ch preserves colimits.

Since colimits in chain complexes are just colimits of the underlying graded abelian groups it suffices to show that for every n the functor

$$\mathrm{Ch}^{un}(X)_n = C(X)_n / A(X)_n$$

preserves colimits in X. We write Ch^{un} as a coequalizer

$$\bigoplus_{T,T'\in\Omega, |T|=|T'|=n} \ \bigoplus_{\tau : T \xrightarrow{\sim} T'} \ \bigoplus_{p\in P_T, p'\in P_{T'}} \mathbb{Z}\langle X_T\rangle \rightrightarrows \bigoplus_{T\in\Omega, |T|=n} \ \bigoplus_{p\in P_T} \mathbb{Z}\langle X_T\rangle$$

where the maps are given on generators by

$$(T, T', \tau, p, p', x) \mapsto (T, p, x) \qquad \text{and} \qquad (T, T', \tau, p, p', x) \mapsto (T', p', \mathrm{sgn}(p', \tau^* p)\tau^*(x)).$$

Now we see that both sides of the coequalizer commute with colimits in X since the direct sum functor and the free abelian group functor commute with colimits. Its also clear that the maps between the two abelian groups commute with colimits since they are (apart from a sign) completely determined by the indexing set. Since coequalizers also commute with colimits this finishes the proof. □

Proposition 7.4.2. *Let* $\Gamma_d : \mathrm{Ch}_{\geq 0} \to$ *dSet be defined by the formula*

$$\Gamma_d(C)_T = \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}}(\mathrm{Ch}(\Omega[T]), C).$$

The pair (Ch, Γ_d) *forms an adjunction.*

Proof. The proof is similar to the proof of Proposition 7.4.1. We again want to show that the functor

$$\mathrm{Ch}(X) = \mathrm{Ch}^{un}(X)/D(X)$$

preserves colimits in X. We consider the following functor

$$\Xi : \mathrm{dSet} \to \mathrm{AbGr} \qquad X \mapsto \bigoplus_{|T|=n} \ \bigoplus_{\tau : T \to T' \text{degeneracy}} \ \bigoplus_{p\in P_T} \mathbb{Z}\langle X'_T\rangle$$

Then there is a natural transformation $\Xi \to \mathrm{Ch}^{un}$ given on generators by

$$(T, \tau, p, x) \mapsto [T, p, \tau^* x]$$

By definition its clear that $\mathrm{Ch}(X)$ is the cokernel of $\Xi(X) \to \mathrm{Ch}^{un}(X)$. Thus the fact that everything clearly commutes with colimits shows the claim. □

Recall that the category $\mathrm{Ch}_{\geq 0}$ of positively graded chain complexes admits two canonical model structures. The projective one and the injective one. In both the weak equivalences are quasi-isomorphisms. In the injective model structure the cofibrations are all monomorphisms and in the projective one the cofibrations are the monomorphisms with levelwise projective cokernel.

Proposition 7.4.3. *The functor* Ch : *dSet* $\to \mathrm{Ch}_{\geq 0}$ *maps boundary inclusions* $\partial\Omega[S] \to \Omega[S]$ *to cofibrations (in either of the model structures). The same is true for the functor* Ch^{un}.

Proof. Let $i : \partial\Omega[S] \to \Omega[S]$ be a boundary inclusion. Because $\partial\Omega[S]_T \to \Omega[S]_T$ are monomorphisms compatible with the relation (7.3), the induced maps $\mathrm{Ch}(i)_n$ and $\mathrm{Ch}^{un}(i)_n$ are monomorphisms between free abelian groups given by inclusion of generators. Hence their cokernels are also free. □

Corollary 7.4.4. *The natural map* $\mathrm{Ch}^{un}(X) \to \mathrm{Ch}(X)$ *is a quasi-isomorphism for every normal dendroidal set* X.

Proof. By Lemma 7.3.2, $D(\Omega[T])$ is acyclic for every tree T. Hence the natural maps $\mathrm{Ch}^{un}(\Omega[T]) \to \mathrm{Ch}(\Omega[T])$ are quasi-isomorphisms. Proposition 7.4.3, Proposition 7.4.1, Proposition 7.4.2 and Lemma 5.5.1 imply the result. □

Proposition 7.4.5. *The functor* Ch : *dSet* $\to \mathrm{Ch}_{\geq 0}$ *maps horn inclusions* $\Lambda^a[T] \to \Omega[T]$ *to trivial cofibrations in either of the model structures. The same is true for the functor* Ch^{un}.

Proof. By Prop. 7.4.3, we only need to see that Ch sends a horn inclusion $i : \Lambda^a[T] \to \Omega[T]$ to a quasi-isomorphism. Let $|T| = n$. Then $\mathrm{Ch}(\Lambda^a[T])_k \to \mathrm{Ch}(\Omega[T])_k$ is an isomorphism for $0 \leq k < n-1$. Hence, $H_k(i)$ is an isomorphism for $0 \leq k \leq n-3$.

Note that $\mathrm{Ch}(\Lambda^a[T])_{n-1}$ is a subgroup of $\mathrm{Ch}(\Omega[T])_{n-1}$ generated by all but one generator, let us denote it $[x_a]$, of $\mathrm{Ch}(\Omega[T])_{n-1}$. The group $\mathrm{Ch}(\Lambda^a[T])_n$ is trivial and $\mathrm{Ch}(\Omega[T])_n$ is generated by one element, call it $[x]$. Then $[x_a] - d([x])$ is in $\mathrm{Ch}(\Lambda^a[T])_{n-1}$. This implies that $d([x_a]) \in d(\mathrm{Ch}(\Lambda^a[T])_{n-1})$, so $H_{n-2}(i)$ is also an isomorphism. Also, $[x_a] - d([x]) \in \mathrm{Ch}(\Lambda^a[T])_{n-1}$ implies $H_{n-1}(i)$ and $H_n(i)$ is an isomorphism. □

From Lemma 5.5.3, Proposition 7.4.3 and Proposition 7.4.5 it directly follows that (Ch, Γ_d) and $(\mathrm{Ch}^{un}, \Gamma_d^{un})$ are Quillen adjunctions.

7.5 The associated spectrum and its homology

In this section we will compare the homology of a dendroidal set to the homology of the associated connective spectrum. Recall that for a spectrum E, its n-th homology group is defined as the n-th homotopy group of the spectrum $E \wedge H\mathbb{Z}$, where $H\mathbb{Z}$ is the Eilenberg-Maclane spectrum.

Theorem 7.5.1. *Let D be a normal dendroidal set. Then the homology $H_n(D)$ is isomorphic to the homology of the associated connective spectrum $\mathcal{K}D$.*

Proof. We will use Corollary 4.9. in [GGN13], which states that the grouplike E_∞-spaces form the "free additive" ∞-category, i.e. two left adjoint ∞-functors $F, G\colon Grp_{E_\infty} \to D$ from the ∞-category of grouplike E_∞-spaces to an additive ∞-category D are equivalent if they coincide on the trivial E_∞-space. Since the ∞-category dSet_n of normal dendroidal sets is equivalent to grouplike E_∞-spaces, the same universal property holds for dSet_n.

We consider the following diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{dSet}_n \arrow[r, shift left] \arrow[d, "\mathrm{Ch}"] & \mathrm{connSp} \arrow[l, shift left] \arrow[d, "-\wedge H\mathbb{Z}"'] \\
Ch_{\geq 0} \arrow[r, shift left] & H\mathbb{Z} - \mathrm{mod} \arrow[l, shift left]
\end{tikzcd}$$

Here connSp denotes the ∞-category of connective spectra and $H\mathbb{Z} - \mathrm{mod}$ is the ∞-category of module spectra in connSp over the ring spectrum $H\mathbb{Z}$. The top row is an equivalence of ∞-categories as a consequence of Theorem 6.3.4. The bottom row is an equivalence of ∞-categories given by the extension of the Dold-Kan correspondence to spectra, Theorem 5.1.6. in [SS02]. The left vertical map is induced by the left Quillen functor Ch studied in the previous sections of this chapter. The right vertical map is given by taking the homology of a spectrum, i.e. by the smash product with $H\mathbb{Z}$.

The ∞-category $H\mathbb{Z}$-mod is an additive ∞-category (see Definition 2.6 in [GGN13]). The dendroidal set η corresponds to the sphere spectrum and its homology is just the spectrum $H\mathbb{Z}$ (as the sphere spectrum is the unit for the smash product). On the other hand the chain complex $\mathrm{Ch}(\eta)$ is just $\mathbb{Z}$ concentrated in degree 0 and under Dold-Kan it corresponds to $H\mathbb{Z}$.

Hence there are two left adjoint ∞-functors from dSet_n to $H\mathbb{Z} - mod$ and since they coincide on η the mentioned result of [GGN13] implies these functors are equivalent. □

Corollary 7.5.2. *A morphism $f : X \to Y$ between normal dendroidal sets is a stable weak equivalence if and only if it is a homology iso, i.e. $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for each n.*

Corollary 7.5.3. *The spectrum associated to the dendroidal set $\Omega[T]/\partial\Omega[T]$ is the n-sphere, i.e. $\Sigma^n\mathbb{S} \cong \Sigma^\infty S^n$.*

Proof. The only spectrum Z such that $H_n(Z) = \mathbb{Z}$ and $H_k(Z) = 0$ for $k \neq n$ is $\Sigma^\infty S^n$. □

Corollary 7.5.4. *The homology of $\Omega[T]$ is given by*

$$H_k(\Omega[T]) = \begin{cases} \mathbb{Z}\langle \ell_T \rangle & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

where ℓ_T is the set of leaves of the tree T.

Proof. The morphism $\sqcup_{\ell_T}\eta \to \Omega[T]$ is a covariant (and hence stable) anodyne extension, so the result follows from Corollary 7.5.2. $\square$

Remark 7.5.5. One can also show the statement of the previous corollary directly using the acyclicity argument 7.7.1. The set of canonical dendrices of $\Omega[T]$ is defined in the same way as the notion of canonical extensions in Chapter 4 and the weight function w is given by the number of possible extensions.

7.6 The homology of A_∞

Let $A_\infty = N_d(Ass)$ be the dendroidal nerve of the operad for associative algebras. Note that A_∞ is the presheaf of planar structures, which we also denoted by P.

Theorem 7.6.1. *The homology of A_∞ vanishes. Therefore the spectrum $\mathcal{K}A_\infty$ is trivial.*

Proof. By the definition, the generators of the free abelian group $Ch^{un}(A_\infty)_n$ are in bijection with the isomorphism classes of planar structures of trees with n vertices. More precisely, for each tree T there is exactly one generator for each orbit of the action of the group $Aut(T)$ on the set of planar structures $P(T)$. Hence we may represent the generators by planar trees with all the edges of the same colour, keeping in mind that isomorphic planar trees are identified.

For example, there for each of the following two shapes the two planar structures get identified, so there is only one generator:

but the following two planar trees are representing two different generators:

We call a generator *canonical* if the leftmost top vertex of such a representative is a stump. For example, in the above pictures, the planar trees on the right represent canonical generators, while the ones on the left represent non-canonical generators.

Let A_n (i.e. B_n) be the set of canonical (i.e. non-canonical) generators of $Ch^{un}(A_\infty)_n$. A bijection between B_n and A_{n+1} is obtained by putting a stump on the leftmost leaf of the chosen representative of a non-canonical generator in B_n.

Obviously, a dendrex with no vertices has no stumps, A_0 is empty. The set B_0 is a singleton, consisting of the tree with one edge. Also A_1 is a singleton containing just the stump. For every generator x we define its weight $w(x)$ as the number of leaves of the planar tree representing it if x is non-canonical and $w(x) = 0$ if x is canonical.

If x is non-canonical, then $\hat{x}$ has exactly one leaf less than x. Every other face of $\hat{x}$ is either canonical (containing the added stump) or it is a non-canonical face obtained by contracting the edge just below the added tree, so it has one leaf less than x. This shows that all the assumptions of Proposition 7.7.1 hold. Hence all homology groups of A_∞ vanish. □

7.7 Acyclicity argument

In this section we prove a proposition which we will use to show acyclicity of certain chain complexes. Namely, one can use this proposition to show that the homology groups of a chain complex generated by degenerate dendrices of a normal dendroidal set, the homology groups of the A_∞ and also the higher homology groups of representable dendroidal sets vanish.

Proposition 7.7.1. *Let $C_\bullet$ be a chain complex such that all C_n are free abelian groups which have a grading*

$$C_n = \bigoplus_{i \in \mathbb{N}_0} C_{n,i}.$$

For $x \in \bigoplus_{i=0}^{m} C_{n,i} \setminus \bigoplus_{i=0}^{m-1} C_{n,i}$ we write $w(x) = m$. Let A_n and B_n be a basis for $C_{n,0}$ and $\bigoplus_{i>0} C_{n,i}$, respectively. Assume there is a bijection between sets B_n and A_{n+1} which sends $x \in B_n$ to $\hat{x} \in A_{n+1}$ and one of the following two statements holds

$$w(x - d(\hat{x})) < w(x) \quad \text{or} \quad w(x + d(\hat{x})) < w(x).$$

Then $H_0(C) = \mathbb{Z}\langle A_0 \rangle$ and $H_n(C) = 0$ for all $n \geq 1$.

Proof. First we show that for every $x \in B_n$ there exists an element $\bar{x} \in C_{n+1,0}$ such that

$$x - d(\bar{x}) \in C_{n,0}. \tag{7.4}$$

Indeed, we can prove this by induction on $w(x)$. If $w(x) = 1$, then we can take $\bar{x}$ to be $\hat{x}$ or $-\hat{x}$ and the statement follows by assumption.

Let $w(x) > 1$ and assume we have proven the statement for all $y \in B_n$ such that $w(y) < w(x)$. We let $x_1 = \pm\hat{x}$, where the sign $\pm$ is such that $w(x - d(x_1)) < w(x)$. So

$$x - d(x_1) = z + y$$

where $z \in C_{n,0}$ and $y \in C_n \setminus C_{n,0}$. Hence y is a finite sum of elements y_i in B_n and $w(y_i) < w(x)$ for all i. By the assumption there are $\bar{y_i} \in C_{n+1,0}$ such that $y_i - d(\bar{y_i}) \in C_{n,0}$. We let $\bar{x} = x_1 + \sum_i \bar{y_i}$.

The set $\{d(\bar{x}) : x \in B_n\}$ is linearly independent, for every n.

Let $\sum_{i=1}^{k} \alpha_i d(\bar{x}_i) = 0$ for some $x_1, ..., x_k \in B_n$. We can write $d\bar{x}_i = x_i + y_i$, where $y_i \in C_{n,0}$ for $i = 1, 2, ..., k$. Hence we have

$$\sum_{i=1}^{k} \alpha_i x_i + \sum_{i=1}^{k} \alpha_i y_i = 0,$$

and we conclude that $\alpha_i = 0$ for all i since B_n is a basis for $C_n \setminus C_{n,0}$.

Since d is linear, the set $\{\bar{x} : x \in B_n\}$ is also linearly independent, for every n.

The restriction $d\colon C_{n,0} \to \operatorname{Im} d$ is surjective.

Let $y = d(a+b)$ be an element of $\operatorname{Im} d$ with $a \in C_{n,0}$ and $b \in C_n \setminus C_{n,0}$. There is an element $\bar{b} \in C_{n+1,0}$ such that $b - d\bar{b} \in C_{n,0}$. Since $d^2 = 0$ we have $y = d(a+b) = d(a+b) - d(d(\bar{b})) = d(a + b - d\bar{b}) \in d(C_{n,0})$. It follows that $\{d(\bar{x} : x \in B_n\}$ is a basis for $\operatorname{Im} d$.

The set $\{\bar{x} : x \in B_n\}$ generates $C_{n+1,0}$. Indeed, the inductive construction of $\bar{x}$ shows that we can write

$$\hat{x} = \pm\bar{x} + \sum_i \pm\hat{z}_i$$

where $z_i \in B_n$ and $w(z_i) < w(x)$. Hence we can show by induction on $w(x)$ that we can express every element $\hat{x}$ as a linear combination of elements of the set $\{\bar{x} : x \in B_n\}$. We are assuming that the set $\{\hat{x} : x \in B_n\}$ is equal to A_{n+1}, hence it generates $C_{n+1,0}$ and the result follows. This implies that the set $\{\bar{x} : x \in B_n\}$ is a basis for $C_{n+1,0}$.

We conclude that the restriction

$$d\colon C_{n+1,0} \to \operatorname{Im} d = \operatorname{span}\{d(\bar{x})\}$$

is an isomorphism for every n. Furthermore, this implies that $\operatorname{Ker} d$ is disjoint with $C_{n+1,0}$ for every n.

Since $d(\bar{x}) \in \operatorname{Ker} d$, the set $\{d(\bar{x}) : x \in B_n\}$ is also disjoint with $C_{n,0}$ and by the construction of $\bar{x}$ we have that $\operatorname{span}\{d(\bar{x})\} \oplus C_{n,0} = C_n$. We also have $\operatorname{Ker} d \oplus C_{n,0} = C_n$ because $C_{n,0} \to \operatorname{Im} d$ is an isomorphism. Since $\operatorname{span}\{d(\bar{x})\} \subseteq \operatorname{Ker} d$, we must have $\operatorname{Ker} d = \operatorname{span}\{d(\bar{x})\} = \operatorname{Im} d$, what we wanted to prove. $\square$

Bibliography

[AHS90] J. Adámek, H. Herrlich, and G.E. Strecker, *Abstract and concrete categories: the joy of cats*, Pure and applied mathematics, Wiley, 1990.

[AR94] J. Adámek and J. Rosicky, *Locally presentable and accessible categories*, Cambridge Tracts in Mathematics, Cambridge University Press, 1994.

[Bar07] C. Barwick, *On (enriched) left Bousfield localizations of model categories*, Preprint arxiv:0708.2067, 2007.

[Ber04] J. E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc (2004).

[BM03] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), no. 4, 805–831.

[BM06] ______, *The Boardman-Vogt resolution of operads in monoidal model categories*, Topology (2006), no. 45, 807–849.

[BM07] ______, *Resolution of coloured operads and rectification of homotopy algebras*, Contemp. Math. (2007), no. 431, 31–58.

[BM11] ______, *On an extension of the notion of Reedy category*, Mathematische Zeitschrift **269** (2011), no. 3-4, 977–1004.

[BN] M. Bašić and T. Nikolaus, *Homology of dendroidal sets*, in preparation.

[BN14] ______, *Dendroidal sets as models for connective spectra*, Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology (2014), 1–35.

[Bro73] K. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc. **186** (1973), 419–458.

[BV73] J.M. Boardman and R.M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, vol. 347, Springer-Verlag, 1973.

[CM11] D.C. Cisinski and I. Moerdijk, *Dendroidal sets as models for homotopy operads*, Journal of Topology **4** (2011), no. 2, 257–299.

[CM13a] ______, *Dendroidal Segal spaces and infinity-operads*, Journal of Topology (2013).

[CM13b] ______, *Dendroidal sets and simplicial operads*, Journal of Topology (2013).

[CP86] J.-M. Cordier and T. Porter, *Vogt's theorems on categories of homotopy coherent diagrams*, Math. Proc. Camb. Phil. Soc. **100** (1986), 65–90.

[Cra95] S.E. Crans, *Quillen Closed Model Structures for Sheaves*, Journal of Pure and Applied Algebra **101** (1995), 35–57.

[DK80a] W.G. Dwyer and D.M. Kan, *Calculating simplicial localizations*, Journal of Pure and Applied Algebra **18** (1980).

[DK80b] ______, *Function complexes in homotopical algebra*, Topology **19** (1980), no. 4, 427 – 440.

[DK80c] ______, *Simplicial localizations of categories*, Journal of Pure and Applied Algebra **17** (1980).

[DS95] W. G. Dwyer and J. Spalinski, *Homotopy Theories and Model Categories*, Handbook of Algebraic Topology (E. James, ed.), 1995, pp. 73–126.

[GGN13] M. Groth, D. Gepner, and T. Nikolaus, *Universality of multiplicative infinite loop space machines*, Preprint arxiv:1305.4550, 2013.

[GK94] Victor Ginzburg and Mikhail Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), no. 1, 203–272.

[GLW11] J. J. Gutiérrez, A. Lukacs, and I. Weiss, *Dold-Kan correspondence for dendroidal abelian groups*, journal of Pure and Applied Algebra (2011), no. 215, 1669–1687.

[GM10] S.I. Gelfand and Y.I. Manin, *Methods of homological algebra*, Springer Monographs in Mathematics, Springer Berlin Heidelberg, 2010.

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, 1967.

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge Univ Press, 2002.

[Heu11a] G. Heuts, *Algebras over infinity-operads*, Preprint arxiv: 1110.1776, 2011.

[Heu11b] ______, *An infinite loop space machine for infinity-operads*, Preprint arxiv: 1112.0625, 2011.

[HHM13] G. Heuts, V. Hinich, and I. Moerdijk, *On the equivalence between lurie's model and the dendroidal model for infinity-operads*, Preprint arxiv: 1305.3568, 2013.

[Hir02] P. S. Hirschhorn, *Model categories and Their Localizations*, American Mathematical Society, 2002.

[Hov07] M. Hovey, *Model categories*, American Mathematical Society, 2007.

[HS98] A. Hirschowitz and C. Simpson, *Descente pour les n-champs (Descent for n-stacks)*, ArXiv Mathematics e-prints (1998).

[Joy08] A. Joyal, *Notes on quasi-categories I*, Preprint, 2008.

[Kan57] D.M. Kan, *On c.s.s. complexes*, American Journal of Mathematics **79** (1957), no. 3, 449–476.

[Kel82] G.M. Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series, vol. 64, Cambridge University Press, Cambridge, 1982.

[Lei04] T. Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004.

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, Princeton University Press, 2009.

[Lur11] ———, *Higher algebra*, Prepublication book draft http://www. math. harvard. edu/lurie, 2011.

[May72] J.P. May, *The geometry of iterated loop spaces*, Lecture notes in mathematics, no. no. 271, Springer-Verlag, 1972.

[May74] ———, *E_∞-spaces, group completions, and permutative categories*, New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972) **11** (1974), 61–93.

[ML98] S. Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[MSS07] M. Markl, S. Shnider, and J.D. Stasheff, *Operads in algebra, topology and physics*, Mathematical surveys and monographs, American Mathematical Society, 2007.

[MT10] I. Moerdijk and B. Toën, *Simplicial methods for operads and algebraic geometry*, Springer, 2010.

[MW07] I. Moerdijk and I. Weiss, *Dendroidal sets*, Algebr. Geom. Topol. **7** (2007), 1441–1470.

[MW09] ______, *On inner Kan complexes in the category of dendroidal sets*, Adv. Math. **221** (2009), no. 2, 343–389.

[Qui67] D. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics 43, Springer Verlag, 1967.

[Qui69] ______, *Rational homotopy theory*, Annals of Mathematics **90** (1969), no. 2, pp. 205–295.

[Ree74] C. Reedy, *Homotopy theory of model categories*, unpublished, 1974.

[Rez91] C. Rezk, *A model for the homotopy theory of homotopy theories*, Trans. Amer. Math. Soc. **3** (1991), no. 353, 973–1007.

[Spi01] M. Spitzweck, *Operads, algebras and modules in model categories and motives*, Ph.D. thesis, Universität Bonn, 2001.

[SS02] S. Schwede and B. Shipley, *Stable model categories are categories of modules*, Topology **42** (2002), 103–153.

[Tho80] R.W. Thomason, *Cat as a closed model category*, Cahiers de topologie et géométrie différentielle catégoriques **21** (1980), no. 3, 305–324.

[Whi95] G.W. Whitehead, *Elements of homotopy theory*, Springer, 1995.

Summary

The main topic of this thesis is the stable homotopy theory of dendroidal sets. This topic belongs to the area of mathematics called algebraic topology. Algebraic topology studies the interaction between the algebraic and topological structures.

Examples of topological spaces with a very rich algebraic structure are (iterated) loop spaces. Loop spaces carry an algebraic structure which is called an A_∞-structure, while infinite loop spaces carry an E_∞-structure. These structures consist of an infinite sequence of operations that satisfy various coherence laws. As it is difficult to grasp all these data, one usually uses topological operads to efficiently describe this information. One can think of operads as carrying "blueprints" for the algebraic structure which is realized in every space with that structure. The characterization results for (iterated) loop spaces using topological operads have been established in the early 1970's by the work of P. May, M. Boardman and R. Vogt. In the 1990's it became evident that it is important to understand the homotopy theory operads.

The theory of dendroidal sets provides a new context for studying operads up to homotopy. Dendroidal sets were introduced in 2007 by I. Moerdijk and I. Weiss. Subsequent work of I. Moerdijk and D.-C. Cisinski shows that dendroidal sets indeed model topological/simplicial operads. An important advantage of dendroidal sets is that the theory is built in a natural way as a generalization of the theory of simplicial sets. The study of dendroidal sets is very combinatorial in its nature since it is based on the notion of trees (graphs with no loops). Also, as a category of presheaves, the category of dendroidal sets has nice categorical properties.

Simplicial sets provide combinatorial models for spaces (think of it in terms of triangulations of spaces given by simplicial approximations) and dendroidal sets provide combinatorial models for infinite loops spaces as spaces together with complicated algebraic structure. In fact, the precise formulation of this idea is one of the main topics of this thesis.

A precise formulation of our results is given in the language of Quillen's model categories. Model categories provide a formalism to study and compare homotopy theories in various contexts (topological spaces, chain complexes, simplicial sets, operads etc.) One of the main results of this thesis is that the category of dendroidal sets admits a model structure such that the underlying homotopy theory is equivalent to the homotopy theory of infinite loop spaces (equivalently, of grouplike E_∞-algebras or connective spectra). We

call this model structure the stable model structure on dendroidal sets.

Constructing a model structure is a tedious job. In our case it requires a great deal of technical combinatorial results about dendroidal sets (i.e. about trees). In order to simplify our arguments, in Chapter 4 we develop a combinatorial technique for proving results about dendroidal anodyne extensions. This technique can be viewed as a result in its own right as one might apply it also in different ways than it is used in the later chapters of the thesis.

We give two constructions of the stable model model structure. The first construction is more elementary and has an advantage of providing a characterization of fibrations between fibrant objects. This construction is based on standard model-theoretical arguments and it is given in Chapter 5.

The second construction, given in Chapter 6, is based on the work of G. Heuts. This approach makes it possible to show that the stable model structure on dendroidal sets is Quillen equivalent to a model structure on E_∞-spaces with grouplike E_∞-spaces as fibrant objects. The equivalence to grouplike E_∞-objects (i.e. connective spectra) might be considered as a solution to the problem of geometric realization of dendroidal sets. Also, these results open new possibilities to investigate the connective part of classical stable homotopy theory.

The results of the thesis presented in Chapter 7 go in that direction. In that final chapter we discuss homology groups of dendroidal sets. This homology theory generalizes the well-known homology theory of simplicial sets (i.e. the singular homology of spaces). The generalization is not straightforward because we work with non-planar trees, but we want to use a certain sign-convention for planar trees. After giving the definition, we establish that these homology groups are homotopy invariant and that they compute the standard homology of the corresponding connective spectrum. The results of Chapters 6 and 7 are joint work with T. Nikolaus.

Samenvatting

Dit proefschrift beschrijft de stabiele homotopietheorie van ‘dendroïdale verzamelingen’. Dit onderwerp behoort tot de algebraïsche topologie, een vakgebied binnen de wiskunde waarin de interactie tussen algebraïsche en topologische structuren wordt bestudeerd.

Voorbeelden van topologische ruimtes met een rijke algebraïsche structuur zijn ruimtes van lussen en hoger-dimensionale bollen in een topologische ruimte. Naarmate de dimensie van deze bollen toeneemt krijgen de ruimtes van bollen een rijkere algebraïsche structuur, te beginnen met de structuur van een A_∞-algebra op lusruimtes en convergerend naar de structuur van een E_∞-algebra op zogenaamde ‘oneindige geïtereerde lusruimtes’. Dergelijke structuren bestaan uit een oneindige verzameling operaties die voldoen aan verscheidene coherentiecondities.

Omdat het moeilijk is om grip op dit soort data te krijgen, wordt doorgaans gebruik gemaakt van topologische operaden om dergelijke structuren efficient te beschrijven. Een operade kan worden gezien als de blauwdruk voor een algebraïsche structuur; de concrete realisaties van de blauwdruk zijn ruimtes met deze algebraïsche structuur. Het werk van P. May, M. Boardman en R. Vogt uit de vroege jaren 70 gebruikt topologische operaden om een karakterisering te geven van (geïtereerde) lusruimtes. In de jaren 90 groeide het besef dat er behoefte was aan een goede homotopietheorie van operaden.

De theorie van dendoïdale verzamelingen biedt een nieuwe context voor het bestuderen van operaden op homotopie na. Dendroïdale verzamelingen werden geïntroduceerd in 2007 door I. Moerdijk en I. Weiss. Later werk van I. Moerdijk en D.-C. Cisinski laat zien dat dendroïdale verzamelingen inderdaad een model bieden voor topologische (of simpliciale) operaden. Het voordeel van dit model is dat het een natuurlijke uitbreiding vormt van de theorie van simpliciale verzamelingen. De theorie van dendroïdale verzamelingen is zeer combinatorisch van aard, aangezien zij is gebaseerd op het begrip van een boom (een graaf zonder lussen). Bovendien heeft de categorie van dendroïdale verzamelingen de goede categorische eigenschappen van een categorie van preschoven.

Simpliciale verzamelingen bieden een combinatorische beschrijving van topologische ruimtes: een simpliciale verzameling kan worden gezien als de beschrijving van een triangulatie van een ruimte. Op dezelfde manier geven dendroïdale verzamelingen een combinatorisch model voor oneindige lusruimtes, samen met hun gecompliceerde algebraïsche structuur. De precieze formulering van dit idee is een van de onderwerpen van dit proefschrift.

Onze resultaten kunnen rigoreus worden geformuleerd in termen van Quillens theorie van modelcategorieën. Modelcategorieën bieden een wiskundig formalisme waarbinnen verschillende soorten homotopietheorieën (zoals topologische ruimtes, ketencomplexen, simpliciale verzamelingen of operaden) kunnen worden bestudeerd en vergeleken. Een van de hoofdresultaten van dit proefschrift is het feit dat de categorie van dendroïdale verzamelingen de structuur kan worden gegeven van een modelcategorie, zodat de onderliggende homotopietheorie equivalent is aan de homotopietheorie van oneindige lusruimtes (of equivalent, de homotopietheorie van groepachtige E_∞-ruimtes of connectieve spectra, d.w.z. spectra zonder homotopie in negatieve graden). We noemen deze modelstructuur de stabiele modelstructuur op de categorie van dendroïdale verzamelingen.

Het is niet eenvoudig om een modelstructuur te construeren. In ons geval hebben we een grote hoeveelheid technische, combinatorische resultaten nodig over dendroïdale verzamelingen (d.w.z. over de combinatoriek van bomen). Ter vereenvoudiging van onze argumenten ontwikkelen we in hoofdstuk 4 een combinatorische methode om resultaten te bewijzen over dendoïdale 'anodyne extensions'. Deze techniek kan op zichzelf al worden gezien als resultaat, aangezien het ook toepasbaar is buiten de context waarin we het in de latere hoofdstukken gebruiken.

We geven twee constructies van de stabiele modelstructuur. De eerste constructie is eenvoudiger en geeft bovendien een karakterisatie van de 'fibrations' tussen 'fibrant objects'. Deze constructie is gebaseerd op standaard model-categorische argumenten en wordt gegeven in hoofdstuk 5.

Hoofdstuk 6 beschrijft een tweede constructie, die is gebaseerd op het werk van G. Heuts. Deze aanpak maakt het mogelijk om te laten zien dat de stabiele modelstructuur op dendroïdale verzamelingen Quillen equivalent is aan een modelstructuur op E_∞-ruimtes, waar de fibrant objects worden gegeven door de groepachtige E_∞-ruimtes. De equivalentie met groepachtige E_∞-ruimtes (of connectieve spectra) kan worden gezien als een meetkundige realisatie van dendroïdale verzamelingen. Bovendien biedt dit nieuwe mogelijkheden om (het connectieve deel van) klassieke stabiele homotopie theorie te bestuderen.

De resultaten uit hoofdstuk 7 geven hier een goed voorbeeld van. In dit laatste hoofdstuk beschrijven we de homologiegroepen van dendroïdale verzamelingen. Deze homologietheorie vormt een uitbreiding van de klassieke homologietheorie van simpliciale verzamelingen (m.a.w. de singuliere homologie van ruimtes). Deze generalisatie is subtieler dan men in eerste instantie zou verwachten: de combinatoriek van planaire bomen suggereert het gebruik van een bepaalde tekenconventie, maar we dienen met niet-planaire bomen te werken. Na de homologiegroepen te hebben gedefinieerd laten we zien dat deze homotopie-invariant zijn en overeenkomen met de standaard homologiegroepen van het bijbehorend connectieve spectrum. De resultaten uit hoofdstukken 6 en 7 zijn verkregen in samenwerking met T. Nikolaus.

Curriculum vitae

Matija Bašić was born on August 21, 1985 in Zagreb, Croatia. He went to elementary school in Sveta Nedelja near Zagreb and after that he finished "V. gimnazija" in Zagreb.

He studied at the Department of Mathematics of the Faculty of Natural Sciences at the University of Zagreb. He was awarded the Dean's prize for academic achievements and the Rector's prize for the article "Derivations of the inner symmetric algebra and the Weyl algebra in the Loday-Pirashvili tensor category". He graduated in 2008 under the supervision of Zoran Škoda with a Diploma thesis titled "Algebras in categories of chain complexes".

In 2008, he enrolled in the PhD programme and started working as a teaching and research assistant at the Department of Mathematics in Zagreb. As a teaching assistant he gave lectures for 10 different courses and as a PhD student he finished 6 PhD courses.

In 2011, he joined the Topology group under the supervision of Ieke Moerdijk and has spent the academic year 2011/12 at Radboud University in Nijmegen. His visit to Nijmegen was partly funded by a scholarship by the Croatian Science Foundation. Since 2011, he works on abstract homotopy theory and dendroidal sets. After a year spent in Nijmegen, he continued to work at the same position in Zagreb and often visits the Radboud University. In 2014, he published a joint paper with Thomas Nikolaus titled "Dendroidal sets as models for connective spectra".

He has been active in mathematical competitions since elementary school. In 2004 he won a bronze medal as a contestant at the International Mathematics Olympiad. Currently he is a member of the National Committee for Mathematical Competitions in Croatia and works with the Croatian IMO and MEMO team.

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In October 2011, I came to Nijmegen and started working on dendroidal sets under the supervision of professor Ieke Moerdijk. After spending an academic year in Nijmegen, I have returned to my teaching position in Zagreb. Nevertheless, I have continued to work on dendroidal sets and I have been meeting with Ieke regularly in Nijmegen, and occasionally in other places. All of this required careful planning and additional effort. I am grateful that Ieke supported my choice to finish the writing of this thesis in a less conventional way and patiently guided me throughout that process.

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The distinction of cases in Theorem 4.3.1 and 4.3.2, the definition of weak simplicial model categories and the presentation of Chapter 5 are based on parts of the article “On the equivalence between Luries model and the dendroidal model for infinity-operads” by Gijs Heuts, Vladimir Hinich and Ieke Moerdijk. I am grateful to Gijs and Ieke for making the early versions of that article available to me.

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